PROOF. Letting $r = 2$ in equation (5), it follows that $E(v_j^r \mid n_{j1}, n_{j2}, \cdots) < 2k^5 - k^4$. Once again, using the identity

$$E(v_j^2 \mid v_1, \cdots, v_{j-1}) = E[E\{v_j^2 \mid v_1, v_2, \cdots, v_{j-1}, n_{j1}, \cdots\} \mid v_1, \cdots, v_{j-1}]$$

and the fact that the probability distribution of $v_j^2 \mid v_1, \cdots, v_{j-1}, n_{j1}, n_{j2}, \cdots$ is a function only of $n_{j1}, n_{j2}, \cdots$, it follows that

$$E(v_j^2 \mid v_1, \cdots, v_{j-1}) < 2k^5 - k^4. \tag{13}$$

Consequently

$$\sum_{j=1}^{\infty} \frac{Ev_j^2 \mid v_1, \cdots, v_{j-1}}{j^2} < \infty,$$

and the theorem is proved.

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ON THE POWER OF A ONE-SIDED TEST OF FIT FOR CONTINUOUS PROBABILITY FUNCTIONS[1]

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Summary. If $F(x)$ is a continuous distribution function of a random variable X, and $F_n(x)$ the empirical distribution function determined by a sample $X_1, X_2, \cdots, X_n$, then the probability $\Pr\{F(x) \leqq F_n(x) + \epsilon \text{ for all } x\}$ is known [1] to be a function $P_n(\epsilon)$, independent of $F(x)$. A closed expression for $P_n(\epsilon)$ and a table of some of its values were presented in [2]. In the present paper $P_n(\epsilon)$ is used to test a hypothesis $F(x) = H(x)$ against an alternative $F(x) = G(x)$. The power of this test is studied and sharp upper and lower bounds for it are obtained for alternatives such that $\sup_{-\infty<x<+\infty}\{H(x) - G(x)\} = \delta$, with preassigned δ. The results of [2] are assumed known.

Received 6/14/52, revised 3/16/53.
[1] Research under the sponsorship of the Office of Naval Research.

1. Description of the test; integral formula for its power. We consider the class (F) of cumulative probability functions $F(x)$, continuous for all real x, and increasing for all x such that $0 < F(x) < 1$. (The assumption that the $F(x)$ are strictly increasing is made for convenience of argument only. Theorems 1 and 2 remain valid if one assumes the $F(x)$ nondecreasing.) We will assume $H(x) \varepsilon (F)$, $G(x) \varepsilon (F)$, and test the hypothesis $F(x) = H(x)$ against the alternative $F(x) = G(x)$ by the following procedure.

To have a test of size α, for sample size n, we use the value $\epsilon_{n,\alpha}$ from Table 1 in [2], obtain an ordered sample $X_1, X_2, \cdots, X_n$ of X, determine the empirical probability function $F_n(x)$, and reject $H(x)$ if and only if the inequality

$$H(x) < F_n(x) + \epsilon_{n,\alpha} \tag{1.1}$$

fails to hold for all real x.

The power of this test is the complementary probability to

$$P = \Pr \{H(x) < F_n(x) + \epsilon_{n,\alpha} \text{ for all } x;\ G(x)\}. \tag{1.2}$$

One verifies easily that (1.1) is satisfied for all x if and only if it is satisfied for all sample points X_i, $i = 1, 2, \cdots, n$, that is if and only if

$$H(X_i) < \frac{i-1}{n} + \epsilon_{n,\alpha}, \qquad \text{for } i = 1, 2, \cdots, n. \tag{1.3}$$

We have, therefore,

$$\begin{aligned} P &= \Pr\left\{H(X_i) < \frac{i-1}{n} + \epsilon_{n,\alpha} \qquad \text{for } i = 1, 2, \cdots, n;\ G(x)\right\} \\ &= \Pr\left\{X_i < H^{(-1)}\left(\frac{i-1}{n} + \epsilon_{n,\alpha}\right) \qquad \text{for } i = 1, 2, \cdots, n;\ G(x)\right\} \\ &= \Pr\left\{G(X_i) < G\left[H^{(-1)}\left(\frac{i-1}{n} + \epsilon_{n,\alpha}\right)\right] \qquad \text{for } i = 1, 2, \cdots, n;\ G(x)\right\}. \end{aligned}$$

Using the notation

$$\begin{aligned} L(V) &= G[H^{(-1)}(V)] && \text{for } 0 < V < 1 \\ L(V) &= \lim_{0 < V \to 0} L(V) && \text{for } V \leqq 0 \\ L(V) &= \lim_{1 > V \to 1} L(V) && \text{for } V \geqq 1. \end{aligned} \tag{1.4}$$

(The meaning and some applications of the expression $G[H^{(-1)}(V)]$ are discussed in [3], Sections 3 and 4) and keeping in mind that $U = G(X)$ has the rectangular distribution $R(u)$ in (0, 1), we conclude

$$P = \Pr\left\{U_i < L\left(\frac{i-1}{n} + \epsilon_{n,\alpha}\right) \qquad \text{for } i = 1, 2, \cdots, n;\ R(u)\right\}$$

where $U_1, U_2, \cdots, U_n$ is an ordered sample of U. The joint probability density of $U_1, U_2, \cdots, U_n$ being equal to $n!$ for $0 \leqq U_1 \leqq U_2 \leqq \cdots \leqq U_n \leqq 1$, and zero elsewhere, this can be written

$$(1.5) \qquad P = n! \int_0^{L(\epsilon)} \int_{U_1}^{L((1/n)+\epsilon)} \cdots \int_{U_{i-1}}^{L((i-1)/n+\epsilon)} \cdots \int_{U_{n-1}}^{L((n-1)/n+\epsilon)} dU_n \cdots dU_i \cdots dU_2 dU_1$$

where ϵ is written in short for $\epsilon_{n,\alpha}$.

If $H(x) = G(x)$, then $L(V) = V$ for $0 < V < 1$ and (1.5) reduces to formula (3.3) in [2]. If $H(x)$ and $G(x)$ are given, it may be possible to evaluate P, and hence the power $1 - P$, from (1.5) by quadrature, or one may compute it by numerical integration. In the general case, however, it is possible to derive from (1.5) inequalities for the power, as will be shown in Sections 2 and 3.

2. Lower bound for the power. For given hypothesis $H(x)$, we consider alternatives $G(x)$ such that

$$(2.1) \qquad \text{l.u.b.} \{H(x) - G(x)\} = \delta$$

and

$$(2.2) \qquad H(X_0) - G(X_0) = \delta.$$

For intuitive reasons one may expect that under these restrictions the power of our test will be close to its minimum when $G(x)$ is close to the function

$$G^*(x) = \begin{cases} H(X_0) - \delta & \text{for } x \leqq X_0 \\ 1 & \text{for } X_0 < x. \end{cases}$$

To verify this conjecture we consider

$$L^*(V) = G^*[H^{(-1)}(V)] = \begin{cases} H(X_0) - \delta & \text{for } 0 < V \leqq H(X_0) \\ 1 & \text{for } H(X_0) < V < 1, \end{cases}$$

and write

$$H(X_0) = V_0, \qquad H(X_0) - \delta = U_0$$

so that, by (2.2), we have

$$(2.3) \qquad L(V_0) = G[H^{-1}(V_0)] = G(X_0) = U_0,$$

and

$$L^*(V) = \begin{cases} U_0 & \text{for } 0 < V \leqq V_0 \\ 1 & \text{for } V_0 < V < 1 \end{cases}$$

Let j be the greatest integer contained in $n(U_0 + \delta - \epsilon)$,

$$(2.4) \qquad j = [n(U_0 + \delta - \epsilon)] = [n(V_0 - \epsilon)].$$

We have

$$\frac{i-1}{n} + \epsilon \leqq \frac{j}{n} + \epsilon \leqq U_0 + \delta = V_0 \qquad \text{for } i - 1 \leqq j,$$

hence, by (2.3),

$$L\left(\frac{i-1}{n} + \epsilon\right) \leqq L(V_0) = U_0 \qquad \text{for } i - 1 \leqq j. \tag{2.5}$$

This shows that replacing in (1.5) the function L by L^* in the upper limits of integration will not decrease these limits, and hence

$$P \leqq n! \int_0^{U_0} \cdots \int_{U_j}^{U_0} \int_{U_{j+1}}^{1} \cdots \int_{U_{n-1}}^{1} dU_n \cdots dU_{j+2}\, dU_{j+1} \cdots dU_1. \tag{2.6}$$

In case $j \leqq 0$, all upper limits of integration in (2.6) are 1 and we have the trivial inequality $P \leqq 1$.

An easy induction shows that the integral in (2.6) is equal to

$$1 - \sum_{i=0}^{j} \binom{n}{i} U_0^i (1 - U_0)^{n-i} = I_{U_0}(j + 1, n - j)$$

where I_p is the incomplete Beta function, so that (2.6) becomes

$$P \leqq 1 - \sum_{i=0}^{j} \binom{n}{i} U_0^i (1 - U_0)^{n-i} = I_{U_0}(j + 1, n - j). \tag{2.7}$$

Summarizing we obtain

THEOREM 1. *If $H(x)$ and $G(x)$ are continuous distribution functions which satisfy* (2.1) *and* (2.2), *then the power of the test described in Section* 1 *has the lower bound*

$$\sum_{i=0}^{j} \binom{n}{i} U_0^i (1 - U_0)^{n-i} = 1 - I_{U_0}(j + 1, n - j) \tag{2.8}$$

where

$$U_0 = H(X_0) - \delta, \qquad j = [n(H(X_0) - \epsilon)]. \tag{2.81}$$

This lower bound, as a function of $H(X_0)$, δ, ϵ, cannot be improved since, for any given $H(x)$ in (F), X_0, ϵ, one can construct a $G(x)$ arbitrarily close to $G^*(x)$.

3. Upper bound for the power. It seems plausible that, for given $H(x)$ and under the restrictions (2.1), (2.2), the power of our test will be close to its maximum when $G(x)$ is close to the function $G^{**}(x) = \max\,[H(x) - \delta, 0]$. We consider $L^{**}(V) = G^{**}[H^{-1}(V)] = \max\,(V - \delta, 0)$ and observe that when (2.1), (2.2) are satisfied we have

$$L(V) \geqq L^{**}(V) \qquad \text{for } 0 < V < 1$$

so that the integral in (1.5) will not be increased if L is replaced by L^{**} in the upper limits of integration. Denoting by $r - 1$ the greatest integer contained in $n(1 - \epsilon + \delta)$,

$$r - 1 = [n(1 - \epsilon + \delta)]$$

we obtain, therefore, the inequality

$$P \geqq n! \int_0^{\epsilon-\delta} \int_{U_1}^{(1/n)+\epsilon-\delta} \cdots \int_{U_{r-1}}^{(r-1)/n+\epsilon-\delta} \int_{U_r}^{1} \cdots \int_{U_{n-1}}^{1} dU_n \cdots dU_{r+1}\, dU_r \cdots dU_2\, dU_1 \qquad \text{for } \epsilon \geqq \delta \tag{3.1}$$

and

$$P \geqq 0 \qquad \text{for } \epsilon < \delta. \tag{3.2}$$

To evaluate (3.1) we observe that the expression on the right side is exactly that of (3.3) and (3.4) in [2], except that ϵ is replaced by $\epsilon - \delta$. This, together with (3.0) in [2], leads to the following theorem.

THEOREM 2. *If $H(x)$ and $G(x)$ satisfy* (2.1) *and* (2.2), *then the power of the test described in Section* 1 *has the upper bound*

$$1 - P_n(\epsilon - \delta) = (\epsilon - \delta) \sum_{i=0}^{[n(1-\epsilon+\delta)]} \binom{n}{i} \left(1 - \epsilon + \delta - \frac{i}{n}\right)^{n-i} \left(\epsilon - \delta + \frac{i}{n}\right)^{i-1} \qquad \text{for } \epsilon \geqq \delta,$$

and the upper bound 1 *for* $\epsilon < \delta$.

These upper bounds cannot be improved since, for any given $H(x)$ in (F), δ, ϵ, it is possible to construct a $G(x)$ in (F) arbitrarily close to $G^{**}(x)$.

4. The case of n large. The lower bound (2.8) for the power may be approximated, for n large, by the normal probability integral and, in view of

$$j = [n(U_0 + \delta - \epsilon)] = nU_0 + n(\delta - \epsilon) - \eta_n, \qquad 0 \leqq \eta_n < 1$$

is approximately equal to

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{1}{\sqrt{U_0(1-U_0)}}\{n^{\frac{1}{2}}(\delta-\epsilon)-n^{-\frac{1}{2}}\eta_n\}} e^{-(s^2/2)}\, ds \\ &> \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{1}{\sqrt{U_0(1-U_0)}}\{n^{\frac{1}{2}}(\delta-\epsilon)-n^{-\frac{1}{2}}\}} e^{-(s^2/2)}\, ds. \end{aligned} \tag{4.1}$$

It was shown by Smirnov [4] that $\epsilon_{n,\alpha}$ is asymptotically equal to $\sqrt{(1/2n)(\log 1/\alpha)}$. Substituting this in (4.1), we obtain for the lower bound of the power the asymptotic expression

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{1}{\sqrt{U_0(1-U_0)}}(n^{\frac{1}{2}}\delta-\sqrt{(1/2)\log(1/\alpha)}-n^{-\frac{1}{2}})} e^{-(s^2/2)}\, ds. \tag{4.2}$$

If only δ is known, but not U_0, then (4.2) may be replaced by its minimum with regard to U_0

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2(n^{\frac{1}{2}}\delta - \sqrt{(1/2)\log(1/\alpha)} - n^{-\frac{1}{2}})} e^{-(s^2/2)} \, ds.$$

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