ON THE CONVOLUTION OF DISTRIBUTIONS

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1. Summary. A systematic approach to distributions having the reproductive property (see [1] p. 171) is attempted, and necessary and sufficient conditions are given. The case of distributions depending on k (> 1) parameters is considered; it need not be a straightforward generalization of the one-parameter case.

2. Additively closed families of distributions. Let $D = D(\lambda)$ be an Abelian semi-group under addition. In particular, denote by $D(I)$, $D(I+)$, $D(r+)$, $D(R+)$, and $D(R+, 0)$ the semi-groups of integers, positive integers, positive rationals, positive reals, and nonnegative reals, respectively. Let $D(r)$, $D(R)$, $D(I+, 0)$ and $D(R+, 0)$ be defined analogously. The abbreviations c.d.f. and c.f. will be used for cumulative distribution function and characteristic function, respectively.

Definition. A one-parameter family of c.d.f.'s $F(x; \lambda)$ with $\lambda \varepsilon D$, and D as above, will be said to be *additively closed* or to *belong to the class* C_1 if, for any two elements $F(x; \lambda_1)$ and $F(x; \lambda_2)$,

$$F(x; \lambda_1) * F(x; \lambda_2) \overset{x}{\equiv} F(x; \lambda_1 + \lambda_2). \tag{1}$$

Among the following results, Theorem 1 is known in one form or another but is required here for a unified presentation. Theorems 2 and 4 are new. Generally, the k-parameter case does not seem to have been considered previously.

Theorem 1. *If (i) $\lambda \varepsilon D(I+)$ or (ii) $\lambda \varepsilon D(r+)$, a necessary and sufficient condition that a family of* c.d.f.'s *$F(x; \lambda)$ be additively closed, that is, that $F(x; \lambda) \varepsilon C_1$, is that the corresponding family of* c.f.'s is *$\phi(t; \lambda) = [f(t)]^{\lambda}$, where $f(t)$ is a* c.f. *not depending on λ. If (iii) $\lambda \varepsilon D(R+)$, and $\phi(t; \lambda)$ is continuous in λ, the same condition is again necessary and sufficient. In cases (ii) and (iii), $f(t)$ is the* c.f. *of an infinitely divisible distribution.*

Proof. The proof of sufficiency is trivial for the ensuing theorems. The three alternatives for λ are considered in turn.

(i), $\lambda \varepsilon D(I+)$. Let $f(t) = \phi(t; 1)$. Translating and iterating (1), we have, for any positive p,

$$\phi(t; p) = \phi(t; 1)\phi(t; 1) \cdots \phi(t; 1) = [f(t)]^p.$$

(ii), $\lambda \varepsilon D(r+)$. We have, from (1), $f(t) = \phi(t; 1) = [\phi(t; 1/p)]^p$. That is, the pth root of $f(t)$ is a c.f. for every positive integral p, whence $f(t)$ is the c.f. of an infinitely divisible (i.d.) distribution and hence never zero. (By the pth root is meant that branch for which $f^{1/p}(0) = 1$, which is unambiguous since

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$f(t) \neq 0$ for real t.) Again applying (1) we see that, for any positive integers p and q,

$$\phi(t; q/p) = [\phi(t; 1/p)]^q = [f(t)]^{q/p}.$$

(iii), $\lambda \varepsilon D(R+)$. Since $\phi(t; \lambda)$ is continuous in λ, it follows from (ii), by taking a sequence of positive rational numbers approaching any real nonnegative λ, that $\phi(t; \lambda) = [f(t)]^\lambda$.

If $\lambda \varepsilon D(R+)$ and the continuity assumption is removed, Theorem 1 is in general untrue. For example, let $F(x; \lambda)$ be a family of normal distributions with variance λ and mean $g(\lambda)$, where $g(\lambda)$ is a discontinuous solution of Cauchy's functional equation $g(x) + g(y) = g(x + y)$. Then

$$\phi(t; \lambda) = \exp\{itg(\lambda) - \tfrac{1}{2}\lambda t^2\}$$

is not of the form $[f(t)]^\lambda$ although $F(x; \lambda) \varepsilon C_1$.

THEOREM 2. *If* $\phi(t; \lambda)$ *for* $\lambda \varepsilon D(R+)$ *is real-valued (for real* t*), a* NSC *that* $F(x; \lambda) \varepsilon C_1$ *is that* $\phi(t; \lambda) = [f(t)]^\lambda$.

PROOF. The set of zeros of $\phi(t; \lambda)$ is independent of λ. For if $\phi(t_0; \lambda_1) = 0$ and $\lambda_2 > \lambda_1$, then

$$\phi(t_0; \lambda_2) = \phi(t_0; \lambda_2 - \lambda_1)\phi(t_0; \lambda_1) = 0. \tag{2}$$

If $\lambda_3 < \lambda_1$ and n is an integer, $[\phi(t_0; \lambda_1/n)]^n = \phi(t_0; \lambda_1) = 0$ whence $\phi(t_0; \lambda_1/n) = 0$ for every positive integer n. But for sufficiently large n, we have $\lambda_3 > \lambda_1/n$. Applying (2), we deduce $\phi(t_0; \lambda_3) = 0$.

For $\lambda = r$, a rational number, we have from Theorem 1 that $\phi(t; r) = [f(t)]^r$ with $f(t)$ never zero. It follows from the above that $\phi(t; \lambda)$ is never zero. Consequently, the properties of c.f.'s that $\phi(0; \lambda) = 1$ and that $\phi(t; \lambda)$ is continuous in t for every λ, show that $\phi(t; \lambda)$ is never negative.

Now $\psi(t; \lambda) = \log \phi(t; \lambda)$ is well defined, and, from the translated form of (1), satisfies Cauchy's functional equation. As

$$\phi(t; \lambda) = |\phi(t; \lambda)| \leqq 1,$$

$\psi(t, \lambda)$ is nonpositive whence the only solution is the continuous one $\psi(t; \lambda) = K_t\lambda$. Thus, for all real $\lambda > 0$,

$$\phi(t; \lambda) = \exp\{K_t\lambda\} = [h(t)]^\lambda.$$

Taking $\lambda = 1$, we have $\phi(t; 1) = h(t) = f(t)$, which proves the theorem.

DEFINITION. Let λ_j be an element of the Abelian semi-group (additive) D_j for $j = 1, 2, \cdots, k$. A k-parameter family of c.d.f.'s will be said to be *additively closed* or to belong to the class C_k if for any two members $F(x; \lambda_1^{(1)}, \cdots, \lambda_k^{(1)})$ and $F(x; \lambda_1^{(2)}, \cdots, \lambda_k^{(2)})$,

$$F(x; \lambda_1^{(1)}, \cdots, \lambda_k^{(1)}) * F(x; \lambda_1^{(2)}, \cdots, \lambda_k^{(2)}) \overset{x}{\equiv} F(x; [\lambda_1^{(1)} + \lambda_1^{(2)}], \cdots, [\lambda_k^{(1)} + \lambda_k^{(2)}]). \tag{3}$$

There may be a set of dormant parameters which are unaffected by the convolution, but these may simply be ignored.

In generalization of Theorem 1, we have:

THEOREM 3. *Let* $F(x; \lambda_1, \cdots, \lambda_k)$ *be a k-parameter family of* c.d.f.'s *with* $\lambda_j \varepsilon D_j$ *where* $D_j = D_j(I+,0)$, $D_j(r+,0)$ *or* $D_j(R+,0)$. *Further, let* $\phi(t; \lambda_1, \cdots, \lambda_k)$ *be continuous in all* λ_j *for which the corresponding* $D_j = D_j(R+,0)$. *Then a* NSC *that* $F \varepsilon C_k$ *is that*

$$\phi(t; \lambda_1, \lambda_2, \cdots, \lambda_k) = \prod_{j=1}^{k} [f_j(t)]^{\lambda_j},$$

where each $f_j(t)$ *is a* c.f. *independent of all* λ_j, *and is* i.d. *providing the corresponding* D_j *is* $D_j(r+,0)$ *or* $D_j(R+,0)$.

PROOF. As in Theorem 1, $\phi(t; 0, \cdots, 0, \lambda_j, 0, \cdots, 0) = G_j(t; \lambda_j) = [f_j(t)]^{\lambda_j}$. Hence,

$$\phi(t; \lambda_1, \cdots, \lambda_k) = \prod_{j=1}^{k} G_j(t; \lambda_j) = \prod_{j=1}^{k} [f_j(t)]^{\lambda_j}.$$

The inclusion of the value zero in each domain D_j immediately implies that each $f_j(t)$ is itself a c.f. The question arises whether this is necessarily so if zero is deleted. Provided the product space $D_1 \times D_2 \times \cdots \times D_k$ is suitably altered, the answer is in the negative.

THEOREM 4. *Let* $F(x; \lambda_1, \lambda_2)$ *be a two-parameter family of* c.d.f.'s *where* $\lambda_1 \varepsilon D(r+)$ *and* $\lambda_2 \varepsilon D(r)$, *with* $\lambda_1 \geqq |\lambda_2|$ *defining the parameter space. A* NSC *that* $F(x; \lambda_1, \lambda_2) \varepsilon C_2$ *is that*

$$\phi(t; \lambda_1, \lambda_2) = \prod_{j=1}^{2} [f_j(t)]^{\lambda_j},$$

where $f_2(t)$ *is not necessarily a* c.f.

PROOF. Since for any positive integer n,

$$[\phi(t; 1/n, 1/n]^n = \phi(t; 1, 1) = r(t), \quad \text{(say)},$$

$r(t)$ is an i.d.c.f., and $\phi(t; p/n, p/n) = [r(t)]^{p/n}$ for any positive integers p and n. Similarly,

$$[\phi(t; 1/m, 0)]^m = \phi(t; 1, 0) = f_1(t), \quad \text{(say)},$$

where $f_1(t)$ is an i.d.c.f. Hence $\phi(t; \lambda_1, 0) = [f_1(t)]^{\lambda_1}$ for $\lambda_1 \varepsilon D(r+)$. Let $f_2(t) = r(t)/f_1(t)$. Then $f_2(t)$ is defined and nonzero for all real t.

Now if $\lambda_2 > 0$ and $\lambda_1 = \lambda_2$, we have

$$\phi(t; \lambda_1, \lambda_2) = \phi(t; \lambda_1, \lambda_1) = [r(t)]^{\lambda_1} = [f_1(t)]^{\lambda_1}[f_2(t)]^{\lambda_2}.$$

If $\lambda_2 > 0$ but $\lambda_1 \neq \lambda_2$,

$$\phi(t; \lambda_1, \lambda_2) = \phi(t; \lambda_1 - \lambda_2, 0)\phi(t; \lambda_2, \lambda_2) = \prod_{j=1}^{2} [f_j(t)]^{\lambda_j}.$$

Furthermore,

$$\phi(t; \lambda_1, \lambda_2)\phi(t; \lambda_1, -\lambda_2) = \phi(t; 2\lambda_1, 0).$$

Substituting in this last equation and solving, we find

$$\phi(t; \lambda_1, -\lambda_2) = [f_1(t)]^{\lambda_1}[f_2(t)]^{-\lambda_2},$$

completing the proof. It is clear from the definition that $f_2(t)$ need not be a c.f.

The following example illustrates Theorem 4. Define $\phi_j(t) = \exp\{\alpha_j(e^{it} - 1)\}$ with $\alpha_1 > 0$ and $\alpha_2 \geqq 0$, and rational for $j = 1$ or 2. Let $\lambda_1 = \alpha_1 + \alpha_2$ and $\lambda_2 = \alpha_1 - \alpha_2$, with

$$\phi(t; \lambda_1, \lambda_2) = \phi_1(t)\phi_2(-t) = [e^{(\cos t)-1}]^{\lambda_1}[e^{i\sin t}]^{\lambda_2}. \tag{4}$$

The parameter space is given by $\lambda_1 \varepsilon D(r+)$ and $\lambda_2 \varepsilon D(r)$, with $\lambda_1 \geqq |\lambda_2|$. Finally, $\exp\{i \sin t\}$ cannot be a c.f. as

$$\exp\{i \sin t\} = 1 + it + \tfrac{1}{2}i^2t^2 + o(t^2), \tag{5}$$

which would imply that the corresponding r.v. had unit mean and zero variance and hence (by the uniqueness theorem for c.f.'s) a c.f. equal to $\exp\{it\}$.

The proof of the following generalization of Theorem 4 is very similar and will not be given.

THEOREM 5. *Let* $F(x; \lambda_1, \lambda_2, \cdots, \lambda_k)$ *be a k-parameter family of* c.d.f.'s, *where* $\lambda_1 \varepsilon D(r+)$ *and* $\lambda_j \varepsilon D(r+, 0)$, *with* $j \geqq 2$ *and* $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_k$ *defining the parameter space. A* NSC *that* $F(x; \lambda_1, \lambda_2, \cdots, \lambda_k) \varepsilon C_k$ *is*

$$\phi(t; \lambda_1, \lambda_2, \cdots, \lambda_k) = \prod_{j=1}^{k} [f_j(t)]^{\lambda_j}.$$

where $f_j(t)$ *is not necessarily a* c.f. *for* $j > 1$.

The last two theorems could be extended to real values of λ under suitable assumptions.

REFERENCES

[1] H. CRAMÉR, "Problems in probability theory," *Ann. Math. Stat.*, Vol. 18 (1947), pp. 165–193.

[2] H. CRAMÉR, *Mathematical Methods of Statistics*, Princeton University Press, 1946.