A "MIXED MODEL" FOR THE ANALYSIS OF VARIANCE[1]

By Henry Scheffé

University of California, Berkeley

1. Summary. A "mixed model" is proposed in which the problem of the appropriate assumptions to make about the joint distribution of the random main effects and interactions is solved by letting this joint distribution follow from more basic and "natural" assumptions about the cell means. The expectations of the mean squares ordinarily calculated turn out, with suitable definition of the variance components, to have the same values as those usually found in more restrictive models, and some of the customary tests and confidence intervals are justified, but some aspects appear to be novel. For example, the over-all test found for the fixed main effects and the associated multiple-comparison method require Hotelling's T^2.

2. Introduction. We consider K replications of a two-way layout with I rows and J columns ($I > 1$, $J > 1$, $K \geqq 1$), the rows corresponding to levels of a "Model I" [4] factor A, whose effects we wish to regard as fixed effects, and the columns corresponding to the levels of a "Model II" factor B, whose effects we wish to regard as random effects. We let y_{ijk} denote the kth measurement in the i, j cell (the intersection of the ith row and jth column). Throughout this paper, i and i', as subscripts or indices of summation, will range over the integers from 1 to I; j, j', and j'' will range from 1 to J, etc., unless otherwise indicated.

As an illustration, we may imagine an experiment involving I different makes of machines and J workers regarded as a sample from a large population of workers. Each worker is put on each machine for K days during the experiment and y_{ijk} is a measurement of the output of the jth worker the kth day he is on the ith machine. It is customary in the analysis of variance to write

$$y_{ijk} = \mu + \alpha_i + b_j + c_{ij} + e_{ijk}, \tag{1}$$

where the general mean μ and the row effects $\{\alpha_i\}$ are constants, about which we may assume without loss of generality that $\sum_i \alpha_i = 0$, and where the column effects $\{b_j\}$, interactions $\{c_{ij}\}$, and "errors" $\{e_{ijk}\}$ are random variables about whose joint distribution certain assumptions are made. The usual assumption that the set $\{e_{ijk}\}$ is statistically independent of the set $\{b_j, c_{ij}\}$ seems acceptable to the writer in many applications, but the further assumptions usually made on the $\{b_j\}$ and $\{c_{ij}\}$ seem to him unsatisfactory,[2] as being *ad hoc*, or too restrictive, or not sufficiently complete. For example, the usual assumption that the $\{c_{ij}\}$ are statistically independent of the $\{b_j\}$ is *ad hoc*, the frequent assumption that

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[2] An example is [12].

all the $\{c_{ij}\}$ are independent is too restrictive, and the additional assumptions stated by those who take $\sum_i c_{ij} = 0$ (all j) are sometimes insufficient even to determine the expected values of the mean squares usually employed.

3. The model. We propose to avoid the unpleasant assumptions as follows: We will assume that

$$y_{ijk} = m_{ij} + e_{ijk}, \tag{2}$$

where the set of errors $\{e_{ijk}\}$ is statistically independent of the set $\{m_{ij}\}$ of "true" cell means.

About the set of errors $\{e_{ijk}\}$, we assume that they are independently and identically distributed with zero means and variance σ_e^2. (This assumption can obviously be lightened without affecting the validity of the expected mean squares and unbiased estimates derived in Section 5, which depends only on the first and second moments of the set $\{m_{ij}, e_{ijk}\}$, and for which it is sufficient that the set $\{e_{ijk}\}$ have zero means, zero correlations, and a common variance σ_e^2.)

The writer hopes that the assumptions about the joint distribution of the set $\{m_{ij}\}$, to be stated below, will be found acceptable. The main effects $\{b_j\}$ and interactions $\{c_{ij}\}$ will then be defined in terms of the $\{m_{ij}\}$ in a natural way, and the joint distribution of the set $\{b_j, c_{ij}\}$ will thus be determined. Some parts of this program and its implications have also been developed earlier by others, as we shall be able to indicate more conveniently at the end of this paper.

Our basic assumption on the rectangular array of the $\{m_{ij}\}$ is that the J columns are distributed independently like a vector random variable m with I components, $m_1, \cdots, m_I$. Thus, in the above illustration of machines and workers, for each worker in the population there is a vector whose I components are his "true means" on the I machines, and the random vector m has the distribution generated by this population of workers, idealized as an infinite population. The J columns of the array $\{m_{ij}\}$ are the J vectors belonging to the J workers in the experiment, assumed to be a random sample from the population of workers.

About the components $\{m_i\}$ of the random vector m, we shall always assume that the I variances are finite. Sometimes we shall also make the normality assumption

($\mathfrak{N}$): The $\{m_i\}$ have a joint normal distribution, and the $\{e_{ijk}\}$ are also jointly normal.

We shall also have occasion to refer to a symmetry assumption

($\mathcal{S}$): The $\{m_i\}$ have equal variances and equal covariances.

We will refer to the assumption ($\mathcal{S}$) as a limiting case in which certain relations become simpler or clearer, but we do *not* recommend it in applications—where there is usually no real symmetry corresponding to this assumption. Thus, in our illustration, two machines might be very similar (perhaps of the same make and model), but very different from the other machines. Further objections to assuming ($\mathcal{S}$) will arise when we consider below the finite analogue of the infinite population of vectors associated with the random vector m.

4. Definition of effects and variance components. The "true" mean for the ith level of the factor A (the reader may find it easier to substitute " 'true' mean for the ith machine") is defined to be

$$\mu_i = E(m_i), \tag{3}$$

and the "true" general mean, to be

$$\mu = \mu_., \tag{4}$$

where the dot notation here and elsewhere signifies that the arithmetic average has been taken over the subscript which the dot replaces, that is, $\mu_. = \sum_i \mu_i / I$. The main effect of the ith level of A, or the ith row effect, is defined to be

$$\alpha_i = \mu_i - \mu, \tag{5}$$

so that $\sum_i \alpha_i = 0$. The "true" meae for the jth level of factor B (for the jth worker in the experiment) is defined to be $m_{.j}$, and the main effect of the jth level of B, or the jth column effect, to be

$$b_j = m_{.j} - \mu. \tag{6}$$

Finally, the interaction effect, c_{ij}, of the jth level of B with the ith level of A is defined to satisfy the equation

$$m_{ij} = \mu + \alpha_i + b_j + c_{ij}, \tag{7}$$

or

$$c_{ij} = m_{ij} - m_{.j} - \alpha_i, \tag{8}$$

so that $\sum_i c_{ij} = 0$ (all j).

We see that the J vectors with $I + 1$ components $b_j, c_{1j}, \cdots, c_{Ij}$ are independently distributed like the random vector with components $b, c_1, \cdots, c_I$ defined in terms of the basic vector m as follows:

$$b = m_. - \mu, \tag{9}$$

$$c_i = m_i - m_. - \alpha_i. \tag{10}$$

We note that b and the $\{c_i\}$ have zero means, and that their variances and covariances depend on the elements of the covariance matrix $(\sigma_{ii'})$ of the vector m in the following way:

$$\operatorname{var}(b) = \sigma_{..}, \tag{11}$$

$$\operatorname{cov}(c_i, c_{i'}) = \sigma_{ii'} - \sigma_{i.} - \sigma_{i'.} + \sigma_{..}, \tag{12}$$

$$\operatorname{cov}(b, c_i) = \sigma_{i.} - \sigma_{..}. \tag{13}$$

The main effects $\{b_j\}$ and interactions $\{c_{ij}\}$ in (1) thus have zero means, and the variances and covariances within a set $\{b_j, c_{1j}, \cdots, c_{Ij}\}$ are given by (11), (12), (13), while the covariance of any member of this set with any member of the set $\{b_{j'}, c_{1j'}, \cdots, c_{Ij'}\}$ is zero for $j \neq j'$.

We shall be led to the appropriate definitions of the "variance components" σ_A^2, σ_B^2, σ_{AB}^2, by way of the analogy with the "finite model", where the vector m can take on only one of a finite number, Q, of values, the qth having components, say, $\mu_{1q}, \cdots, \mu_{Iq}$. For the corresponding $I \times Q$ rectangular array, the usual definitions, chosen to give the simplest formulas for the expectations of the mean squares customarily computed, are

$$\sigma_A^2 = (I-1)^{-1} \sum_i (\mu_{i\cdot} - \mu_{\cdot\cdot})^2, \tag{14}$$

$$\sigma_B^2 = (Q-1)^{-1} \sum_q (\mu_{\cdot q} - \mu_{\cdot\cdot})^2, \tag{15}$$

$$\sigma_{AB}^2 = (I-1)^{-1} (Q-1)^{-1} \sum_i \sum_q (\mu_{iq} - \mu_{i\cdot} - \mu_{\cdot q} + \mu_{\cdot\cdot})^2. \tag{16}$$

If we regard our previous infinite model as a limiting case of the finite model as $Q \to \infty$, we see that the analogues of the above formulas are to be found by replacing μ_{iq}, $\mu_{i\cdot}$, $\mu_{\cdot q}$, $\mu_{\cdot\cdot}$, $(Q-1)^{-1}\sum_q$ by m_i, μ_i, $m_\cdot$, μ, E, respectively, and we are led to the following definitions, which we shall adopt for the infinite model:

$$\sigma_A^2 = (I-1)^{-1} \sum_i \alpha_i^2, \tag{17}$$

$$\sigma_B^2 = \operatorname{var}(b), \tag{18}$$

$$\sigma_{AB}^2 = (I-1)^{-1} \sum_i \operatorname{var}(c_i). \tag{19}$$

The variance components σ_B^2 and σ_{AB}^2 may be expressed in terms of the elements of the covariance matrix $(\sigma_{ii'})$ of the vector m, from (11) and (12),

$$\sigma_B^2 = \sigma_{\cdot\cdot}, \tag{20}$$

$$\sigma_{AB}^2 = (I-1)^{-1} \sum_i (\sigma_{ii} - \sigma_{\cdot\cdot}). \tag{21}$$

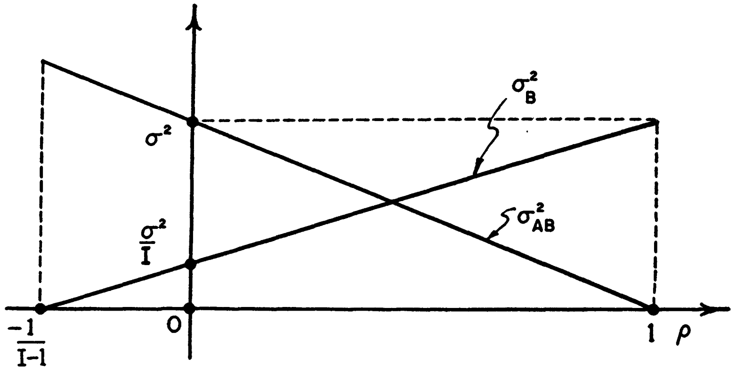
We note that $\sigma_B^2 = 0$ if and only if $b = 0$ (we omit the phrase "with probability one" here and elsewhere where it obviously applies), that is, if and only if the basic vector m has a degenerate distribution satisfying $\sum_i m_i = \text{constant} = I\mu$. Also, $\sigma_{AB}^2 = 0$ if and only if $\operatorname{var}(c_i) = 0$ for all i, or $c_i = 0$ for all i, or $m_i = m_\cdot + \alpha_i$; that is, except for additive constants $\{\alpha_i\}$, the random variables m_i are *identical* (not just identically distributed). Some further insight into our definitions of the random main and interaction effects and their variance components may be obtained by considering the symmetric case ($\mathcal{S}$) where $\sigma_{ii'} = \rho\sigma^2$ if $i \neq i'$, $\sigma_{ii} = \sigma^2$. Then, from (20) and (21),

$$\sigma_B^2 = \sigma^2 [1 + \rho(I-1)]/I, \tag{22}$$

$$\sigma_{AB}^2 = \sigma^2 (1 - \rho), \tag{23}$$

where $-(I-1)^{-1} \leqq \rho \leqq 1$. These relations are shown graphically in Fig. 1.

The previously mentioned objection to assuming ($\mathcal{S}$) in the infinite model is



Variance components σ_B^2 and σ_{AB}^2 in symmetric case (S)

FIG. 1

that its analogue in the finite model is the fulfillment of the following $\frac{1}{2}I(I + 1)-2$ conditions: If $g_{ii'}$ denotes

$$(24) \qquad \sum_q (\mu_{iq} - \mu_{i\cdot})(\mu_{i'q} - \mu_{i'\cdot}),$$

then all g_{ii} are equal, and all $g_{ii'}$ with $i \neq i'$ are equal. There would seem to be nothing in most applications to justify this.

5. Expected mean squares and point estimates. We shall consider the customary sums of squares—namely, those for rows, columns, interaction, and error, which we shall denote by $(SS)_A$, $(SS)_B$, $(SS)_{AB}$, $(SS)_e$, respectively—and the corresponding mean squares,

$$(25) \qquad (MS)_A = (I - 1)^{-1}(SS)_A,$$

$$(26) \qquad (MS)_B = (J - 1)^{-1}(SS)_B,$$

$$(27) \qquad (MS)_{AB} = (I - 1)^{-1}(J - 1)^{-1}(SS)_{AB},$$

$$(28) \qquad (MS)_e = I^{-1}J^{-1}(K - 1)^{-1}(SS)_e,$$

where

$$(29) \qquad (SS)_A = JK\sum_i (y_{i\cdot\cdot} - y_{\cdot\cdot\cdot})^2,$$

$$(30) \qquad (SS)_B = IK\sum_j (y_{\cdot j\cdot} - y_{\cdot\cdot\cdot})^2,$$

$$(31) \qquad (SS)_{AB} = K\sum_i \sum_j (y_{ij\cdot} - y_{i\cdot\cdot} - y_{\cdot j\cdot} + y_{\cdot\cdot\cdot})^2,$$

$$(32) \qquad (SS)_e = \sum_i \sum_j \sum_k (y_{ijk} - y_{ij\cdot})^2.$$

In addition, we shall need the contribution to $(SS)_{AB}$ from the ith row,

$$(33) \qquad (SS)_{AB,i} = K\sum_j (y_{ij\cdot} - y_{i\cdot\cdot} - y_{\cdot j\cdot} + y_{\cdot\cdot\cdot})^2,$$

and its mean square

$$(MS)_{AB,i} = (J - 1)^{-1}(SS)_{AB,i}. \tag{34}$$

In deriving the expected mean squares we will utilize the following three formulas for a set of independently and identically distributed random variables, $x_1, \cdots, x_N$, with variance σ_x^2:

$$\mathrm{var}\,(x.) = N^{-1}\sigma_x^2, \tag{35}$$

$$\mathrm{var}\,(x_n - x.) = (1 - N^{-1})\sigma_x^2, \tag{36}$$

$$\sum_n E(x_n - x.)^2 = (N - 1)\sigma_x^2. \tag{37}$$

It is convenient to define now

$$\hat{\alpha}_i = y_{i\cdots} - y_{\cdots}. \tag{38}$$

We have from (1),

$$\hat{\alpha}_i = \alpha_i + c_{i\cdot} + e_{i\cdot\cdot} - e_{\cdots}, \tag{39}$$

since $c_{\cdot j} = 0$ and hence $c_{\cdot\cdot} = 0$. It follows that

$$E(\hat{\alpha}_i) = \alpha_i \tag{40}$$

and

$$\begin{aligned} \mathrm{var}\,(\hat{\alpha}_i) &= \mathrm{var}\,(c_{i\cdot}) + \mathrm{var}\,(e_{i\cdot\cdot} - e_{\cdots}) \\ &= J^{-1}\,\mathrm{var}\,(c_i) + (1 - I^{-1})\,\mathrm{var}\,(e_{i\cdot\cdot}), \end{aligned} \tag{41}$$

from (35) and (36). Again from (35),

$$\mathrm{var}\,(\hat{\alpha}_i) = J^{-1}\,[\mathrm{var}\,(c_i) + K^{-1}\,(1 - I^{-1})\sigma_e^2]. \tag{42}$$

Writing

$$(SS)_A = JK\sum_i \hat{\alpha}_i^2, \tag{43}$$

we may substitute (42) in

$$E(SS)_A = JK\sum_i\,[\mathrm{var}\,(\hat{\alpha}_i) + \alpha_i^2] \tag{44}$$

to get

$$E(SS)_A = K\sum_i \mathrm{var}\,(c_i) + (I - 1)\,\sigma_e^2 + JK\sum_i \alpha_i^2. \tag{45}$$

Using the definitions (17) and (19), we then find that

$$E(MS)_A = JK\sigma_A^2 + K\sigma_{AB}^2 + \sigma_e^2. \tag{46}$$

After substituting (1) into (30) we have

$$(SS)_B = IK\sum_j (b_j - b. + e_{\cdot j\cdot} - e_{\cdots})^2, \tag{47}$$

and so from (37),

$$(48)\qquad E(SS)_B = IK(J-1)\operatorname{var}(b_j + e_{\cdot j\cdot}) = IK(J-1)(\sigma_B^2 + I^{-1}K^{-1}\sigma_e^2);$$

hence

$$(49)\qquad E(MS)_B = IK\sigma_B^2 + \sigma_e^2.$$

Substitution of (1) in (33) gives

$$(50)\qquad (SS)_{AB,i} = K\sum_j (c_{ij} - c_{i\cdot} + e_{ij\cdot} - e_{i\cdot\cdot} - e_{\cdot j\cdot} + e_{\cdot\cdot\cdot})^2;$$

whence

$$(51)\qquad E(SS)_{AB,i} = K\sum_j E(c_{ij} - c_{i\cdot})^2 + K\sum_j E(e_{ij\cdot} - e_{i\cdot\cdot} - e_{\cdot j\cdot} + e_{\cdot\cdot\cdot})^2.$$

Call the last term a_i. It is clear that the value of a_i does not depend on i, and it is known from Model I theory that $\sum_i a_i = (I-1)(J-1)\sigma_e^2$. Thus, $a_i = (1 - I^{-1})(J-1)\sigma_e^2$. By (37), the first term on the right of (51) may be written $K(J-1)\operatorname{var}(c_i)$. Hence,

$$(52)\qquad E(SS)_{AB,i} = (J-1)[K\operatorname{var}(c_i) + (1 - I^{-1})\sigma_e^2],$$

$$(53)\qquad E(MS)_{AB,i} = K\operatorname{var}(c_i) + (1 - I^{-1})\sigma_e^2.$$

Summing (52) over i and dividing by $(I-1)(J-1)$, we get

$$(54)\qquad E(MS)_{AB} = K\sigma_{AB}^2 + \sigma_e^2.$$

Finally, if we rewrite (32) as

$$(55)\qquad (SS)_e = \sum_i\sum_j\sum_k (e_{ijk} - e_{ij\cdot})^2,$$

we see that for $K > 1$

$$(56)\qquad E(MS)_e = \sigma_e^2.$$

We shall use the noun "estimate" always to mean "unbiased estimate." The above formulas for the expected mean squares lead to the following estimates of σ_B^2, σ_{AB}^2, σ_e^2, respectively, if $K > 1$:

$$(57)\qquad \hat\sigma_B^2 = (IK)^{-1}[(MS)_B - (MS)_e],$$

$$(58)\qquad \hat\sigma_{AB}^2 = K^{-1}[(MS)_{AB} - (MS)_e],$$

$$(59)\qquad \hat\sigma_e^2 = (MS)_e.$$

An estimate of α_i is the $\hat\alpha_i$ defined by (38); an estimate of its variance (42) is $J^{-1}K^{-1}(MS)_{AB,i}$. An estimate of $\mu_i = \mu + \alpha_i$ is $y_{i\cdot\cdot}$; an estimate of its variance is $J^{-1}\hat\tau_{ii}$, where $\hat\tau_{ii}$ is defined by (62) below. An estimate of $\alpha_i - \alpha_{i'}$ is $y_{i\cdot\cdot} - y_{i'\cdot\cdot}$; an estimate of its variance is

$$S_{ii'} = J^{-1}(J - 1)^{-1}\sum_j (y_{ij\cdot} - y_{i'j\cdot} - y_{i\cdot\cdot} + y_{i'\cdot\cdot})^2. \tag{60}$$

In order to estimate the covariance matrix $(\sigma_{ii'})$ of the basic vector m, we note that the J columns of cell means $\{y_{ij\cdot}\}$ are distributed independently like a random vector $u = m + v$, where v is statistically independent of m and has the distribution of the vector with components $e_{1j\cdot}, \cdots, e_{Ij\cdot}$ (which distribution does not depend on j). It follows that the covariance matrix of u is $(\hat{\tau}_{ii'})$, where

$$\tau_{ii'} = \sigma_{ii'} + \delta_{ii'}K^{-1}\sigma_e^2, \tag{61}$$

and $\delta_{ii'} = 1$ if $i = i'$, 0 if $i \neq i'$. An estimate of $\tau_{ii'}$ is the sample covariance of the ith row of cell means $\{y_{ij\cdot}\}$ with the i'-th row,

$$\hat{\tau}_{ii'} = (J - 1)^{-1}\sum_j (y_{ij\cdot} - y_{i\cdot\cdot})(y_{i'j\cdot} - y_{i'\cdot\cdot}), \tag{62}$$

and hence if $K > 1$, an estimate of $\sigma_{ii'}$ is

$$\hat{\sigma}_{ii'} = \hat{\tau}_{ii'} - \delta_{ii'}K^{-1}\hat{\sigma}_e^2. \tag{63}$$

We remark that if we estimate σ_B^2 and σ_{AB}^2 by substituting the estimates (63) in (20) and (21), we get the same estimates as before in (57) and (58).

6. Distribution theory under the normality assumption. Under the normality assumption $(\mathfrak{N})$ of Section 3, the four sums of squares $(SS)_A$, $(SS)_B$, $(SS)_{AB}$, $(SS)_e$ are pairwise independent, except for the pair $(SS)_B$, $(SS)_{AB}$. We shall prove this for the pair $(SS)_A$, $(SS)_{AB}$; the independence of the other pairs may be verified similarly.

Let us write

$$(SS)_A = JK\sum_{i'} L_{i'}^2, \tag{64}$$

$$(SS)_{AB} = K\sum_i \sum_j L_{ij}^2, \tag{65}$$

where

$$L_{i'} = A_{i'} + B_{i'}, \tag{66}$$

$$L_{ij} = A_{ij} + B_{ij}, \tag{67}$$

$$A_{i'} = \alpha_{i'} + c_{i'\cdot}, \tag{68}$$

$$B_{i'} = e_{i'\cdot\cdot} - e_{\cdot\cdot\cdot}, \tag{69}$$

$$A_{ij} = c_{ij} - c_{i\cdot}, \tag{70}$$

$$B_{ij} = e_{ij\cdot} - e_{i\cdot\cdot} - e_{\cdot j\cdot} + e_{\cdot\cdot\cdot}. \tag{71}$$

Then it suffices because of the joint normality of the set $\{L_{i'}, L_{ij}\}$ to prove cov $(L_{i'}, L_{ij}) = 0$ for all i', i, j. Now, any B just defined is independent of any A because of our assumption that the set $\{e_{ijk}\}$ is independent of the set $\{m_{ij}\}$. Furthermore, $B_{i'}$ and B_{ij} are orthogonal by the familiar Model I theory. Hence, it remains only to show cov $(A_{i'}, A_{ij}) = 0$:

$$(72)\quad \operatorname{cov}(A_{i'}, A_{ij}) = E[c_{i'}.(c_{ij} - c_{i}.)] = E[J^{-1}\sum_{j'} c_{i'j'}(c_{ij} - J^{-1}\sum_{j''} c_{ij''})]$$
$$= J^{-1}\sum_{j'} E(c_{i'j'}c_{ij}) - J^{-2}\sum_{j'}\sum_{j''} E(c_{i'j'}c_{ij''})$$
$$= J^{-1}E(c_{i'}c_{i}) - J^{-1}E(c_{i'}c_{i}) = 0,$$

since $E(c_{ij}c_{i'j'}) = \delta_{jj'}E(c_i c_{i'})$.

The above proof shows also that $\hat{\alpha}_i$ is statistically independent of $(SS)_{AB,i}$, since $\hat{\alpha}_i = L_i$ and $(SS)_{AB,i} = K\sum_j L_{ij}^2$.

From (55) it follows that $(SS)_e$ is distributed as σ_e^2 times χ^2 with $IJ(K - 1)$ d.f. To see that $(SS)_B$ is distributed as $E(MS)_B$ times χ^2 with $J - 1$ *d.f.*, write $f_j = b_j + e_{\cdot j\cdot}$ in (47), so that $(SS)_B = IK\sum_j (f_j - f_{\cdot})^2$, where the set $\{f_j\}$ are independently $N(0, \sigma_f^2)$ (normal with mean 0 and variance σ_f^2) with $\sigma_f^2 = \sigma_B^2 + I^{-1}K^{-1}\sigma_e^2$, and hence $(SS)_B$ is $IK\sigma_f^2$ times χ^2 with $J - 1$ d.f. Similarly, putting $g_j = c_{ij} + e_{ij\cdot} - e_{\cdot j\cdot}$ in (50), we find $(SS)_{AB,i}$ is $E(MS)_{AB,i}$ times χ^2 with $J - 1$ d.f. It may be shown that for $I > 2$, $(SS)_A$ and $(SS)_{AB}$ are not, in general, distributed as a constant times noncentral (which includes central) χ^2. However, under the hypothesis H_{AB} that $\sigma_{AB}^2 = 0$, all $c_{ij} = 0$, so $(SS)_{AB}$ becomes simply

$$(73)\qquad K\sum_i\sum_j (e_{ij\cdot} - e_{i\cdot\cdot} - e_{\cdot j\cdot} + e_{\cdots})^2,$$

which is known from Model I theory to be distributed as σ_e^2 times χ^2 with $(I - 1)(J - 1)$ d.f.

The obvious consequence of our assumptions, that the J columns of cell means $\{y_{ij\cdot}\}$ are independently distributed like an I-variate normal vector with means $\mu_1, \cdots, \mu_I$ and covariance matrix $(\tau_{ii'})$ given by (61), we shall utilize in the next section.

7. Tests and confidence intervals. Suppose first that $K > 1$. Then the χ^2-distribution of $(SS)_e/\sigma_e^2$ affords confidence intervals for σ_e^2 in the usual way.

Since the quotient of $(MS)_B/(IK\,\sigma_B^2 + \sigma_e^2)$ by $(MS)_e/\sigma_e^2$ has the F-distribution with $J - 1$ and $IJ(K - 1)$ d.f., confidence intervals for σ_B^2/σ_e^2 are available as well as tests of the hypothesis that $\sigma_B^2 = 0$, or, more generally, that $\sigma_B^2/\sigma_e^2 \leqq c$, a given constant. The test at the α level of significance consists of rejecting the hypothesis if and only if $(MS)_B/(MS)_e \geqq (IKc + 1)F_\alpha$, where F_α is the upper α point of the F-distribution. The power of the test can be expressed in terms of the (central) F-distribution.

The hypothesis $H_{AB} : \sigma_{AB}^2 = 0$ may be tested with the statistic $(MS)_{AB}/(MS)_e$, which, under H_{AB}, has the F-distribution with $(I - 1)(J - 1)$ and $IJ(K - 1)$ d.f. Since this statistic is distributed as the quotient of a linear combination (with coefficients in general unequal) of independent χ^2 variables by another independent χ^2 variable, the power of the test is not expressible in terms of the noncentral F-distribution, but it could be approximated by use of a central F-distribution by using methods of Box [2].

We now drop the restriction $K > 1$. Even through $(MS)_A$ and $(MS)_{AB}$ are statistically independent and under the hypothesis H_A: all $\alpha_i = 0$ have the same expected values, their quotient does not, in general, have the F-distribution under H_A. A test of H_A based on Hotelling's T^2 statistic is given in the next paragraph. However, confidence intervals for a particular α_i, a particular μ_i, or a particular difference $\alpha_i - \alpha_{i'}$ (none of these selected according to the outcome of the experiment) can be based on the t-distribution with $J - 1$ d.f. of the respective quotients

$$J^{1/2}K^{1/2}(\hat{\alpha}_i - \alpha_i)/(MS)_{AB,i}^{1/2}, \tag{74}$$

$$J^{1/2}(y_{i\cdot\cdot} - \mu_i)/\hat{\tau}_{ii}^{1/2}, \tag{75}$$

$$[(\hat{\alpha}_i - \hat{\alpha}_{i'}) - (\alpha_i - \alpha_{i'})]/S_{ii'}^{1/2}, \tag{76}$$

where the denominators are defined by (34), (62), and (60).

We assume now that $J \geqq I$. To calculate Hotelling's T^2 statistic for H_A, and, in case we find it significant, to make multiple comparisons, we construct a rectangular table with $R = I - 1$ rows and J columns, the entry in the rth row and jth column being

$$d_{rj} = y_{rj\cdot} - y_{Ij\cdot}, \tag{77}$$

and we compute the R means $\{d_{r\cdot}\}$ and the $\frac{1}{2}R(R + 1)$ sums of products (which, divided by $J(J - 1)$, are estimates of the covariances of the $\{d_{r\cdot}\}$)

$$a_{rr'} = \sum_j (d_{rj} - d_{r\cdot})(d_{r'j} - d_{r'\cdot}) = \sum_j d_{rj}d_{r'j} - Jd_{r\cdot}d_{r'\cdot}. \tag{78}$$

The T^2 statistic is (except for a constant factor)

$$F = J(J - I + 1)(I - 1)^{-1}Q, \tag{79}$$

where Q is the quadratic form

$$Q = \sum_r \sum_{r'} a^{rr'}d_{r\cdot}d_{r'\cdot}, \tag{80}$$

and $(a^{rr'})$ is the matrix inverse to $(a_{rr'})$. It is not necessary actually to compute the inverse matrix, since Q may be written in a form given by Rao [10] in terms of two determinants of order R calculated from $(a_{rr'})$,

$$Q = \frac{|a_{rr'} + d_{r\cdot}d_{r'\cdot}|}{|a_{rr'}|} - 1. \tag{81}$$

The statistic F in (79) has under H_A the F-distribution with $I - 1$ and $J - I + 1$ d.f., so that if F'_α denotes the upper α point of the F-distribution with these numbers of d.f., H_A is rejected at the α level of significance if and only if $F > F'_\alpha$.

The above form of the T^2 test appears to lack symmetry, since the Ith row plays a distinguished role. It is easy to see that if instead of the $\{d_{r\cdot}\}$, any basis is used for the $(I - 1)$-dimensional space spanned by the differences $\{y_{i\cdot\cdot} - y_{i'\cdot\cdot}\}$, the same test would be obtained. A symmetric form of Q (and of the noncentrality parameter δ^2 below) was given by Hsu [7], but this form would involve more numerical calculation.

The power of the T^2 test of H_A may be expressed [7] in terms of the noncentral F-distribution: The statistic (79) is distributed as noncentral F with $I - 1$ and $J - I + 1$ d.f. and noncentrality parameter δ^2, whose value will be given below, that is, as

$$(I - 1)^{-1}(J - I + 1)\left[(x_1 + \delta)^2 + \sum_{\nu=2}^{I-1} x_\nu^2\right]\left(\sum_{\nu=I}^{J} x_\nu^2\right)^{-1}, \tag{82}$$

where the $\{x_\nu\}$ are independently $N(0, 1)$. The noncentrality parameter δ^2 has the value

$$\delta^2 = \sum_r \sum_{r'} \alpha^{rr'} \delta_r \delta_{r'}\,, \tag{83}$$

where $(\alpha^{rr'})$ is the matrix inverse to that with elements

$$\alpha_{rr'} = \operatorname{cov}(d_{r\cdot}\,, d_{r'\cdot}) = J^{-1}(\tau_{rr'} - \tau_{rI} - \tau_{r'I} + \tau_{II}), \tag{84}$$

and

$$\delta_r = \alpha_r - \alpha_I = \mu_r - \mu_I\,. \tag{85}$$

In his paper in 1931 on the T^2 test, Hotelling [6] gave an associated confidence ellipsoid. Recently, the writer [11] published a method of multiple comparison derived from the confidence ellipsoid associated with the F-test for equality of means in Model I. The same method, when based on Hotelling's confidence ellipsoid, tells us the following: Let θ be any contrast among the $\{\alpha_i\}$ or $\{\mu_i\}$, $\theta = \sum_i h_i\alpha_i = \sum_i h_i\mu_i$, where $\{h_i\}$ is any set of known constants satisfying $\sum_i h_i = 0$. Let $\hat\theta$ be the estimate $\hat\theta = \sum_i h_i y_{i\cdot\cdot} = \sum_r h_r d_{r\cdot}$, so that its variance $\sigma_{\hat\theta}^2 = \sum_r \sum_{r'} h_r h_{r'} \alpha_{rr'}$ is estimated by

$$\hat\sigma_{\hat\theta}^2 = J^{-1}(J - 1)^{-1} \sum_r \sum_{r'} h_r h_{r'} a_{rr'}\,. \tag{86}$$

Then, for the totality of contrasts $\{\theta\}$, the probability is $1 - \alpha$ that they simultaneously satisfy

$$\hat\theta - S\hat\sigma_{\hat\theta} \leqq \theta \leqq \hat\theta + S\hat\sigma_{\hat\theta}\,, \tag{87}$$

where the constant S is calculated from F'_α, the upper α point of the F-distribution with $I - 1$ and $J - I + 1$ d.f., by

$$S^2 = (I - 1)(J - 1)(J - I + 1)^{-1}F'_\alpha\,. \tag{88}$$

Whenever the T^2 test rejects H_A at significance level α, there will exist contrasts θ for which the interval (87) does not cover zero (and conversely). However, it may occasionally happen in applications that none of the contrasts thus found to be "significantly different from zero" is of any practical interest.

An interesting interpretation of the quantities $\hat\sigma_{\hat\theta}$ needed in (87), which yields an alternative way of calculating them not requiring calculation of the $\{a_{rr'}\}$ or $\{d_{rj}\}$, is the following[3]: Let $\hat\theta_j$ be the estimate of θ from the jth column, $\hat\theta_j = \sum_i h_i y_{ij\cdot}$, so $\hat\theta = \hat\theta_\cdot$; then

[3] Pointed out to me by Professor J. W. Tukey.

$$\hat{\sigma}_{\hat{\theta}}^2 = J^{-1}(J-1)^{-1} \sum_j (\hat{\theta}_j - \hat{\theta})^2. \tag{89}$$

However, if the calculations for the T^2 test of H_A have already been made, use of (87) may be faster.

8. Concluding remarks. The T^2 test of H_A and the associated method of multiple comparison are valid under less restrictive assumptions about the errors $\{e_{ijk}\}$. For instance, it would be sufficient that they be independently $N(0, \sigma_i^2)$, or, more generally, that the J vectors with components $e_{1j\cdot}, \cdots, e_{Ij\cdot}$ be independently distributed like a normal random vector with zero means and an arbitrary covariance matrix. The test and comparison method should be fairly insensitive to violation of the normality assumption ($\mathfrak{N}$), from consideration of the asymptotic distribution when $J \to \infty$.

A common practice in the analysis of variance is to employ as statistic to test a hypothesis the quotient of two independent mean squares whose expected values are equal under the hypothesis, and to refer this statistic to the F-tables with the numbers of d.f. equal to the ranks of the quadratic forms in the mean squares. According to this practice, $(MS)_A/(MS)_{AB}$ would be treated as though it had the F-distribution with $I - 1$ and $(I - 1)(J - 1)$ d.f. under H_A. A justification of this would be welcomed by the practitioner, because the computations are simpler and more familiar than those with Hotelling's T^2, but until numerical investigations are made which indicate the errors involved are tolerable, the practice should be suspect in the present case. The exact distribution of the statistic under H_A depends on unknown parameters. The distribution has been treated by McCarthy [8], but in a form that does not seem useful for $I > 3$. Some general theory for the distribution of ratios of statistically independent quadratic forms in jointly normal variables has recently been given by Box [2], and the above distribution falls under an application he made to another problem ([3] pp. 489–490), where he approximates it by an F-distribution with reduced numbers of d.f. However, these numbers of d.f. would depend on the covariance matrix (τ_{ij}) whose elements are defined by (61), and if we were to estimate these numbers from the data—with somewhat questionable effects on the resulting test and multiple comparisons method—it would require computation of the whole estimated covariance matrix $(\hat{\tau}_{ij})$ defined by (62). The amount of numerical work involved would then be comparable to that for the above exact methods utilizing the T^2 statistic.

An interesting practical conclusion from the present model is that the number J of levels of the Model II factor should be at least a few more than the number I of levels of the Model I factor, since the F statistic for the T^2 test has $J - I + 1$ d.f. in the denominator.

The writer acknowledges his inspiration from a paper by Tukey [13] in which the expected mean squares in the mixed model fall out as limiting cases of those obtained in sampling a finite model similar to the above with sampling of both rows and columns, as the population number of columns becomes infinite, and

with the population number of rows equal to the sample number. The effect of sampling the rows is to make all permutations of the rows equally probable and thus impose the symmetry condition $(\mathcal{S})$. However, this does not affect the expected mean squares we derived for A, B, AB, and e, since the formulas for the corresponding sums of squares are invariant under permutation of the rows. Wilk and Kempthorne [14] have recently calculated expected mean squares for a model somewhat resembling Tukey's, closer to the above in that only columns are sampled, but differing more in that the error term is generated solely by the actual randomization used to assign the "treatment combinations" to experimental units from a finite population, with the consequent introduction of treatment-unit interactions: If these are neglected the expected mean squares of Wilk and Kempthorne agree with Tukey's.

A multivariate normal model[4] for randomized blocks was studied by McCarthy [8] as an approximation to Neyman's [9] more realistic model reflecting the randomization actually used in the assignment of the varieties to the plots in each block. A test, implicitly assuming such a multivariate normal model for randomized blocks, and employing Hotelling's T^2 was recently proposed by Graybill [5]. A multivariate model for the analysis of variance was also considered by Box [1] in a different situation where the condition $(\mathcal{S})$ was tenable, and he included among other tests one of $(\mathcal{S})$. The application of Hotelling's T^2 statistic to test the equality of the components of the vector of means in samples from a multivariate normal population is due to Hsu [7].

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[4] When I discussed my results with Dr. Jerome Cornfield, he informed me that he and Dr. Max Halperin had also considered a multivariate model for the present problem and had been led to the T^2 test. Professor J. L. Hodges, Jr., formulated the above multivariate model before I did. A model equivalent to the above under the assumptions $(\mathfrak{N})$ and $(\mathcal{S})$ was proposed earlier by Mr. Leon Herbach in an unpublished paper, in which he derived the expectations and distributions of the usual mean squares and tests of the usual hypotheses.

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