SOME NOTES ON THE APPLICATION OF SEQUENTIAL METHODS IN THE ANALYSIS OF VARIANCE[1]

By N. L. Johnson

University of North Carolina and University College, London

Summary. Sequential tests of linear hypotheses in the systematic linear model are studied. Methods of overcoming difficulties in the construction of tests when there is a random model are considered.

1. Introduction. The original methods [1] of constructing sequential tests required problems to be formulated as discrimination between two simple hypotheses. In cases where composite hypotheses were involved, a more or less arbitrary weighting function was introduced so that the problem could in effect be reduced to discrimination between simple hypotheses. Recent work by Barnard [2] and Cox [3] has made it possible to extend the sphere of application of sequential tests to a number of cases where composite hypotheses are to be compared. Barnard refers to unpublished work by C. M. Stein on this problem and there is a remark in [4] which implies that both Stein and M. A. Girshick approach the problem from the same angle as Cox.

It is the purpose of these notes to discuss some points of detail arising in the application of sequential methods to the particular type of composite hypotheses associated with the analysis of variance. Tests of the general linear hypothesis in systematic (parametric) models will be discussed first, followed by a discussion of component of variance models for simple special cases.

2. The general linear hypothesis. It will be helpful to start with a brief resumé of the general linear hypothesis and its likelihood ratio test in the case of samples of fixed size [5], [6].

It is assumed that

(i) $\mathbf{x} = (x_1, \cdots, x_N)$ are N independent normal variables,
(ii) $\mathcal{E}(\mathbf{x}) = \boldsymbol{\theta}\mathbf{C}'$,

where

$$\boldsymbol{\theta} = (\theta_1, \cdots, \theta_s) = (\theta_1, \cdots, \theta_{s-q} \mid \theta_{s-q+1}, \cdots, \theta_s) = (\boldsymbol{\theta}_{(1)}, \boldsymbol{\theta}_{(2)})$$

$$\mathbf{C} = \begin{pmatrix} c_{11} & \cdots & c_{1s} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ c_{N1} & \cdots & c_{Ns} \end{pmatrix} = (\mathbf{C}_{(1)}, \mathbf{C}_{(2)}) \qquad (s < N),$$

Received 11/29/52.

[1] This research was supported in part by The United States Air Force under Contract AF 18(600)-83 monitored by the Office of Scientific Research.

where $\mathbf{C}$ is partitioned between the $(s - q)$th and $(s - q + 1)$st columns, as is $\boldsymbol{\theta}$

(iii) var $x_i = \sigma^2$ $(i = 1, \cdots, N)$,

(iv) The θ's and σ^2 are unknown parameters; the c's are known constants, defined by the design of the experiments.

The hypothesis to be tested is $H_0:\boldsymbol{\theta}_{(2)} = (0, 0, \cdots, 0) = \mathbf{0}$. A likelihood ratio criterion for this problem is provided by any monotonic function of $G = S_b/S_a$, where

S_a is the minimum of $(\mathbf{x} - \boldsymbol{\theta}\mathbf{C}')(\mathbf{x} - \boldsymbol{\theta}\mathbf{C}')'$ with respect to $\boldsymbol{\theta}$

$S_a + S_b$ is the minimum of $(\mathbf{x} - \boldsymbol{\theta}_{(1)}\mathbf{C}'_{(1)})(\mathbf{x} - \boldsymbol{\theta}_{(1)}\mathbf{C}'_{(1)})'$ with respect to $\boldsymbol{\theta}_{(1)}$.

It can be shown [4], [7] that the probability density function of G satisfies

$$p(G \mid \boldsymbol{\theta}_{(2)}\sigma^{-1}) = e^{-\frac{1}{2}\lambda}p(G \mid \mathbf{0})M(\tfrac{1}{2}(N - s + q), \tfrac{1}{2}q; \tfrac{1}{2}\lambda G(1 + G)^{-1}) \tag{1}$$

where

$$\lambda\sigma^2 = \boldsymbol{\theta}_{(2)}\mathbf{C}'_{(2)}(\mathbf{I} - \mathbf{C}_{(1)}(\mathbf{C}'_{(1)}\mathbf{C}_{(1)})^{-1}\mathbf{C}'_{(1)})\mathbf{C}_{(2)}\boldsymbol{\theta}'_{(2)}$$

and

$$M(X, Y; u) = \sum_{j=0}^{\infty} \frac{\Gamma(Y)\Gamma(X + j)}{\Gamma(X)\Gamma(Y + j)} \cdot \frac{u^j}{j!}$$

is a confluent hypergeometric function.

3. Sequential analysis for the systematic model. Now consider a sequential form of experiment in which successive x's, or sets of x's, are measured until a decision is reached. As the experiment is continued, so will $\mathbf{C}$ grow by the addition of further rows. The way in which it is decided to obtain each successive observation (or set of observations) will determine the numerical values of the c's in the successive rows of $\mathbf{C}$. In these notes only those cases where the design is predetermined (i.e., where the c's are not random variables depending on the results of earlier measurements) will be considered, although it would appear, intuitively, that determination of the c's on the basis of results already observed would lead to improved procedures.

At each stage in the experiment a value $G^{[N]}$ may be calculated. (The superscript $[N]$ means "pertaining to sample size N," and will be omitted when confusion is not likely to be incurred by such omission.) The distributions of the corresponding random variables will depend on $\boldsymbol{\theta}_{(2)}\sigma^{-1}$ through the parameters $\lambda^{[N]}$. It might be hoped to use the sequence $G^{[N]}$ in a test to discriminate between the hypotheses

$$H_j:\boldsymbol{\theta}_{(2)}\sigma^{-1} = \boldsymbol{\Delta}_j (j = 1, 2).$$

(Evidently by taking $\boldsymbol{\Delta}_1 = \mathbf{0}$ and choosing $\boldsymbol{\Delta}_2$ suitably a sequential test discriminating between H_1 and H_2 could be compared with the likelihood ratio test of H_0.)

Such a sequential test could certainly be obtained if

$$(2) \qquad p(G^{[2]}, \cdots, G^{[N]}) = p(G^{[N]} \mid \boldsymbol{\theta}_{(2)}\sigma^{-1})f(G^{[2]}, \cdots, G^{[N]})$$

where $f(G^{[2]}, \cdots, G^{[N]})$ does not depend on $\boldsymbol{\theta}_{(2)}\sigma^{-1}$. Cox [3] gives conditions under which (2) is true.

If it is possible to pick out a subsequence $G^{[N_n]}$ such that each of the terms in the corresponding subsequence $\lambda^{[N_n]}$ depend only on the same scalar function of $\boldsymbol{\theta}_{(2)}\sigma^{-1}$ then it can be shown that Cox's conditions apply, and so

$$p(G^{[N_1]}, \cdots, G^{[N_n]} \mid \boldsymbol{\theta}_{(2)}\sigma^{-1}) = p(G^{[N_n]} \mid \boldsymbol{\theta}_{(2)}\sigma^{-1})f(G^{[N_1]}, \cdots, G^{[N_n]}).$$

Since $p(G^{[N_n]} \mid \boldsymbol{\theta}_{(2)}\sigma^{-1})$ depends only on $\lambda^{[N_n]}$ which is itself required to depend only on some scalar function $\phi(\boldsymbol{\theta}_{(2)}\sigma^{-1})$ of $\boldsymbol{\theta}_{(2)}\sigma^{-1}$ we may write

$$p(G^{[N_1]}, \cdots, G^{[N_n]} \mid \phi) = p(G^{[N_n]} \mid \phi)f(G^{[N_1]}, \cdots, G^{[N_n]}).$$

It follows that

(a) all hypotheses (about $\boldsymbol{\theta}_{(2)}\sigma^{-1}$) which specify the same value for ϕ will be, for present purposes, equivalent,

(b) a sequential test for discriminating between the hypotheses

$$H':\phi = \phi' \text{ and } H'':\phi = \phi''$$

will be specified by instructions of the form

$$\text{(I)} \qquad \text{“Accept } H' \text{ if } \frac{p(G^{[N_n]} \mid \phi'')}{p(G^{[N_n]} \mid \phi')} \leqq \frac{\beta}{1-\alpha}.$$

$$\text{Accept } H'' \text{ if } \frac{p(G^{[N_n]} \mid \phi'')}{p(G^{[N_n]} \mid \phi')} \geqq \frac{1-\beta}{\alpha}.$$

Otherwise take a further set of $N_{n+1} - N_n$ observations in accordance with the prescribed pattern"

provided this sequence of operations terminates with probability one when either H' or H'' is true.

The prescribed pattern will, of course, be such that $\lambda^{[N_n]}$ depends only on $\phi(\boldsymbol{\theta}_{(2)}\sigma^{-1})$.

4. Limiting form of the test. Decision to take a further set of observations will be made if

$$\frac{\beta}{1-\alpha} < \frac{p(G \mid \phi'')}{p(G \mid \phi')} < \frac{1-\beta}{\alpha}$$

where we have, for convenience, dropped the superscript $[N]$. From (1) this is equivalent to

$$(2) \qquad \begin{aligned} A + \tfrac{1}{2}(\lambda'' - \lambda') &< \log M(\tfrac{1}{2}(N_n - s + q), \tfrac{1}{2}q; \tfrac{1}{2}\lambda'' y^2) \\ &- \log M(\tfrac{1}{2}(N_n - s + q), \tfrac{1}{2}q; \tfrac{1}{2}\lambda' y^2) < B + \tfrac{1}{2}(\lambda'' - \lambda') \end{aligned}$$

where

$$A = \log \frac{\beta}{1 - \alpha}; \quad B = \log \frac{1 - \beta}{\alpha}; \quad y = G^{\frac{1}{2}}(1 + G)^{-\frac{1}{2}}.$$

The application of tests of this type (for $q > 1$) requires tables of $M(X, Y; u)$ rather more extensive than are known to the author at present. We can, however, make certain deductions about inequalities (2), incidentally showing that in certain important cases, the sequence does, indeed, terminate with probability one when either H' or H'' is true.

It will be assumed from now on that $\phi'' > \phi'$ and $\lambda'' > \lambda'$. As u increases from 0 to 1,

$$[\log M(X, Y; \tfrac{1}{2}\lambda'' u) - \log M(X, Y; \tfrac{1}{2}\lambda' u)]$$

increases from 0 to

$$[\log M(X, Y; \tfrac{1}{2}\lambda'') - \log M(X, Y; \tfrac{1}{2}\lambda')]$$

(if $X > 0$ and $Y > 0$). Hence (2) is equivalent to

$$\underline{G}(N_n, \alpha, \beta, \lambda', \lambda'') < G < \bar{G}(N_n, \alpha, \beta, \lambda', \lambda'') \tag{3}$$

$\underline{G}$ and $\bar{G}$ being fixed numbers defined by the quantities in brackets. (N.B. If $A + \frac{1}{2}(\lambda'' - \lambda') > 0$ then $\underline{G} = 0$, while if $B + \frac{1}{2}(\lambda'' - \lambda') \geqq \log M(X, Y; \frac{1}{2}\lambda'') - \log M(X, Y; \frac{1}{2}\lambda')$ then $\bar{G} = \infty$).

Now consider the special case where successive sets of k observations are taken, so that $N_n = kn$, and each set is arranged in the same pattern, so that identical sets of rows are added to **C** at each stage. In this case it is possible to take $\phi = \lambda^{[k]}/k$ and then $\lambda^{[N_n]} = \lambda^{[kn]} = nk\phi$ is a function of ϕ only. (ϕ may be thought of as "noncentrality per unit observation"). (2) now becomes

$$\begin{aligned} A + \tfrac{1}{2}nk(\phi'' - \phi') < \log M(\tfrac{1}{2}(nk - s + q), \tfrac{1}{2}q; \tfrac{1}{2}nk\phi'' y^2) \\ - \log M(\tfrac{1}{2}(nk - s + q), \tfrac{1}{2}q; \tfrac{1}{2}nk\phi' y^2) \\ < B + \tfrac{1}{2}nk(\phi'' - \phi'). \end{aligned} \tag{4}$$

Now Perron [8] has shown that

$$M(X, Y; u) = \frac{\Gamma(Y)}{2\sqrt{\pi}} e^{\frac{1}{2}u}(Xu)^{\frac{1}{4} - \frac{1}{2}Y} e^{2\sqrt{Xu}}(1 + O(X^{-\frac{1}{2}})).$$

Hence from (4) if n is large and $\phi' \neq 0$, we have

$$\begin{aligned} A + \tfrac{1}{2}nk(\phi'' - \phi') + O(n^{-\frac{1}{2}}) < \tfrac{1}{4}(q - 1) \log (\phi'/\phi'') + \tfrac{1}{4}nk(\phi'' - \phi')y^2 \\ + n[k\{k - (s - q)n^{-1}\}]^{\frac{1}{2}}(\sqrt{\phi''} - \sqrt{\phi'})y < B + \tfrac{1}{2}nk(\phi'' - \phi') + O(n^{-\frac{1}{2}}). \end{aligned} \tag{5}$$

Remembering that $\phi'' > \phi'$, (5) may be rearranged to give

$$\begin{aligned} 2 + \frac{(q - 1) \log(\phi''/\phi') + 4A}{nk(\phi'' - \phi')} + O(n^{-\frac{3}{2}}) < y^2 + \frac{4}{\sqrt{\phi''} + \sqrt{\phi'}} \\ \cdot\left(1 - \frac{s - q}{kn}\right)^{\frac{1}{2}} y < 2 + \frac{(q - 1) \log(\phi''/\phi') + 4B}{nk(\phi'' - \phi')} + O(n^{-\frac{3}{2}}). \end{aligned} \tag{6}$$

This implies

$$y^2 + \frac{4}{\sqrt{\phi''} + \sqrt{\phi'}} y - 2 < O(n^{-1}).$$

As n tends to infinity, y tends to $\phi^{\frac{1}{2}}(1 + \phi)^{-\frac{1}{2}}$ and y^2 tends to $\phi(1 + \phi)^{-1}$ almost certainly. Unless

$$\frac{\phi}{1 + \phi} + \frac{4}{\sqrt{\phi''} + \sqrt{\phi'}} \frac{\sqrt{\phi}}{\sqrt{(1 + \phi)}} = 2 \tag{7}$$

for either $\phi = \phi'$ or $\phi = \phi''$ it is clear that the sequence (I) must terminate with probability one in either case. It is of interest to note that large samples are not likely to be reached unless (7) is satisfied, at any rate approximately. From (7)

$$\phi^{\frac{1}{2}}(1 + \phi)^{-\frac{1}{2}} = 2[\{1 + \tfrac{1}{4}(\sqrt{\phi''} + \sqrt{\phi'})^2\}^{\frac{1}{2}} - 1](\sqrt{\phi''} + \sqrt{\phi'})^{-1}. \tag{8}$$

We may note that if both ϕ'' and ϕ' are small (8) gives $\sqrt{\phi} \doteqdot \frac{1}{2}(\sqrt{\phi''} + \sqrt{\phi'})$ which is not unexpected on intuitive grounds.

The argument above applies to the case $\phi' \neq 0$. If $\phi' = 0$ then instead of (5) we obtain

$$A + \tfrac{1}{2}nk\phi'' + 0(n^{-\frac{1}{2}}) < \tfrac{1}{4}(q - 1) \log k\phi'' + \tfrac{1}{4}nk\phi''y^2 + nk\{1 - (s - q)/nk\}^{\frac{1}{2}}\sqrt{\phi''}y < B + \tfrac{1}{2}nk\phi'' + 0(n^{-\frac{1}{2}})$$

leading finally to (7) with $\phi' = 0$.

5. One-way classification. The special case of one-way classification into k groups will now be considered. At each stage we decide whether or not to take a further set of k observations, one from each group. The usual systematic model

$$x_{ti} = a + b_t + z_{ti} \qquad (t = 1, \cdots, k; i = 1, \cdots, n; \textstyle\sum_t b_t = 0) \tag{9}$$

is included in the general linear model described in Section 2. Then, in the notation of Section 3, $N_n = nk$, $s = k$ and the hypothesis $b_t = 0$ $(t = 1, \cdots, k)$ implies $q = k - 1$. Further

$$G^{[nk]} = n \sum_t (\bar{x}_t - \bar{x})^2 / \sum_t \sum_i (x_{ti} - \bar{x}_t)^2,$$

$$\lambda^{[nk]} = n \sum_t b_t^2/\sigma^2, \phi = \sum_t b_t^2/k\sigma^2.$$

Hence the sequence $G^{[2n]}, G^{[3n]} \cdots$, may be used in a sequential test based on discrimination between H': $\sum b_t^2/k\sigma^2 = \phi'$ and H'': $\sum b_t^2/k\sigma^2 = \phi''$ and constructed as in (I).

6. Random model in the one-way classification. It is well known that sometimes a model different from (9) may be used. This is

$$x_{ti} = a + u_t + z_{ti} \tag{10}$$

where the u's are normal variables, each with expected value zero and standard deviation σ_R, mutually independent and also independent of the z's. This model is often called the "random" or "component of variance" model. In this case

$$(11) \qquad p(G^{[nk]} \mid \delta) = \frac{(1 + n\delta)^{\frac{1}{2}k(n-1)}}{B(\frac{1}{2}(k-1), \frac{1}{2}k(n-1))} \frac{G^{[nk]\frac{1}{2}(k-3)}}{(1 + n\delta + G^{[nk]})^{\frac{1}{2}(kn-1)}}$$

where $\delta = \sigma_R^2/\sigma^2$. This depends only on δ, which plays the part played by ϕ in the systematic model. Cox's result shows that the sequence $G^{[2n]}, G^{[3n]}, \cdots$, may be used in a sequential test based on discrimination between $H_R':\delta = \delta'$ and $H_R'':\delta = \delta''$ provided the procedure specified terminates with probability one.

The procedure will be

(II) "Accept H_R' if $f(G, \delta', \delta'') \leqq \beta/(1 - \alpha)$.

Accept H_R'' if $f(G, \delta', \delta'') \geqq (1 - \beta)/\alpha$.

Otherwise measure one further item in each of the k groups"

where

$$f(G, \delta', \delta'') = \left(\frac{1 + n\delta''}{1 + n\delta'}\right)^{\frac{1}{2}k(n-1)} \left(\frac{1 + n\delta' + G}{1 + n\delta'' + G}\right)^{\frac{1}{2}(kn-1)}.$$

Under this procedure sampling will be continued whenever $\underline{G}_R < G < \bar{G}_R$ where

$$(12) \quad \underline{G}_R = n(\delta''\underline{\epsilon} - \delta')(1 - \underline{\epsilon})^{-1} - 1; \bar{G}_R = n(\delta''\bar{\epsilon} - \delta')(1 - \bar{\epsilon})^{-1} - 1,$$

$$\underline{\epsilon} = \left(\frac{\beta}{1 - \alpha}\right)^{2/(kn-1)} \left(\frac{1 + n\delta'}{1 + n\delta''}\right)^{k(n-1)/(kn-1)},$$

$$\bar{\epsilon} = \left(\frac{1 - \beta}{\alpha}\right)^{2/(kn-1)} \left(\frac{1 + n\delta'}{1 + n\delta''}\right)^{k(n-1)/(kn-1)}.$$

It can easily be shown that, if $\delta'' > \delta' > 0$

$$\lim_{n\to\infty} \underline{G}_R = [2A + (k - 1) \log (\delta''/\delta')]/k(\delta'^{-1} - \delta''^{-1}),$$

$$\lim_{n\to\infty} \bar{G}_R = [2B + (k - 1) \log (\delta''/\delta')]/k(\delta'^{-1} - \delta''^{-1})$$

so that $\lim_{n\to\infty}\underline{G}_R \neq \lim_{n\to\infty}\bar{G}_R$.

On the other hand if $\delta' = 0$ (the case considered by Cox) $\lim_{n\to\infty}\underline{G}_R = \lim_{n\to\infty}\bar{G}_R = 0$.

The random model can be regarded as a mixture of systematic models in which the quantity $\sum b_t^2/k\sigma^2 = \phi$ is distributed as $\delta\cdot(\chi_{k-1}^2/k)$. For any systematic model such that $\lim_{n\to\infty}\underline{G}_R < \phi < \lim_{n\to\infty}\bar{G}_R$ there is a nonzero probability that $\underline{G}_R < G^{[nk]} < \bar{G}_R$ for all n. If $\delta \neq 0$ there will be a nonzero proportion of systematic models, in the mixture constituting the random model, for which this is the case. Hence, the sequential procedure outlined above for $\delta'' > \delta' > 0$ will not conclude with probability one unless $\delta = 0$.

Now consider the test based on $\delta'' > \delta' = 0$. This will terminate with probability one both when $\delta = \delta''$ and when $\delta = 0$, and so can be used as a sequential test. Some values of $\underline{G}_R$ and $\bar{G}_R$ for this case are given in Tables Ia and Ib. As might be expected the procedure is not very practicable for small values of k. For example, in neither of the cases covered by the tables is it possible to obtain

TABLE Ia

$\delta'' = 1, \quad \delta' = 0, \quad \alpha = \beta = 0.05$

(i) *Values of* $\underline{G}_R$

n \ k	2	3	4	5	6	7	8	9	10	11	12	15	20	30	60	∞
2	—	—	—	—	—	.086	.204	.308	.398	.478	.549	.720	.918	1.146	1.414	1.732
3	—	—	—	—	.086	.180	.256	.319	.372	.417	.455	.545	.640	.744	.855	.974
4	—	—	—	.037	.126	.194	.249	.292	.328	.359	.385	.444	.505	.569	.636	.707
5	—	—	—	.066	.137	.190	.232	.265	.292	.315	.334	.378	.423	.469	.517	.566
6	—	—	—	.080	.138	.181	.215	.242	.264	.282	.297	.331	.366	.403	.439	.477
7	—	—	.015	.086	.135	.172	.200	.222	.240	.256	.268	.297	.325	.355	.385	.416
8	—	—	.026	.088	.131	.162	.187	.206	.221	.234	.245	.269	.294	.319	.344	.370
9	—	—	.033	.088	.126	.154	.175	.192	.206	.217	.226	.247	.269	.290	.312	.335
10	—	—	.038	.087	.121	.146	.165	.180	.192	.202	.211	.229	.248	.267	.287	.306
11	—	—	.041	.086	.117	.139	.156	.170	.181	.190	.197	.214	.231	.248	.265	.283
12	—	—	.043	.084	.112	.133	.148	.161	.171	.179	.186	.201	.216	.232	.248	.263
15	—	—	.045	.078	.101	.117	.129	.139	.147	.153	.159	.171	.183	.195	.207	.220
20	—	.004	.044	.069	.086	.098	.107	.115	.121	.125	.130	.138	.147	.156	.165	.174
30	—	.011	.039	.056	.067	.076	.082	.087	.090	.094	.096	.102	.108	.114	.120	.126
60	—	.013	.028	.037	.042	.047	.050	.052	.054	.056	.057	.060	.063	.066	.069	.072

(ii) *Values of* $\bar{G}_R$

n \ k	2	3	4	5	6	7	8	9	10	11	12	15	20	30	60	∞
2	—	—	—	—	29.314	12.456	8.377	6.542	5.500	4.828	4.359	3.537	2.918	2.435	2.049	1.732
3	—	7.457	3.962	2.909	2.403	2.106	1.911	1.773	1.671	1.591	1.528	1.398	1.279	1.169	1.068	.974
4	4.610	2.376	1.764	1.479	1.315	1.208	1.133	1.078	1.035	1.001	.974	.915	.859	.806	.755	.707
5	2.214	1.440	1.158	1.014	.926	.867	.825	.793	.768	.748	.731	.696	.662	.629	.597	.566
6	1.465	1.042	.873	.783	.724	.685	.657	.635	.618	.604	.593	.569	.545	.522	.499	.477
7	1.100	.823	.705	.640	.599	.571	.550	.534	.522	.511	.503	.485	.468	.450	.433	.416
8	.883	.682	.595	.545	.514	.492	.476	.463	.454	.446	.439	.425	.411	.397	.384	.370
9	.739	.585	.516	.477	.451	.434	.421	.411	.403	.396	.391	.380	.368	.357	.346	.335
10	.636	.513	.457	.424	.404	.389	.378	.370	.363	.358	.354	.344	.334	.325	.316	.306
11	.559	.457	.411	.383	.366	.354	.344	.337	.332	.327	.323	.315	.307	.299	.291	.283
12	.500	.414	.373	.350	.335	.324	.317	.311	.306	.302	.298	.291	.284	.277	.270	.263
15	.380	.323	.296	.280	.270	.262	.257	.252	.249	.246	.244	.239	.234	.229	.224	.220
20	.274	.239	.222	.212	.206	.201	.198	.195	.193	.191	.190	.187	.183	.180	.177	.174
30	.178	.160	.151	.146	.143	.140	.138	.137	.136	.135	.134	.133	.131	.129	.127	.126
60	.089	.084	.081	.079	.078	.077	.076	.076	.076	.075	.075	.074	.074	.073	.073	.072

a decision in favor of $H'(\delta = 0)$ with $n \leqq 60$ if $k = 2$. In such cases it will probably be a good scheme to curtail testing at some convenient value of n.

7. Alternative procedures. An alternative method of procedure is to keep n constant and to decide, at each stage, whether to choose another *group* at random and take a sample of n from it. This method, which may sometimes be practicable, has the advantage that it has in general a probability of one of concluding even when based on a nonzero value for δ'. Since the ratio $p(G \mid \delta'')/p(G \mid \delta')$ will have the same mathematical expression as in (II) it follows that the appropriate values of $\underline{G}_R$, $\bar{G}_R$ are those given by (12). Further, in the case when $\delta' = 0$ tables such as Table I may be used in carrying out the test, proceeding along the rows of the table (i.e., increasing k) instead of down the columns (i.e., increasing n). By analogy with the case of samples of fixed size it would be expected that this

TABLE Ib

$\delta'' = 1, \quad \delta' = 0, \quad \alpha = \beta = 0.01$

(i) *Values of* $\underline{G}_R$

n \ k	2	3	4	5	6	7	8	9	10	11	12	15	20	30	60	∞
2	—	—	—	—	—	—	—	—	.057	.168	.216	.405	.635	.909	1.273	1.732
3	—	—	—	—	—	—	.030	.104	.167	.221	.269	.382	.507	.646	.800	.974
4	—	—	—	—	—	.022	.088	.143	.189	.228	.261	.339	.421	.510	.605	.707
5	—	—	—	—	—	.057	.110	.153	.188	.218	.243	.302	.363	.427	.495	.566
6	—	—	—	—	.018	.074	.117	.152	.181	.206	.226	.273	.321	.371	.423	.477
7	—	—	—	—	.034	.082	.119	.149	.173	.193	.210	.249	.289	.330	.372	.416
8	—	—	—	—	.044	.086	.118	.143	.164	.182	.196	.229	.263	.298	.334	.370
9	—	—	—	.001	.050	.087	.115	.138	.156	.171	.184	.213	.242	.272	.303	.335
10	—	—	—	.009	.054	.087	.112	.132	.149	.162	.174	.199	.225	.252	.279	.306
11	—	—	—	.015	.056	.086	.109	.127	.142	.154	.164	.187	.210	.234	.258	.283
12	—	—	—	.020	.057	.085	.106	.122	.136	.147	.156	.177	.198	.219	.241	.263
15	—	—	—	.028	.058	.080	.096	.109	.120	.129	.136	.152	.169	.185	.202	.220
20	—	—	—	.032	.055	.071	.084	.093	.101	.108	.113	.125	.137	.149	.162	.174
30	—	—	.010	.032	.047	.058	.066	.073	.078	.082	.086	.094	.102	.110	.118	.126
60	—	—	.013	.025	.033	.038	.042	.046	.048	.050	.052	.056	.060	.064	.068	.072

(ii) *Values of* $\bar{G}_R$

n \ k	2	3	4	5	6	7	8	9	10	11	12	15	20	30	60	∞
2	—	—	—	—	—	—	—	47.044	19.175	12.533	9.549	5.999	4.157	3.032	2.275	1.732
3	—	—	14.911	6.572	4.459	3.499	2.952	2.599	2.352	2.171	2.031	1.757	1.520	1.314	1.134	.974
4	57.058	5.470	3.150	2.342	1.934	1.689	1.524	1.407	1.319	1.251	1.196	1.083	.978	.881	.791	.707
5	7.645	2.548	1.792	1.457	1.263	1.139	1.052	.988	.939	.900	.868	.801	.738	.678	.620	.566
6	2.887	1.675	1.271	1.070	.950	.870	.814	.766	.738	.711	.690	.644	.600	.558	.517	.477
7	1.964	1.255	.990	.852	.767	.710	.669	.638	.613	.594	.578	.543	.510	.478	.446	.416
8	1.490	1.007	.814	.711	.647	.603	.571	.546	.527	.512	.500	.472	.446	.420	.395	.370
9	1.203	.843	.694	.612	.561	.525	.500	.480	.464	.452	.442	.419	.398	.376	.355	.335
10	1.009	.727	.606	.539	.496	.467	.446	.429	.416	.406	.397	.378	.360	.342	.324	.306
11	.870	.640	.539	.482	.446	.421	.403	.389	.378	.369	.361	.345	.329	.314	.298	.283
12	.765	.572	.486	.437	.406	.384	.368	.356	.347	.339	.332	.318	.304	.290	.277	.263
15	.564	.436	.377	.344	.322	.306	.295	.286	.279	.274	.269	.259	.249	.239	.229	.220
20	.394	.315	.278	.256	.242	.232	.224	.219	.214	.210	.207	.201	.194	.187	.181	.174
30	.249	.208	.185	.173	.165	.159	.155	.152	.149	.147	.145	.141	.137	.133	.130	.126
60	.120	.104	.096	.091	.088	.086	.084	.083	.082	.081	.080	.078	.077	.075	.074	.072

latter procedure should give, in general, a lower average sample number (of individuals) than the method based on increasing n.

It is interesting to note [9] that, while it is impossible to construct a fixed size sample test using G as criterion if $\delta' \neq 0$ unless k exceeds a certain minimum value depending on α, β and δ''/δ', for a sufficiently large k a fixed sample size test *is* available. This suggests that there should be sequential tests, having the required properties, available for a somewhat wider range of values of k.

8. Further alternative procedures. Owing to the method of construction of our tests it is not possible to use Wald's approximate formula for average sampling number, which applies to a sequence of identically distributed and independent random variables. It, therefore, seems worthwhile to note that the hypotheses discussed in earlier sections can be subjected to sequential tests of standard type based on sequences of independent random variables. This was pointed out to me by A. G. Baker, but the idea has also been used by O. J. Carpenter [10].

In the systematic case such a test can be constructed as follows. Successive samples of size $m(\geqq 2)$ are taken from each of the k groups. A value, g, of G is calculated separately for each such sample of size km and the sequence of independent g's so obtained used in a sequential procedure constructed in the usual manner. Such a procedure will not, however, lead to independent g's in the random model. If a new set of k *groups* is chosen at random each time, and an observation taken from each, then the successive g's will be independent. If we denote the ith value obtained by g_i the procedure will be

(III) "After the nth set of km observations;

Accept $H':\delta = \delta'$ if

$$\sum_{i=1}^{n} \log \frac{1 + m\delta' + g_i}{1 + m\delta'' + g_i} \leqq \frac{2A}{km - 1} + \frac{nk(m - 1)}{km - 1} \log \frac{1 + m\delta'}{1 + m\delta''}.$$

Accept $H'':\delta = \delta''$ if

$$\sum_{i=1}^{n} \log \frac{1 + m\delta' + g_i}{1 + m\delta'' + g_i} \geqq \frac{2B}{km - 1} + \frac{nk(m - 1)}{km - 1} \log \frac{1 + m\delta'}{1 + m\delta''}.$$

Otherwise take a further sample of n from each group."

In this case, using Wald's approximate formula for average sample number, the approximate expected number of individuals to be observed is $mk\{A(1 - \alpha) + B\alpha\}/E_{m,k}(\delta')$ if H' is true, $mk\{B(1 - \beta) + A\beta\}/E_{m,k}(\delta'')$ if H'' is true where

$$E_{m,k}(\delta) = \mathcal{E}\left[\log \frac{p(g \mid \delta'')}{p(g \mid \delta')} \,\middle|\, \delta\right] = \tfrac{1}{2}k(m - 1) \log \frac{1 + m\delta''}{1 + m\delta'} + \tfrac{1}{2}(km - 1)\mathcal{E}\left[\log \frac{1 + m\delta' + g}{1 + m\delta'' + g} \,\middle|\, \delta\right].$$

The leading terms of a useful approximate expression for $E_{m,k}(\delta)$, (obtained by the method of statistical differentials) are

$$E_{m,k}(\delta) = \tfrac{1}{2}k(m - 1) \log \frac{1 + m\delta''}{1 + m\delta'} + \tfrac{1}{2}(km - 1)\Bigg[\log (R'/R'') + \frac{m^2(m - 1)k(k - 1)}{km + 1}\left\{\frac{(\delta - \delta'')^2}{R''^2} - \frac{(\delta - \delta')^2}{R'^2}\right\} - \frac{8m^3(m - 1)k(k - 1)(km - 2k + 1)}{3(km + 1)(km + 3)}\left\{\frac{(\delta - \delta'')^3}{R''^3} - \frac{(\delta - \delta')^3}{R'^3}\right\}\Bigg]$$

where $R' = km - 1 + m(k - 1)\,\delta + m(m - 1)k\delta'$, $R'' = km - 1 + m(k - 1)\,\delta + m(m - 1)k\delta''$.

Some calculations in the case $\delta' = 0$, $\delta'' = 1$ indicate that, when $\delta = \delta' = 0$, the average sampling number will be minimized by taking $m = 4$.

REFERENCES

[1] A. WALD, *Sequential Analysis*, John Wiley and Sons, 1947.

[2] G. A. BARNARD, "The frequency justification of certain sequential tests," *Biometrika*, Vol. 39 (1952), pp. 144–150.

[3] D. R. COX, "Sequential tests for composite hypotheses," *Proc. Cambridge Philos. Soc.*, Vol. 48 (1952), pp. 290–299.

[4] K. J. ARNOLD, *Introduction to Tables to Facilitate Sequential t-tests* National Bureau of Standards, A.M.S. 7 (1951).

[5] S. KOLODZIECZYK, "On an important class of statistical hypotheses," *Biometrika*, Vol. 27 (1935), pp. 165–190.

[6] F. N. DAVID, AND N. L. JOHNSON, "A method of investigating the effect of nonnormality and heterogeneity of variance on tests of the general linear hypothesis," *Ann. Math. Stat.* Vol. 22 (1951), pp. 382–392.

[7] P. L. HSU, "Analysis of variance from the power function standpoint," *Biometrika*, Vol. 32 (1941), pp. 62–69.

[8] O. PERRON, "Über das Verhalten einer Ausgearteten Hypergeometrischen Reihe bei unbegrenzten Wachstum eines Parameters," *J. für reine und ang. Math.* Vol. 151 (1921), pp. 63–78.

[9] N. L. JOHNSON, "Alternative systems in the analysis of variance," *Biometrika*, Vol. 35 (1948), pp. 80–89.

[10] O. J. CARPENTER, Ph.D. Dissertation, Iowa State College.