[2] W. G. Cochran, "The distribution of the largest of a set of estimated variances as a fraction of their total," *Ann. Eugenics*, Vol. 11 (1941), pp. 47–52.
[3] Eisenhart, Hastay, and Wallis (Editors), *Selected Techniques of Statistical Analysis*, McGraw-Hill Book Co., 1947, pp. 390–391.
[4] A. Wald, *Statistical Decision Functions*, John Wiley and Sons, 1950, pp. 123–130.

AN EXTENSION OF THE BUFFON NEEDLE PROBLEM

By Nathan Mantel

Biometrics Section, National Cancer Institute[1]
Bethesda, Maryland

1. Introduction. An empirical determination of the value of π can be made from the relationship[2]

$$E = 2L_1L_2/(\pi A), \tag{1}$$

where E is the expected number of intersections of a group of line segments of total length L_1 with a group of line segments of total length L_2, both groups being distributed over an area A. This relationship applies under the following conditions.

(i) The arrangement of the two groups of line segments on the area A must be independent of each other, but the individual line segments of a group may have a systematic arrangement relative to each other.

(ii) The arrangement of at least one of the two groups of line segments on the area A must be at random. The randomness must be such that the probability of a specified point on a line segment falling into a sub-area of A is proportional to its area and the segment may assume any angle relative to some base line with equal probability.

Two applications of this relationship to the estimation of π are considered below.

2. The Buffon needle problem using a parallel line system. Consider an area A on which is superimposed a series of equally spaced parallel lines (without loss of generality we shall take the common distance between them to be unity), on which a straight line of length $L \leqq 1$ is allowed to fall at random. At each fall the line must either intersect the series of parallel lines only once, or not at all. Thus the expected number of intersections, E, is the probability, P, of an intersection occurring at a fall. And since for this system the total length of the

Received 6/12/52, revised 12/22/52.

[1] National Institutes of Health, Public Health Service, Department of Health, Education and Welfare.

[2] This relationship is developed in passing by Cornfield and Chalkley, "A problem in geometric probability," *J. Wash. Acad. Sci.*, July, 1951.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics*. www.jstor.org

parallel lines is A, (border effects would result in a total length different from A, for areas with dimensions which are large relative to the distance between parallel lines these border effects would be trivial.) $P = 2L/\pi$. Thus from an empirical determination of the probability, $\hat{P}$, can be made an empirical determination,

$$\hat{\pi} = 2L/\hat{P} \tag{2}$$

This determination is subject to sampling variation. Taking into account that, on the basis of N falls, the standard error of $\hat{P}$ is $\sqrt{P(1 - P)/N}$ it follows that, asymptotically,

$$SE\hat{\pi} \cong \pi\sqrt{(\pi - 2L)/2LN}. \tag{3}$$

This formula indicates that more precise estimates of π can be made by using a longer line relative to the spacing of the parallel lines.

Various empirical determinations of the value of π have been made and published making use of the foregoing relationship, the experimental results serving simultaneously as an empirical demonstration of the correctness of Bernoulli's theorem. Curiously enough, virtually all the results published have been closer to the expected value than should be expected, with some significantly too close. Apparently only those experiments which gave good results have been published. However, one example in the literature gives patent indication of having been terminated when the results obtained were good. Thus Lazzerini's experiment in 1901 with 3,408 falls provided an estimate of π equal to 3.1415929, having an error of only 0.0000003. Terminating the experiment one fall sooner or later would inevitably have lost half the decimal places of accuracy.

3. The Cartesian grid system. Consider an area A on which are superimposed two series of equally unit-spaced parallel lines, the two series being at right angles to each other. The expected number of intersections with this system of a straight line of length L falling at random is $4L/\pi$ for all values of L. The estimate of π yielded by N falls with an empirical average of $\bar{c}$ intersections per fall is

$$\hat{\pi} = 4L/\bar{c} \tag{4}$$

with standard error of estimate

$$SE\hat{\pi} \cong \frac{\pi^2\sigma_c}{4L\sqrt{N}} \tag{5}$$

where σ_c is the standard deviation of the number of intersections at a fall. This standard deviation can be evaluated either theoretically or empirically for any value of L.

The theoretical evaluation of σ_c for large L is of interest. Consider an L so large ($L \gg 1$) that certain marginal effects can be disregarded. (These marginal effects arise from the actual location of the end of the line within the square in which it falls, they would slightly increase the value of σ_c^2 over what is shown here, but would have no effect on $E(c)$.) For any given angle θ, at which the line falls, there would be $L \mid \sin \theta \mid$ vertical intersections and $L \mid \cos \theta \mid$ horizontal intersections. Then the expected number of intersections is given by

$$(6) \qquad E(c) = \frac{2}{\pi} \int_0^{\pi/2} L(\sin \theta + \cos \theta)\, d\theta = 4L/\pi.$$

The expected square of the number of intersections is given by

$$(7) \qquad E(c^2) = \frac{2}{\pi} \int_0^{\pi/2} L^2(\sin \theta + \cos \theta)^2\, d\theta = L^2\left(1 + \frac{2}{\pi}\right).$$

These yield

$$(8) \qquad \sigma_c^2 = E(c^2) - E^2(c) = L^2\left(1 + \frac{2}{\pi} - \frac{16}{\pi^2}\right)$$

and substituting in (5)

$$(9) \qquad SE\hat{\pi} \cong \pi\sqrt{(\pi^2 + 2\pi - 16)/16N}$$

for large L.

The quantity inside the square root sign is numerically equal to $.0095/N$. This compares with the value for the quantity inside the square root sign for (3) of $.5708/N$ for $L = 1$ (the most efficient value of L for that situation) and would indicate that more information about the value of π is yielded by a single fall in this system (with large L) than by 60 falls in the parallel line system with $L = 1$.

4. An alternative estimate. The preceding section has covered the estimation of π from the average number of intersections per fall. Equation (8) would suggest that an estimate can be made from the variation in number of intersections from fall to fall. Let $V = \hat{\sigma}_c^2/L^2$, where $\hat{\sigma}_c$ is the sample standard deviation of intersections per fall. Then equation (8) yields as an estimate of π the solution to $(1 - V)X^2 + 2X - 16 = 0$,

$$(10) \qquad \hat{\pi} = \frac{-1 + \sqrt{1 + 16(1 - V)}}{1 - V}.$$

How good are estimates so obtained? For any sample, V must lie between 0 (all falls give same number of intersections) and $\frac{1}{4}(3 - 2\sqrt{2})$ (half the falls parallel or perpendicular to system, remainder at an angle $\pi/4$ or $3\pi/4$; that is, half the lines have the minimum number of intersections and half have the maximum number). But corresponding to $V = 0$, $\hat{\pi} = 3.1231$, and corresponding to $V = \frac{1}{4}(3 - 2\sqrt{2})$, $\hat{\pi} = 3.1752$. This can be considered a demonstration that π must lie between 3.1231 and 3.1752, and indicates that the procedure will give satisfactory estimates.

Table I gives approximate 90 per cent probability limits for estimates of π based on 101 falls by the three methods considered. For estimates based on the average number of intersections the limits are given by $\pi \pm 1.645\, \sigma_{\hat{\pi}}$. For the estimate based on variation in the number of intersections, the limits are the estimates corresponding to $V = (1/1.28)(1 + 2/\pi - 16/\pi^2) = .0121$ and $V = 1.24(1 + 2/\pi - 16/\pi^2) = .0192$ where 1/1.28 and 1.24 are the 5th and 95th percentiles of the distribution of $F_{100,\infty}$ respectively.

TABLE I

Estimates of π based on 101 falls; 90% probability limits

Buffon needle case	
$L = 1$	2.75 to 3.53
Cartesian grid system	
Mean number of intersections	3.09 to 3.19
Variation in number of intersections	3.138 to 3.146

The estimate based on variation in the number of intersections is relatively insensitive to counting and measurement errors. Thus a 10 per cent error in measuring L will produce only $1/10$ of 1 per cent error in the estimate of π. A similar error in measuring L will produce a 10 per cent error in the estimate of π by the other methods. It should be remarked that the situation here is unusual in that the sample variance provides a much better estimate of the true mean number of intersections than does the sample mean. This is in contrast with the case of the Poisson distribution for which the sample mean provides the best estimate of the population variance.

REFERENCES

[1] E. Kasner and J. Newman, *Mathematics and the Imagination*, Simon and Schuster, 1940, pp. 246–247.

[2] J. V. Uspensky, *Introduction to Mathematical Probability*, McGraw-Hill Book Co., 1937, pp. 112–115, 251–257.

A HIGHER ORDER COMPLETE CLASS THEOREM[1]

By Lionel Weiss

University of Virginia and Cornell University

1. Introduction. The purpose of this note is to show that one can prove complete class theorems in which the risk for each possible distribution is not only a scalar, as is usual in the Wald theory, but actually a vector with as many com-

Received 5/7/53.

[1] Research under contract with the Office of Naval Research.