THE OPERATING CHARACTERISTIC OF THE CONTROL CHART FOR SAMPLE MEANS[1]

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1. Summary. In this paper we derive the operating characteristic of the control chart for sample means when process standards are unspecified. Under the null hypothesis the distribution of the process is $N(\mu, \sigma^2)$, where μ and σ are fixed but unknown. Under the alternative the process mean is a random variable with a $N(\mu, \theta^2\sigma^2)$ distribution. Exact results are obtained for cases ranging from two samples of size 2 to four samples of 10. Bounds on the operating characteristic are obtained in particular cases ranging from five samples of 5 to 25 samples of 10.

2. Introduction. The usual procedure in constructing control charts from past data consists of the following steps [1].

(a) Classify the total number, N say, of observations to be drawn from the process into m samples of size n according to some rational method of subgrouping.

(b) For each sample, calculate a mean and a range, plotting these values on separate charts in the order drawn.

(c) Using prescribed formulae based on the data collected and the sample size, calculate upper and lower control limits for each chart.

(d) If all the plotted points fall within the control limits on both charts, accept the hypothesis that the process is in a state of statistical control. Otherwise, reject this hypothesis.

In what follows we shall confine our attention to the control chart for means. Letting x_{ij} denote the jth observation in the ith random sample of size n, we will be concerned with the following linear model

$$x_{ij} = \mu_i + \epsilon_{ij}, \qquad i = 1, 2, \cdots, m; j = 1, 2, \cdots, n,$$

where

(1) the ϵ_{ij} are statistically independent and distributed according to $N(0, \sigma^2)$;

(2) the μ_i are statistically independent and distributed according to $N(\mu, \theta^2\sigma^2)$;

(3) the μ_i are statistically independent of the ϵ_{ij};

(4) μ and σ are fixed but unknown.

Let $\bar{x}_i$ and r_i denote the mean and range, respectively, of the ith sample $x_{i1}, \cdots, x_{in}$. The usual control chart procedure prescribes that the limits on the $\bar{x}$-chart be set at $\bar{\bar{x}} \pm A_2\bar{r}$, where

$$\bar{\bar{x}} = \frac{1}{mn}\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}, \qquad \bar{r} = \frac{1}{m}\sum_{i=1}^{m} r_i,$$

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and A_2 is defined by $E(A_2\bar{r}) = 3\sigma/\sqrt{n}$ when the underlying distribution is normal. What we seek then is the probability that the inequalities

$$\bar{\bar{x}} - A_2\bar{r} < \bar{x}_i < \bar{\bar{x}} + A_2\bar{r}, \qquad i = 1, 2, \cdots, m,$$

be satisfied simultaneously. Defining

$$(2.1) \quad \beta(k) = P\left(-\frac{k}{3}A_2\bar{r} < \bar{x}_1 - \bar{\bar{x}} < \frac{k}{3}A_2\bar{r}, \cdots, -\frac{k}{3}A_2\bar{r} < \bar{x}_m - \bar{\bar{x}} < \frac{k}{3}A_2\bar{r}\right)$$

we wish to evaluate $\beta(3)$ under the hypotheses

$$H_0 : \theta = 0;$$

$$H_1 : \theta > 0.$$

The null hypothesis, H_0, is the hypothesis of statistical control. Our choice of the alternative $\theta > 0$ is motivated by practical considerations. Since shifts in the mean of an industrial process might occur at any time and be of any magnitude, an alternative which ordered the μ_i in a particular way would be of limited interest. By treating the μ_i as random variables we are taking into account the "average effect" of m independent assignable causes of varying magnitude. Although we are unable to specify the size of any particular shift, a measure of the size of the μ_i as a group is given by the parameter θ.

3. Taylor's series expansion of $\beta(3)$. The general method employed in calculating $\beta(3)$ is first to obtain

$$(3.1) \quad \begin{aligned} \beta_0(k) &= P\left(-k\frac{\sigma}{\sqrt{n}} < \bar{x}_1 - \bar{\bar{x}} < k\frac{\sigma}{\sqrt{n}}, \cdots, -k\frac{\sigma}{\sqrt{n}} < \bar{x}_m - \bar{\bar{x}} < k\frac{\sigma}{\sqrt{n}}\right) \\ &= P\left(-k < \frac{\bar{x}_1 - \bar{\bar{x}}}{\sigma/\sqrt{n}} < k, \cdots, -k < \frac{\bar{x}_m - \bar{\bar{x}}}{\sigma/\sqrt{n}} < k\right) \end{aligned}$$

as a function of k for fixed m. We note that $\beta_0(k)$ is of considerable interest in its own right, since $\beta_0(3)$ represents the probability that the $\bar{x}$-chart will show control when σ is known.

To evaluate $\beta(3)$ we multiply the conditional probability that (for fixed $\bar{r}$) $-A_2\bar{r} < \bar{x}_i - \bar{\bar{x}} < A_2\bar{r}$ $(i = 1, 2, \cdots, m)$ by the pdf of $\bar{r}$ and integrate on $\bar{r}$ from 0 to ∞. For fixed $\bar{r}$ we have

$$\begin{aligned} &P(-A_2\bar{r} < \bar{x}_1 - \bar{\bar{x}} < A_2\bar{r}, \cdots) \\ &\qquad = P\left(-\frac{A_2\bar{r}}{\sigma/\sqrt{n}} < \frac{\bar{x}_1 - \bar{\bar{x}}}{\sigma/\sqrt{n}} < \frac{A_2\bar{r}}{\sigma/\sqrt{n}}, \cdots\right) = \beta_0\left(\frac{\sqrt{n}\,A_2\bar{r}}{\sigma}\right). \end{aligned}$$

Let $g_{mn}(\bar{r}; \sigma)$ denote the pdf of the average range of m random samples of n from a $N(\mu, \sigma^2)$ universe. Then

$$\beta(3) = \int_0^\infty g_{mn}(w; \sigma)\beta_0\left(\sqrt{n}\,A_2\frac{w}{\sigma}\right) dw.$$

Letting $g_{mn}(w; 1) = g_{mn}(w)$, we have

$$g_{mn}(w; \sigma) = \frac{1}{\sigma} g_{mn}\left(\frac{w}{\sigma}\right).$$

Thus

$$\beta(3) = \int_0^\infty \frac{1}{\sigma} g_{mn}\left(\frac{w}{\sigma}\right) \beta_0\left(\sqrt{n}\, A_2 \frac{w}{\sigma}\right) dw.$$

Finally, letting $\bar{r} = w/\sigma$ yields

$$\beta(3) = \int_0^\infty g_{mn}(\bar{r}) \beta_0(\sqrt{n} A_2 \bar{r})\, d\bar{r}. \tag{3.2}$$

We now expand $\beta_0(\sqrt{n}\, A_2 \bar{r})$ in a Taylor's series with remainder about the point $\bar{r} = d_2$, where $d_2\sigma$ denotes the expected value of the range in a random sample of n from a $N(\mu, \sigma^2)$ population. Clearly

$$\beta_0(\sqrt{n} A_2 \bar{r}) = \beta_0(3) + b_1(\bar{r} - d_2) + \cdots + \frac{b_p}{p!}(\bar{r} - d_2)^p + R'(\xi), \tag{3.3}$$

where the b_i are the ith derivatives of $\beta_0(\sqrt{n}\, A_2 \bar{r})$ with respect to $\bar{r}$, evaluated at $\bar{r} = d_2$, and

$$R'(\xi) = \frac{b_{p+1}(\xi)}{(p+1)!}(\bar{r} - d_2)^{p+1}, \qquad \xi = d_2 + \alpha(\bar{r} - d_2), \qquad 0 \leqq \alpha \leqq 1. \tag{3.4}$$

This expansion is valid since all the b_i are continuous. Now, taking the expectation of both sides of (3.3) yields

$$\int_0^\infty g_{mn}(\bar{r}) \beta_0(\sqrt{n} A_2 \bar{r})\, d\bar{r} = \beta_0(3) \int_0^\infty g_{mn}(\bar{r})\, d\bar{r} + b_1 \int_0^\infty (\bar{r} - d_2) g_{mn}(\bar{r})\, d\bar{r} + \cdots + \int_0^\infty R'(\xi) g_{mn}(\bar{r})\, d\bar{r}$$

But since the first integral on the right side is unity and the second zero, we have

$$\beta(3) = \beta_0(3) + \frac{b_2}{2!}\mu_2(\bar{r}) + \cdots + \frac{b_p}{p!}\mu_p(\bar{r}) + R, \tag{3.5}$$

where the $\mu_i(\bar{r})$ are central moments of average range, and

$$R = \int_0^\infty \frac{b_{p+1}(\xi)}{(p+1)!}(\bar{r} - d_2)^{p+1} g_{mn}(\bar{r})\, d\bar{r}. \tag{3.6}$$

For the actual numerical computation of $\beta(3)$, relation (3.5) is used in the form

$$\beta(3) = \beta_0(3) + \frac{b_2}{2!}\frac{\mu_2(r)}{m} + \frac{b_3}{3!}\frac{\mu_3(r)}{m^2} + \frac{b_4}{4!}\frac{1}{m^3}[\mu_4(r) + 3(m-1)\mu_2^2(r)] + \cdots, \tag{3.7}$$

where the $\mu_i(r)$ are the ith central moments of the range in samples of n from a $N(\mu, 1)$ population. In each case, a sufficient number of terms of (3.7) is used to insure that the remainder, R, is small.

4. Derivation of $\beta_0(k)$. As indicated in the previous section, our first task is to obtain an expression for $\beta_0(k)$. It is apparent that it can be written as

$$\beta_0(k) = P\left(\frac{\bar{x}_{(m)} - \bar{\bar{x}}}{\sigma/\sqrt{n}} < k, \frac{\bar{\bar{x}} - \bar{x}_{(1)}}{\sigma/\sqrt{n}} < k\right), \tag{4.1}$$

where $\bar{x}_{(m)}$ and $\bar{x}_{(1)}$ are the largest and smallest sample means, respectively. To obtain a tractable expression for this probability we shall first investigate the joint distribution of $\frac{\bar{x}_{(m)} - \bar{\bar{x}}}{\sigma/\sqrt{n}}$ and $\frac{\bar{\bar{x}} - \bar{x}_{(1)}}{\sigma/\sqrt{n}}$. For this purpose let us consider the joint pdf of the ordered variates $\bar{x}_{(1)} < \bar{x}_{(2)} < \cdots < \bar{x}_{(m)}$. Since $\bar{x}_i = \mu_i + \bar{\epsilon}_i$, where $\bar{\epsilon}_i = \sum_{j=1}^{n} \epsilon_{ij}/n$, we see that the distribution of $\bar{x}_i$ is normal with mean μ and variance $(\theta^2 + 1/n)\sigma^2$. Assuming $\mu = 0$ (with no loss of generality) we have

$$f(\bar{x}_{(1)}, \cdots, \bar{x}_{(m)}) = \frac{m!}{(2\pi)^{m/2}\left(\frac{1 + n\theta^2}{n}\right)^{m/2}\sigma^m} \exp\left[-\frac{n}{2(1 + n\theta^2)\sigma^2}\sum_{i=1}^{m} \bar{x}_{(i)}^2\right]$$

as the joint pdf of $\bar{x}_{(1)} < \bar{x}_{(2)} < \cdots < \bar{x}_{(m)}$. To obtain the joint distribution of $\frac{\bar{x}_{(m)} - \bar{\bar{x}}}{\sigma/\sqrt{n}}$ and $\frac{\bar{\bar{x}} - \bar{x}_{(1)}}{\sigma/\sqrt{n}}$, let

$$\frac{\sigma}{\sqrt{n}}\sqrt{2}\, v_1 = -\bar{x}_{(2)} + \bar{x}_{(3)},$$

$$\frac{\sigma}{\sqrt{n}}\sqrt{3\cdot 2}\, v_2 = -\bar{x}_{(2)} - \bar{x}_{(3)} + 2\bar{x}_{(4)},$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\sigma}{\sqrt{n}}\sqrt{(m-3)(m-2)}\, v_{m-3} = -\bar{x}_{(2)} - \bar{x}_{(3)} - \cdots - \bar{x}_{(m-2)} + (m-3)\,\bar{x}_{(m-1)},$$

$$\frac{\sigma}{\sqrt{n}}\sqrt{(m-1)(m-2)}\, v_{m-2} = -(m-2)\bar{x}_{(1)} + \bar{x}_{(2)} + \cdots + \bar{x}_{(m-1)},$$

$$\frac{\sigma}{\sqrt{n}}\sqrt{m(m-1)}\, v_{m-1} = -\bar{x}_{(1)} - \bar{x}_{(2)} - \cdots - \bar{x}_{(m-1)} + (m-1)\,\bar{x}_{(m)},$$

$$\frac{\sigma}{\sqrt{n}}\sqrt{m}\, v_m = \bar{x}_{(1)} + \bar{x}_{(2)} + \cdots + \bar{x}_{(m-1)} + \bar{x}_{(m)}.$$

This transformation gives

$$g(v_1, \cdots, v_m) = \frac{m!}{(2\pi)^{m/2}(1 + n\theta^2)^{m/2}} \exp\left[-\frac{1}{2(1 + n\theta^2)}\sum_{i=1}^{m} v_i^2\right]$$

as the joint pdf of the v_i, defined over the region S given by

$$S\begin{cases} v_1 > 0, \quad v_i < \sqrt{\dfrac{i+2}{i}}\, v_{i+1}, & (i = 1, \cdots, m-4), \\ \sqrt{m(m-2)}\, v_{m-1} > \sqrt{(m-1)(m-3)}\, v_{m-3} + v_{m-2}, \\ \sqrt{\dfrac{m-1}{m-2}}\, v_{m-2} > \dfrac{v_1}{\sqrt{2}} + \dfrac{v_2}{\sqrt{3\cdot 2}} + \cdots + \dfrac{v_{m-3}}{\sqrt{(m-2)(m-3)}}. \end{cases}$$

Now, we integrate out v_m and let

$$v_i = u_i, \qquad (i = 1, \cdots, m-3),$$

$$v_{m-2} = -\frac{u}{\sqrt{(m-1)(m-2)}} + \sqrt{\frac{m-1}{m-2}}\, v,$$

$$v_{m-1} = \sqrt{\frac{m}{m-1}}\, u,$$

where in terms of the original variables

$$(4.2) \qquad \begin{cases} u = \dfrac{\bar{x}_{(m)} - \bar{\bar{x}}}{\sigma/\sqrt{n}}, \\ v = \dfrac{\bar{\bar{x}} - \bar{x}_{(1)}}{\sigma/\sqrt{n}}. \end{cases}$$

The Jacobian of this transformation is $|J| = \sqrt{m/(m-2)}$. The joint density of $u_1, u_2, \cdots, u_{m-3}, u, v$ is finally obtained as

$$(4.3) \qquad h_m(u_1, \cdots, u_{m-3}, u, v) = \frac{m!\sqrt{\dfrac{m}{m-2}}}{(2\pi)^{(m-1)/2}(1+n\theta^2)^{(m-1)/2}} \cdot \exp\left[-\frac{1}{2(1+n\theta^2)}\sum_{i=1}^{m-3} u_i^2\right] \exp\left[-\frac{\Lambda_m}{2(1+n\theta^2)}\right]$$

over S', where

$$\Lambda_m = \frac{m-1}{m-2}(u^2+v^2) - \frac{2}{m-2}uv,$$

$$S' \text{ is } \begin{cases} u_1 > 0,\ u > 0,\ v > 0,\ u_i < \sqrt{\dfrac{i+2}{i}}\, u_{i+1}, & (i = 1, \cdots, m-4), \\ \dfrac{u_1}{\sqrt{2}} + \dfrac{u_2}{\sqrt{3\cdot 2}} + \cdots + \dfrac{u_{m-3}}{\sqrt{(m-2)(m-3)}} < \dfrac{1}{m-2}[(m-1)v - u], \\ \sqrt{(m-2)(m-3)}\, u_{m-3} < (m-1)u - v. \end{cases}$$

For the cases $m = 2$ through $m = 4$, $\beta_0(k)$ is obtained by integrating (4.3) over the range $u_1, \cdots, u_{m-3} \in S'$, $u < k$, $v < k$. $\beta(3)$ is then evaluated from the series (3.7). For cases where $m > 4$, bounds on the desired probabilities are obtained by methods described in Section 5*D*.

5. Exact evaluation of $\beta_0(k)$ and $\beta(3)$.

A. Case $m = 2$. In this case, $\beta_0(k)$ becomes simply

$$(5.1) \qquad \begin{aligned} \beta_0(k) &= P\left(-k < \frac{\bar{x}_1 - \bar{x}_2}{2\sigma/\sqrt{n}} < k\right) \\ &= \Phi\left(\frac{\sqrt{2}k}{\sqrt{1+n\theta^2}}\right) - \Phi\left(-\frac{\sqrt{2}k}{\sqrt{1+n\theta^2}}\right), \end{aligned}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$. The series expansion for $\beta(3)$ becomes

$$(5.2)\quad \beta(3) = \beta_0(3) + \frac{b_2}{4}\mu_2(r) + \frac{b_3}{24}\mu_3(r) + \frac{b_4}{192}(\mu_4(r) + 3\mu_2^2(r)) + \cdots,$$

where

$$b_i = \left[\frac{d_i}{d\bar{r}^i}\beta_0(\sqrt{n}\, A_2 \bar{r})\right]_{\bar{r}=d_2}.$$

The moments $\mu_i(r)$ are obtained from Hartley and Pearson [2]. Expression (5.2) is used to evaluate $\beta(3)$ for $n = 5$ and $n = 10$. For the sake of completeness the case $n = 2$ is also included. In this case, the pdf of the range is

$$h(r) = \frac{1}{\sqrt{\pi}\sigma} e^{-(r^2/(4\sigma^2))} \qquad (r > 0).$$

Hence, by the standard convolution (see [4], p. 191),

$$g_{22}(\bar{r}; \sigma) = 2\int_0^{2\bar{r}} h(2\bar{r} - t)\, h(t)\, dt,$$

which reduces to

$$g_{22}(\bar{r}; \sigma) = \frac{4}{\sqrt{2\pi}\sigma} e^{-(\bar{r}^2/(2\sigma^2))}\left[\Phi\left(\frac{\bar{r}}{\sigma}\right) - \Phi\left(-\frac{\bar{r}}{\sigma}\right)\right].$$

Hence

$$\begin{aligned}\beta(3) &= \int_0^\infty g_{22}(\bar{r})\beta_0(\sqrt{2}A_2\bar{r})\, d\bar{r}\\ &= \int_0^\infty \frac{4}{\sqrt{2\pi}} e^{-\frac{1}{2}\bar{r}^2}[\Phi(\bar{r}) - \Phi(-\bar{r})]\\ &\qquad \cdot\left[\Phi\left(\frac{2A_2\bar{r}}{\sqrt{1+2\theta^2}}\right) - \Phi\left(-\frac{2A_2\bar{r}}{\sqrt{1+2\theta^2}}\right)\right] d\bar{r}.\end{aligned}$$

$1 - \beta(3)$ is then evaluated by numerical integration.

The results for $m = 2$ are summarized in Tables I and II below.

B. Case $m = 3$. We find from (4.3) that the joint density of u and v is given by

$$(5.3)\quad f_3(u, v) = \frac{3!\sqrt{3}}{2\pi(1 + n\theta^2)} \exp\left[-\frac{1}{2(1 + n\theta^2)}(u^2 - uv + v^2)\right] \text{ over } S',$$

where

$$S' \text{ is } \begin{cases} 0 < v < 2u, \\ 0 < u < 2v. \end{cases}$$

Then

$$\beta_0(k) = 2\int_0^k du \int_{\frac{1}{2}u}^{u} f_3(u, v)dv,$$

by virtue of the fact that $f_3(u, v)$ is symmetric about the line $u = v$. This integral reduces to

$$(5.4) \qquad \beta_0(k) = 6\sqrt{\frac{3}{\pi}}\int_0^{k/(2\sqrt{1+n\theta^2})} e^{-3t^2}F_2(t)\,dt,$$

where $F_2(x)$ is defined by

$$F_2(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

$F_n(x)$ is the cdf of the extreme deviation from the sample mean in samples of n from a $N(0, 1)$ population. This function has been tabulated by Grubbs [3]. $\beta_0(3)$

TABLE I

$\beta_0(3)$ *for* $m = 2$

θ	$n = 2$	$n = 5$	$n = 10$
0	1.00	1.00	1.00
.5	1.00	1.00	.98
1.0	.99	.92	.80
1.5	.93	.77	.62
2.0	.84	.64	.49
2.5	.75	.55	.40
3.0	.67	.47	.34

TABLE II

$\beta(3)$ *for* $m = 2$

θ	$n = 2$	$n = 5$	$n = 10$
0	.96	1.00	1.00
.5	.94	.98	.96
1.0	.89	.89	.79
1.5	.82	.75	.61
2.0	.75	.63	.49
2.5	.67	.54	.40
3.0	.61	.46	.34

is evaluated from (5.4) by numerical integration and $\beta(3)$ obtained from the series (3.7). The results for this case are summarized in Tables III and IV.

C. Case $m = 4$. From (4.3) we obtain the joint pdf of u_1, u, and v as

$$(5.6) \qquad h_4(u_1, u, v) = \frac{4!\sqrt{\frac{4}{2}}}{(2\pi)^{\frac{3}{2}}(1 + n\theta^2)^{3/2}}\, e^{-[u_1^2/(2(1+n\theta^2))]}e^{-[\Lambda_4/(2(1+n\theta^2))]} \quad \text{over } S,$$

where

$$\Lambda_4 = \tfrac{3}{2}(u^2 + v^2) - uv,$$

$$S \text{ is} \begin{cases} u_1 > 0,\ u > 0,\ v > 0, \\ \sqrt{2}\,u_1 < 3v - u, \\ \sqrt{2}\,u_1 < 3u - v. \end{cases}$$

To obtain $f_4(u, v)$, the joint density of u and v, we integrate over the range of u_1, with the following result

$$f_4(u, v) = \begin{cases} C_1 e^{-[\Lambda_4/(2(1+n\theta^2))]}\displaystyle\int_0^{(3v-u)/\sqrt{2}} e^{-[u_1^2/(2(1+n\theta^2))]}\,du, & \text{for } v < u < 3v, \\ C_1 e^{-[\Lambda_4/(2(1+n\theta^2))]}\displaystyle\int_0^{(3u-v)/\sqrt{2}} e^{-[u_1^2/(2(1+n\theta^2))]}\,du, & \text{for } u < v < 3u, \end{cases}$$

where

$$C_1 = \frac{4!\sqrt{\frac{4}{2}}}{[2\pi(1 + n\theta^2)]^{3/2}}.$$

But

$$\int_0^{(3v-u)/\sqrt{2}} e^{-u_1^2/(2(1+n\theta^2))}\,du_1 = \sqrt{2}\sqrt{1 + n\theta^2}\int_0^{(3v-u)/(2\sqrt{1+n\theta^2})} e^{-t^2}\,dt$$

$$= \sqrt{\frac{\pi}{2}}\sqrt{1 + n\theta^2}\,F_2\left(\frac{3v - u}{2\sqrt{1 + n\theta^2}}\right).$$

TABLE III

$\beta_0(3)$ *for* $m = 3$

θ	$n = 2$	$n = 5$	$n = 10$
0	1.00	1.00	1.00
.5	.99	.96	.88
1.0	.92	.71	.48
1.5	.74	.45	.27
2.0	.56	.30	.17
2.5	.42	.21	.11
3.0	.32	.15	.08

TABLE IV

$\beta(3)$ *for* $m = 3$

θ	$n = 5$	$n = 10$
0	.99	1.00
.5	.94	.86
1.0	.69	.48
1.5	.45	.27
2.0	.30	.17
2.5	.21	.11
3.0	.15	.08

In exactly the same manner

$$\int_0^{(3u-v)/\sqrt{2}} e^{-[u_1^2/(2(1+n\theta^2))]}\,du_1 = \sqrt{\frac{\pi}{2}}\sqrt{1 + n\theta^2}\,F_2\left(\frac{3u - v}{2\sqrt{1 + n\theta^2}}\right).$$

Therefore

$$f_4(u, v) = C_1\sqrt{\frac{\pi}{2}}\sqrt{1 + n\theta^2}\,e^{-[A_4/(2(1+n\theta^2))]}\,G(u, v),$$

where

$$G(u, v) = \begin{cases} F_2\left(\dfrac{3u - v}{2\sqrt{1 + n\theta^2}}\right) & \text{for} \quad u < v < 3u, \\ F_2\left(\dfrac{3v - u}{2\sqrt{1 + n\theta^2}}\right) & \text{for} \quad v < u < 3v. \end{cases}$$

Then

$$\beta_0(k) = 2\int_0^k du \int_{u/3}^u f_4(u, v)\,du,$$

since $f_4(u, v)$ is symmetric in u and v. After some reduction we find

$$\beta_0(k) = 12 \frac{\sqrt{\frac{4}{3}}}{\sqrt{2\pi}} \int_0^{2k/(3\sqrt{1+n\theta^2})} e^{-(3t^2)/2} F_3(t)\, dt, \tag{5.7}$$

where

$$F_3(t) = \frac{3\sqrt{3}}{2\sqrt{\pi}} \int_0^t e^{-(3x^2/4)} F_2\left(\frac{3x}{2}\right) dx.$$

As before, $\beta_0(k)$ is evaluated by integrating (5.7) numerically and $\beta(3)$ is obtained from the series expansion. The results for this case are summarized in Tables V and VI.

TABLE V

$\beta_0(3)$ *for* $m = 4$

θ	$n = 2$	$n = 5$	$n = 10$
0	1.00	1.00	1.00
.5	.98	.91	.76
1.0	.82	.52	.27
1.5	.56	.25	.11
2.0	.34	.12	.05
2.5	.22	.07	.03
3.0	.14	.05	.02

TABLE VI

$\beta(3)$ *for* $m = 4$

θ	$n = 5$	$n = 10$
0	.99	1.00
.5	.87	.74
1.0	.51	.27
1.5	.26	.11
2.0	.13	.05
2.5	.07	.03
3.0	.05	.02

D. Case $m > 4$. In this section we derive upper and lower bounds for $\beta_0(k)$ and $\beta(3)$. Recalling that $\beta_0(k) = P(u < k, v < k)$ we immediately obtain the upper bound $\beta_0(k) \leqq P(u < k)$. But since $u = (\bar{x}_{(m)} - \bar{\bar{x}})/(\sigma/\sqrt{n})$, we see that $u/\sqrt{1 + n\theta^2}$ is distributed as the extreme deviation from the sample mean in samples of m from a $N(0, 1)$ population. Hence, using Grubbs' notation [3]

$$\beta_0(k) \leqq F_m\left(\frac{k}{\sqrt{1 + n\theta^2}}\right). \tag{5.8}$$

It follows that

$$\beta(3) = \int_0^\infty g_{mn}(\bar{r})\beta_0(\sqrt{n}\, A_2\bar{r})\, d\bar{r} \leqq \int_0^\infty g_{mn}(\bar{r}) F_m\left(\frac{\sqrt{n}\, A_2\bar{r}}{\sqrt{1 + n\theta^2}}\right) d\bar{r}. \tag{5.9}$$

(5.9) is then expanded in a Taylor's series with remainder by the method of Section 3.

To obtain a lower bound, we have from elementary probability considerations $P(u < k \text{ or } v < k) = P(u < k) + P(v < k) - P(u < k, v < k)$, or

$$P(u < k, v < k) = P(u < k) + P(v < k) - P(u < k \text{ or } v < k). \tag{5.10}$$

Since the last term in (5.10) cannot exceed unity we have

$$P(u < k, v < k) = \beta_0(k) \geqq P(u < k) + P(v < k) - 1.$$

But

$$P(u < k) = P(v < k) = F_m\left(\frac{k}{\sqrt{1 + n\theta^2}}\right).$$

TABLE VII

Bounds on $\beta_0(3)$

θ	$m = 5$		$m = 10$	
	$n = 5$	$n = 10$	$n = 5$	$n = 10$
0	1.00	1.00	.98 .99	.98 .99
.5	.87 .94			
1.5		.00 .12	.00 .06	.00 .01
2.0	.00 .14	.00 .05	.00 .01	
2.5	.00 .07	.00 .02		
3.0	.00 .04	.00 .01		

θ	$m = 15$		$m = 20$	
	$n = 5$	$n = 10$	$n = 5$	$n = 10$
0	.97 .98	.97 .98	.96 .98	.96 .98
.75			.00 .26	.00 .04
1.0	.00 .13	.00 .02	.00 .06	
1.25			.00 .01	
1.5	.00 .01			

θ	$m = 25$	
	$n = 5$	$n = 10$
0	.95 .97	.95 .97
.25	.80 .91	
.75	.00 .18	.00 .02
1.00	.00 .03	

Therefore

$$\beta_0(k) \geqq 2F_m\left(\frac{k}{\sqrt{1 + n\theta^2}}\right) - 1. \tag{5.11}$$

The corresponding lower bound on $\beta(3)$ becomes

$$\beta(3) \geqq \int_0^\infty g_{mn}(\bar{r}) \left[2F_m\left(\frac{\sqrt{n}\, A_2 \bar{r}}{\sqrt{1 + n\theta^2}}\right) - 1\right] d\bar{r}, \tag{5.12}$$

which is again evaluated by a Taylor expansion. The results given in Tables VII and VIII do not include certain intermediate values of θ in the range 0.25 to 1.50 because the lower bounds (5.11) and (5.12) are very weak in this range.

TABLE VIII

Bounds on $\beta(3)$

θ	$m = 5$		$m = 10$	
	$n = 5$	$n = 10$	$n = 5$	$n = 10$
0	.99	1.00	.97 .99	.98 .99
.5	.81 .99			
1.5		.00 .12	.00 .06	.00 .01
2.0	.00 .14	.00 .05	.00 .01	
2.5	.00 .07	.00 .02		
3.0	.00 .04	.00 .01		

θ	$m = 15$		$m = 20$	
	$n = 5$	$n = 10$	$n = 5$	$n = 10$
0	.96 .98	.96 .98	.94 .97	.95 .98
.75			.00 .27	.00 .04
1.0	.00 .13	.00 .02	.00 .06	
1.25			.00 .01	
1.5	.00 .01			

	$m = 25$	
θ	$n = 5$	$n = 10$
0	.94 .97	.94 .97
.25	.80 .90	
.75	.00 .18	.00 .02
1.00	.00 .03	

An important feature of Tables VII and VIII is the close agreement between corresponding values of $\beta_0(3)$ and $\beta(3)$. Our ignorance of σ is of little consequence so far as these results are concerned.

6. Bounds on the remainder term. In each of the Taylor expansions used in Section 5, there is a remainder term of the form

$$R = \int_0^\infty \frac{b_{p+1}(\xi)}{(p+1)!} (\bar{r} - d_2)^{p+1} g_{mn}(\bar{r}) \, d\bar{r}.$$

In order to determine a bound on R, we note that

$$|R| \leqq \int_0^\infty \frac{|b_{p+1}(\xi)|}{(p+1)!} |\bar{r} - d_2|^{p+1} g_{mn}(\bar{r})\, d\bar{r}$$

$$\leqq \frac{\text{Max } |b_{p+1}(\xi)|}{(p+1)!} \int_0^\infty |\bar{r} - d_2|^{p+1} g_{mn}(\bar{r})\, d\bar{r}.$$

If we restrict p to be an odd integer, we have

$$|R| \leqq \frac{\text{Max } |b_{p+1}(\xi)|}{(p+1)!} \mu_{p+1}(\bar{r}). \tag{6.1}$$

In practically all cases given in the preceding tables, a sufficient number of terms of the series (3.7) is used to insure that the bound (6.1) does not exceed 0.005. In isolated instances where this is not possible, terms up to and including $b_6/6!\, \mu_6(\bar{r})$ are employed.

7. Acknowledgment. The author is indebted to Professor E. G. Olds for suggesting this problem and to both Professor Olds and Mr. R. B. Murphy for their kind guidance and encouragement in the course of study.

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