ON THE PROBLEM OF CONSTRUCTION OF ORTHOGONAL ARRAYS[1]

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1. Summary. A method of constructing orthogonal arrays of an arbitrary strength t is formulated. This method is a modification of the method based on differences, formulated by R. C. Bose [1] for the purpose of constructing orthogonal arrays of strength 2. It is shown further that each of the multifactorial designs of R. L. Plackett and J. P. Burman [2], in which each factor takes on two levels, provide a scheme for constructing orthogonal arrays of strength 3, consisting of the maximum possible number of rows.

An orthogonal array (36, 13, 3, 2) is constructed. The method used for its construction cannot lead to a number of constraints greater than 13. It is known however [3] that 16 is an upper bound for the number of constraints in this case; the problem as to whether this bound can actually be attained remains unsolved.

2. Introduction. The theory of orthogonal arrays was developed by R. C. Bose and K. A. Bush [3].[2] Following their definition a $k \times N$ matrix A with entries from a set Σ of $s \geqq 2$ elements, is called an orthogonal array of size N, k constraints, s levels, strength t; if each $t \times N$ submatrix of A contains all possible $t \times 1$ column vectors with the same frequency λ. Such an array is denoted by the symbol (N, k, s, t), and the number λ is called the index of the array. Clearly $N = \lambda s^t$.

Hotelling [4] considered orthogonal arrays of strength two and two levels from the point of application of factorial designs to chemistry. His work was continued by Mood [5]. Plackett and Burman [2] studied orthogonal arrays, in their terminology multifactorial designs, from the point of view of an application in physical and industrial research. Their work provided a complete solution to the problem suggested by Hotelling. Some of the designs constructed by Plackett and Burman were analysed by Kempthorne [6], and Brownlee and Loraine [7]. They pointed out that in the cases considered the main effects are confounded with the first order interactions; hence the designs are inadequate when the assumption that there is no interaction between the factors is unrealistic. These remarks can be extended to all the designs constructed by

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[2] R. C. Bose asked me to mention that the theory of orthogonal arrays was started by C. R. Rao. The references to his papers are: (1) C. R. Rao, "Hypercubes of strength, d' leading to confounded designs in factorial experiments," *Bull. Calcutta Math. Soc.*, Vol. 38 (1946), pp. 67–78. (2) C. R. Rao, "Factorial experiments derivable from combinatorial arrangements of arrays," *J. Roy. Stat. Soc., Suppl.*, Vol. 9 (1947), pp. 67–78.

Plackett and Burman, in the sense that in each of them the main effects are confounded with at least some of the degrees of freedom belonging to the first order interactions. However this deficiency can be removed by constructing certain designs of strength greater than or equal to 3. All the designs of Plackett and Burman in which s is equal to 2, yield such extensions. Moreover, one more factor can be accommodated in these extended designs.

3. Construction of arrays of strength 3 from arrays of strength 2.

THEOREM 1. *Let S be an ordered set of s elements $e_0, e_1, \cdots, e_{s-1}$. For any integer t consider the s^t different ordered t-tuples of the elements of S. They can be divided into s^{t-1} sets, each consisting of s t-tuples and closed under cyclic permutations of the elements of S. Denote these sets by S_i, $i = 1, 2, \cdots, s^{t-1}$. Suppose that it is possible to find a scheme of r rows with elements belonging to S*

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{matrix} \qquad (n = \lambda s^{t-1})$$

such that in every t-rowed submatrix the number of elements belonging to each S_i is the same, say, equal to λ; then one can use this scheme in order to construct an orthogonal array $(\lambda s^t, r, s, t)$. If in addition this scheme consists of an array of strength $t - 1$, then one can construct an orthogonal array $(\lambda s^t, r + 1, s, t)$.

PROOF. The sets $S_i (i = 1, \cdots, s^{t-1})$ may be, for example, defined as follows. Consider the s^{t-1} distinct $(t - 1)$-tuples of the elements of S and let the first t-tuple of each S_i be the vector $(e, e_{i_1}, e_{i_2}, \cdots, e_{i_{t-1}})$ where e is an arbitrarily chosen element of S and the remaining elements of the vector form one of the $s^{t-1} (t - 1)$-tuples made to correspond to the set S_i. The additional $s - 1$ t-tuples of each of the sets S_i are obtained from the first by cyclic permutation of the elements of S.

An array $(\lambda s^t, r, s, t)$ can now be constructed. Let its first λs^{t-1} columns be identical with the scheme satisfying the conditions of the theorem. Then the array is completed by adjoining to these columns all the transformations of the scheme consisting of cyclic permutations of the elements of S.

If the scheme consists of an array of strength $t - 1$, then an additional row can be added of which, for example, the first λs^{t-1} elements are equal to the first element of S, the next λs^{t-1} to the second element of S, and so on until all the elements of S are exhausted.

THEOREM 2. *If s is 2, then any orthogonal array of strength 2 forms a scheme, satisfying the conditions of Theorem 1, for the construction of an array of strength 3.*

PROOF. Denote the elements of the array by 0 and 1, and the index of the array of strength 2 by λ. Let

$$S_1 = \begin{Bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{Bmatrix}, \quad S_2 = \begin{Bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{Bmatrix}, \quad S_3 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}, \quad S_4 = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{Bmatrix}.$$

The theorem will be proved if we show that every three-rowed matrix of the scheme contains λ elements belonging to each S_i, $i = 1, 2, 3, 4$. Let $x_{ijk}(i, j, k = 0, 1)$ denote the number of columns of any three-rowed matrix which contains i in the first row, j in the second row and k in the third row. Then clearly

$$\sum_i x_{ijk} = \sum_j x_{ijk} = \sum_k x_{ijk} = \lambda,$$

from which it is easy to deduce the theorem. For example, $x_{000} + x_{010} = x_{010} + x_{110}$ or $x_{000} = x_{110}$. Also, $x_{110} + x_{111} = \lambda$ so that $x_{000} + x_{110} = \lambda$.

We will illustrate Theorems 1 and 2 by the following example. Consider the orthogonal array (12, 11, 2, 2) constructed by Plackett and Burman [2].

$$\begin{matrix}
0&1&1&0&1&1&1&0&0&0&1&0\\
0&1&0&1&1&1&0&0&0&1&0&1\\
0&0&1&1&1&0&0&0&1&0&1&1\\
0&1&1&1&0&0&0&1&0&1&1&0\\
0&1&1&0&0&0&1&0&1&1&0&1\\
0&1&0&0&0&1&0&1&1&0&1&1\\
0&0&0&0&1&0&1&1&0&1&1&1\\
0&0&0&1&0&1&1&0&1&1&1&0\\
0&0&1&0&1&1&0&1&1&1&0&0\\
0&1&0&1&1&0&1&1&1&0&0&0\\
0&0&1&1&0&1&1&1&0&0&0&1
\end{matrix}$$

It is seen that this orthogonal array of strength 2 satisfies the conditions of Theorem 2. Now we construct the orthogonal array (24, 12, 3, 2) by adjoining to this scheme its transformation obtained by interchanging zero and one. The 12th row will consist of 12 zeros and 12 ones.

$$\begin{matrix}
0&1&1&0&1&1&1&0&0&0&1&0&1&0&0&1&0&0&0&1&1&1&0&1\\
0&1&0&1&1&1&0&0&0&1&0&1&1&0&1&0&0&0&1&1&1&0&1&0\\
0&0&1&1&1&0&0&0&1&0&1&1&1&1&0&0&0&1&1&1&0&1&0&0\\
0&1&1&1&0&0&0&1&0&1&1&0&1&0&0&0&1&1&1&0&1&0&0&1\\
0&1&1&0&0&0&1&0&1&1&0&1&1&0&0&1&1&1&0&1&0&0&1&0\\
0&1&0&0&0&1&0&1&1&0&1&1&1&0&1&1&1&0&1&0&0&1&0&0\\
0&0&0&0&1&0&1&1&0&1&1&1&1&1&1&1&0&1&0&0&1&0&0&0\\
0&0&0&1&0&1&1&0&1&1&1&0&1&1&1&0&1&0&0&1&0&0&0&1\\
0&0&1&0&1&1&0&1&1&1&0&0&1&1&0&1&0&0&1&0&0&0&1&1\\
0&1&0&1&1&0&1&1&1&0&0&0&1&0&1&0&0&1&0&0&0&1&1&1\\
0&0&1&1&0&1&1&1&0&0&0&1&1&1&0&0&1&0&0&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1
\end{matrix}.$$

The described method of constructing orthogonal arrays of strength 3 renders, in case s equals 2, the maximum possible number of rows. This follows from Theorem 2A proved by Bose and Bush [3] which reads: "For any orthogonal array $(\lambda s^3, k, s, 3)$ of strength 3, the number of constraints k satisfies the inequality $k \leqq [(\lambda s^2 - 1)/(s - 1)] + 1$." For $s = 2$ the inequality reduces to $k \leqq 4\lambda$.

4. An array of strength 2. An orthogonal array (36, 13, 3, 2) will be constructed. The construction is based on the method of differences formulated by Bose [1] and Bose and Bush [3]. It will be more convenient to use here the first formulation. It reads as follows: Let M be a module consisting of e elements. Suppose it is possible to find a scheme of $r + 1$ rows

$$\begin{array}{cccc} a_{01}, & a_{02}, & \cdots, & a_{0n} \\ a_{11}, & a_{12}, & \cdots, & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1}, & a_{r2}, & \cdots, & a_{rn} \end{array}$$

such that (1) each row contains λs elements belonging to M; (2) among the differences of corresponding elements of any two rows, each element of M occurs exactly λ times, then we can use the scheme to construct an orthogonal array $(\lambda s^2, r + 2, s, 2)$. The following 12×12 scheme satisfying the conditions of the theorem, was found by trial and error:

$$\begin{array}{cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&2&2&2&2\\
0&0&1&2&2&2&0&1&1&1&2&0\\
1&0&0&2&2&1&0&2&2&0&1&1\\
1&2&0&0&0&2&2&1&1&0&2&1\\
0&0&2&1&0&1&2&2&0&1&2&1\\
0&2&1&0&1&2&0&2&0&2&1&1\\
1&0&2&0&1&2&2&0&2&1&1&0\\
0&2&0&1&2&2&1&0&2&1&0&1\\
2&0&0&1&0&2&1&2&1&2&1&0\\
2&0&1&0&2&1&2&0&1&2&0&1\\
0&1&0&0&2&0&2&2&1&1&1&2
\end{array}.$$

Twelve rows of the (36, 13, 3, 2) array are obtained by adjoining to this scheme its two transformations consisting of cyclic permutations of the elements zero, one and two. The 13th row will be added by putting, for example, four zeros, four ones, and four twos under each of the schemes in the same order.

Two questions now naturally arise. The first is whether it is possible, using the same method of construction, to build a scheme consisting of a number of rows greater than 12. The second is whether it is possible to use a scheme of 12 rows to construct an orthogonal array consisting of more than 13 rows. Both questions will be answered in the negative.

The proof is based on an algebraic property of orthogonal arrays pointed out by Bose and Bush [3].

Let n_{ij} denote the number of columns that have j coincidences, that is, j

elements equal, with the ith column. A necessary condition for an array to be an orthogonal array (N, k, s, t) is that whatever be the number h such that $0 \leqq h \leqq t$, the following equalities hold:

$$\sum_{j=0}^{k} n_{ij} C_h^j = C_h^k(\lambda s^{t-h} - 1) \qquad \text{for } i = 1, 2, \cdots, N.$$

In the case considered h takes on the values 0, 1, and 2 and the condition reduces to

$$(*) \begin{cases} \sum_{j=0}^{k} n_{ij} = \lambda s^2 - 1 \\ \sum_{j=0}^{k} j n_{ij} = k(\lambda s - 1) \\ \sum_{j=0}^{k} j(j-1) n_{ij} = k(k-1)(\lambda - 1) \qquad i = 1, 2, \cdots, N. \end{cases}$$

Consider now the first $r + 1$ rows of an array constructed with the aid of the theorem of Bose. It is easy to see that $n_{i0} \geqq s - 1$ for all i. Clearly these inequalities hold also for any subarray extracted out of the first $r + 1$ rows. This means that in the case considered $n_{i0} \geqq 2$ as long as we deal with the first 12 rows of the above constructed array. On the other hand for $k = 12$, $n_{i0} \leqq 2$ because otherwise the square of the deviation of the value $j = 0$ from the mean of the j's would exceed the total sum of squares of the deviations from the mean. Hence for $k = 12$, $n_{i0} = 2$ for all i. Furthermore, $n_{i0} = 2$ implies $n_{i4} = 33$ and $n_{ij} = 0$ for $j \neq 0, 4$ and all i. This can be shown by applying equalities (*) and noticing that if $k = 12$ and $n_{i0} = 2$, then $Q = \sum_{j=1}^{k} (j - 4)^2 n_{ij} = 0$.

Such an array could not include a scheme consisting of 13 rows, because the solutions $n_{i0} = 2$, $n_{i4} = 22$, $n_{i5} = 11$ and $n_{ij} = 0, j \neq 0, 4, 5$ do not satisfy the equations (*) for $s = 3$, $\lambda = 4$, $k = 13$. This answers the first question.

To answer the second question notice that if for $k = 12$, $\lambda = 4$, $s = 3$ the solution of the equations (*) are $n_{i0} = 2$, $n_{i4} = 33$, $n_{ij} = 0$ for $j \neq 0, 4$ and all i, then for $k = 13$ the corresponding solutions are $n_{i1} = 2$, $n_{i4} = 24$, $n_{i5} = 9$ and $n_{ij} = 0$ for $j \neq 1, 4, 5$. This means every set of three columns belonging to the subarray of 12 rows and closed under the cyclic transformations of the elements of the array has the same element in the 13th row. It is seen that this condition together with the condition that each element of the array has to appear twelve times in each row will suffice in order to construct the 13th row.

Let us see now whether one could add a 14th row to this array. It is enough to consider the solutions of equations (*) for the unknown values of n_{i4}, n_{i5}, n_{i6} provided that $n_{i2} = 2$. This follows from the following reasoning. For 12 rows $n_{i0} = 2$, $n_{i4} = 33$ and $n_{ij} = 0$ for $j \neq 0, 4$. We may identify the two columns which have no coincidences with the ith column as l and l'. For 13 rows $n_{ij} = 2$ and clearly these columns must be l and l'. Hence by first considering rows $1, 2, \cdots, 13$ and then rows $1, 2, \cdots, 12$ and 14 it follows that columns

l and l' will have the same 2-tuple in rows 13 and 14 that column i has. Hence $n_{i1} = 2$. For $k = 14$, $n_{i2} = 2$ implies $n_{i4} = 16$, $n_{i5} = 16$, $n_{i6} = 1$, $n_{ij} = 0$ for $j \neq 2, 4, 5, 6$. Consider i equal to 1. We may assume that the last two elements of the first column are equal to zero. Then because $n_{i2} = 2$, the last two elements of the 13th and 25th columns will be equal to zero. Since $n_{i6} = 1$ there exists one more column different from the 1st, 13th and 25th which has the last two elements equal to zero. Let us denote this column by i'. By our assumption, $n_{i'2} = 2$; hence two more columns will have to have the last two elements equal to zero, consequently the assumptions that $\lambda = 4$ and the array is of strength 2 would be incompatible for $k = 14$.

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