SUFFICIENT STATISTICS AND SELECTION DEPENDING ON THE PARAMETER

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1. Summary. The concept of "functional sufficiency" or "f-sufficiency" for a class of density functions is introduced and conditions given under which it corresponds to sufficiency as defined by Halmos and Savage [2]. A minimal f-sufficient statistic is defined and shown to exist, and its construction is given.

The minimal f-sufficient statistic for a class of densities for which the region of positive density varies with the parameter is shown to be equivalent to the combination of a "statistic of selection" and the minimal f-sufficient statistic for a class of densities for which the region of positive density is fixed. The construction of sufficient statistics in this latter case subject to certain conditions of regularity has been treated by Koopman [1].

If the parameter is a parameter of selection from a fixed distribution, then the statistic of selection is the minimal f-sufficient statistic. If in addition the regions of positive density are monotone and indexed monotonely by a real parameter, then the statistic of selection is sufficient according to the Halmos and Savage definition. Three examples are given to illustrate the results.

2. Introduction. The following notation follows closely that introduced in [2] by Halmos and Savage. Let X be a general space and S a Borel class over it; then (X, S) is called a measurable space. Since X will be considered the space of a chance quantity, a statistic T is a function over X, that is, a transformation from X into a measurable space (T, S'). The transformation is measurable if the inverse transformation of the Borel sets S' are elements of S.

Let $\mathfrak{M} = \{\mu\}$ be a class of probability measures over the space X. If μ and ν are two measures on X, ν is absolutely continuous with respect to μ, in symbols $\nu \ll \mu$, if $\nu(E) = 0$ for every $E \in S$ for which $\mu(E) = 0$. A set of measures $\mathfrak{M}$ on X will be called dominated if there exists a measure λ on X such that for each $\mu \in \mathfrak{M}$ we have $\mu \ll \lambda$, in notation $\mathfrak{M} \ll \lambda$.

In this paper we consider only dominated sets of measures $\mathfrak{M} \ll \lambda$. The Radon-Nikodym Theorem gives us the following statement:

For any dominated set of measures $\mathfrak{M} \ll \lambda$, there exists a set of nonnegative real-valued functions over X, $\{f_\mu(x)\}$, such that

$$\mu(E) = \int_E f_\mu(x)\, d\lambda(x)$$

for every $\mu \in \mathfrak{M}$ and $E \in S$. The functions $f_\mu(x)$ are unique except on a set of λ-measure zero. We shall refer to $f_\mu(x)$ as a density function relative to the fixed measure λ.

For any such class of measures and a corresponding set of densities, this paper presents theorems concerning sufficient statistics.

3. Definition of sufficient statistic. The concept of sufficient statistic was given a rigorous formulation in [2]. In words the definition is as follows.

T is sufficient for $\mathfrak{M}$ if there exists for almost all (μT^{-1}) values of T a conditional probability function which is independent of $\mu \in \mathfrak{M}$.

Corollary 1 to Theorem 1 in [2] gives an equivalent condition for a sufficient statistic:

The statistic T is sufficient for a dominated set of measures if and only if $f_\mu(x)$ factors in the form $f_\mu(x) = g_\mu(T)h(x)$ (except on a set of μ-measure zero), where $g_\mu(T)$ is measurable, g and h are nonnegative, and h and gh are integrable with respect to λ.

For the purposes of this paper we introduce the concept of functional sufficiency with respect to a class of density functions $\{f_\mu(x)\}$ over a fixed measure λ. For these density functions μ can be considered as a parameter or index of the family. *A function $T(x)$ is functionally sufficient (f-sufficient) for $\{f_\mu(x)\}$ if and only if $f_\mu(x) = g_\mu(T)h(x)$, where $g_\mu(T)$, $h(x)$ are nonnegative.* For the study of f-sufficiency the normalization and integrability conditions on $f_\mu(x)$ are not important; however, the property that $f_\mu(x) \geqq 0$ will be assumed throughout. *A function $T(x)$ is a minimal f-sufficient statistic*[1] *for $\{f_\mu(x)\}$ if no function of $T(x)$ (other than 1-1) is f-sufficient for $\{f_\mu(x)\}$.*

LEMMA 1. *For any $\{f_\mu(x)\}$ there exists a minimal f-sufficient statistic.*

PROOF. The minimal f-sufficient statistic will be given as a mapping of the elements of X onto a class of disjoint subsets covering X. The mapping is from the elements of a set to the set itself. Thus we need only define the disjoint sets covering X. Define sets as follows:

$$E_0 = \{x \mid f_\mu(x) \text{ is independent of } \mu\},$$

$$E_{x'} = \{x \mid \exists h(x) \text{ such that } f_\mu(x) = h(x)f_\mu(x')\}.$$

By this procedure the whole space X can be covered. Also a simple argument shows that the sets are disjoint.

We must now show that these sets provide us with an f-sufficient statistic. Let T index the sets and $T(x)$ be the mapping from x to the set containing x. Then we have either $f_\mu(x) = h_0(x)$, or $f_\mu(x) = h_T(x)f_\mu(x')$, where $x' = x'(T)$. But this implies $f_\mu(x) = g_\mu(T)h(x)$, and hence T is f-sufficient.

For any function $T'(T)$ other than 1-1 there would exist at least two sets E', E'' giving the same value to the function T'. Over E', E'' the representation $f_\mu(x) = g_\mu(T')h(x)$ is impossible, since it would imply that the sets were identical. Thus T is a minimal f-sufficient statistic.

[1] The concept of a minimal f-sufficient statistic was obtained independently of [4]. The partition of X induced by the minimal f-sufficient statistic is essentially the partition produced by the operator ϑ introduced by Lehmann and Scheffé.

LEMMA 2. *There is a unique minimal f-sufficient statistic and it is a function of any other f-sufficient statistic.*

PROOF. Let $T(x)$ be any f-sufficient statistic: then $f_\mu(x) = g_\mu(T)h(x)$. Consider the sets E corresponding to the partition of X induced by the function $T(x)$. Within any one E of the sets

$$f_\mu(x) = g_\mu(E)h(x).$$

Hence E is contained in a set of the partition induced by the minimal f-sufficient statistic defined in Lemma 1 and therefore that minimal f-sufficient statistic is a function of $T(x)$. This with the definition of minimal f-sufficient statistic establishes uniqueness.

LEMMA 3. *Any sufficient statistic for a dominated set of measures $\mathfrak{M} \ll \lambda$ is an f-sufficient statistic for an equivalent set of densities (relative to λ).*

PROOF. Let T be any sufficient statistic for $\mathfrak{M}$. By Corollary 1 to Theorem 1 in [2], there exists a set of densities $\{f_\mu(x)\}$ such that $f_\mu(x) = g_\mu(T)h(x)$ except on a set of μ-measure zero. An equivalent set of densities is $\{g_\mu(T)h(x)\}$ and these factor for all values of x. Hence T is f-sufficient for $\{f_\mu(x)\}$.

LEMMA 4. *An f-sufficient statistic T for $\{f_\mu(x)\}$ is sufficient for $\{\mu\} = \mathfrak{M}$ if and only if $f_\mu(x) = g_\mu(T)h(x)$ where $T(x)$ is measurable, $g_\mu(T)$ is measurable, and h and gh are integrable with respect to λ.*

PROOF. Follows immediately from Corollary 1, Theorem 1 [2].

The structure of f-sufficiency, unlike sufficiency, is not invariant under changes in $\{f_\mu(x)\}$ which do not alter the set $\mathfrak{M}$. However for applications the conditions in Lemma 4 will usually be fulfilled, and consequently any procedure for obtaining f-sufficient statistics will assist the study of sufficient statistics.

4. Selection. Consider a set of densities $\{f_\mu(x)\}$ over a space X. Considering μ as an index or parameter of the class, we define a parameter of selection.

μ is a "parameter of selection" for $\{f_\mu(x)\}$ if

$$f_\mu(x) = \phi_{S_\mu}(x)f(x)g(\mu)\,,$$

where $S_\mu = \{x \mid f_\mu(x) > 0\}$ and $\phi_S(x) = 0, 1$, according as $x \notin S, \in S$.

If there is a class of measures $\mathfrak{M}$ which are essentially truncations of, or selections from a fixed measure, then there will exist an equivalent set of densities for which μ is a "parameter of selection."

We now define a "statistic of selection" for any set of densities $\{f_\mu(x)\}$. It is a function mapping X into the space of subsets of X.

$T(x)$, a statistic of selection, is defined by

$$T(x) = \bigcap S_\mu\,,$$

where the intersection is over all $\mu \in \mathfrak{M}^*(x) = \{\mu \mid x \in S_\mu\} = \{\mu \mid f_\mu(x) > 0\}$.

We also define a characteristic function for sets A and $B \subset X$:

$$\psi(A, B) = 0 \text{ if } A \not\subset B,$$
$$= 1 \text{ if } A \subset B.$$

5. F-sufficiency and the statistic of selection. A set of measures $\mathfrak{M}$ is said to be homogeneous if $\nu \ll \mu$ and $\mu \ll \nu$ for every pair $\mu, \nu \in \mathfrak{M}$. Thus a dominated set of measures with densities $\{f_\mu(x)\}$ is homogeneous if S_μ is independent of μ. In such a case the Halmos and Savage criterion of sufficiency is simplified, and if working over the real space subject to certain continuity conditions on the $f_\mu(x)$ the problem of sufficiency has been treated by Koopman [1].

The theorem in this section reduces the problem of finding an f-sufficient (or minimal f-sufficient) statistic for a set of densities corresponding to a dominated set of measures to that of finding an f-sufficient (or minimal f-sufficient) statistic for the special type of densities (mentioned above) corresponding to a homogeneous set of measures. Application of these results for sufficiency then rests on Lemmas 3 and 4 and the fact that in applications sufficiency and f-sufficiency will usually be the same.

THEOREM 1. *For a class of densities $f_\mu(x)$ relative to a fixed measure λ on X, the minimal f-sufficient statistic is the combination of the statistic of selection and the minimal f-sufficient statistic for a class of densities for which S_μ is independent of μ (the carrier independent of the parameter).*

PROOF. With no loss of generality we can assume $\bigcup S_\mu = X$, where the union is over all $\mu \in \mathfrak{M}$.

We first show the equivalence of the two characteristic functions $\phi_{S_\mu}(x)$ and $\psi(T(x), S_\mu)$, where $T(x)$ is the statistic of selection.

$$\phi_{S_\mu}(x) = 0, 1$$

implies

$$x \notin S_\mu, \qquad x \in S_\mu,$$

which implies

$$T(x) \not\subset S_\mu, \qquad T(x) \subset S_\mu,$$

which implies

$$\psi(T, S_\mu) = 0, 1.$$

Hence

$$\phi_{S_\mu}(x) \equiv \psi(T(x), S_\mu).$$

The density function $f_\mu(x)$ can be written as follows:

$$\begin{aligned} f_\mu(x) &= \phi_{S_\mu}(x) f_\mu(x) \\ &= \psi(T, S_\mu) f_\mu(x) \\ &= \psi(T, S_\mu) \bar{f}_\mu(x), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \bar{f}_\mu(x) &= f_\mu(x) \text{ if } x \in S_\mu, \\ &= g_\mu(x) \text{ if } x \notin S_\mu. \end{aligned}$$

$g_\mu(x)$ is chosen everywhere greater than zero and of a functional form introduced later in the proof; it can as shown in the proof be chosen independent of μ, that is $g_\mu(x) = g(x)$.

The functions $\bar{f}_\mu(x)$ are everywhere positive and hence the carrier $\bar{S}_\mu$ is independent of the parameter μ. The choice of function $g_\mu(x)$ or $g(x)$ still has a high degree of arbitrariness, and will be made in application in such a way as to facilitate the determination of a minimal f-sufficient statistic for $\{\bar{f}_\mu(x)\}$.

Let T', T'' be minimal f-sufficient statistics for $\{f_\mu(x)\}$, $\{\bar{f}_\mu(x)\}$, then

$$\begin{aligned} f_\mu(x) &= g_\mu(T')h(x), \\ \bar{f}_\mu(x) &= \bar{g}_\mu(T'')\bar{h}(x). \end{aligned} \tag{5.2}$$

Substituting (5.2) in (5.1), we obtain

$$f_\mu(x) = \psi(T, S_\mu)\bar{g}_\mu(T'')\bar{h}(x).$$

Thus (T, T'') is an f-sufficient statistic for $\{f_\mu(x)\}$, and we need only establish that it is minimal.

For the last section of the proof, we shall need to know that $T(x)$ is constant valued for all x for which T' has a fixed value. This is true since T is a function of T':

$$T = \bigcap S_\mu$$

where the intersection is over all $\mu \in \mathfrak{M}^*(x) = \{\mu \mid f_\mu(x) > 0\} = \{\mu \mid g_\mu(T') > 0\}$.

As in Lemma 1, we study the functions $T'(x)$ and $(T(x), T''(x))$ by considering the partitions they induce on the space X. Any set of the partition is thus the set of points for which the function takes a constant value. For the partition induced by (T, T''), we shall show that any set is also a set of the partition corresponding to $T'(x)$. Thus T' and (T, T'') can be put into 1-1 correspondence and hence (T, T'') is minimal f-sufficient.

Let E be any set of the partition of X corresponding to (T, T''); then

$$\begin{aligned} E_{x'} &= \{x \mid T(x) = T(x');\ \exists h(x) \ni \bar{f}_\mu(x) = h(x)\bar{f}_\mu(x') \text{ for all } \mu\} \\ &= \left\{x \,\middle|\, \begin{array}{l} T(x) = T(x');\ \exists h(x) \ni f_\mu(x) = h(x)f_\mu(x') \text{ for all } \mu \text{ for which} \\ S_\mu \supset T(x), \text{ and } g_\mu(x) = h(x)g_\mu(x') \text{ for all } \mu \text{ for which } S_\mu \not\supset T(x) \end{array}\right\} \\ &= \left\{x \,\middle|\, \begin{array}{l} T(x) = T(x');\ \exists h(x) \ni f_\mu(x) = h(x)f_\mu(x') \text{ for all } \mu, \text{ and} \\ g_\mu(x) = h(x)g_\mu(x') \text{ for all } \mu \text{ for which } S_\mu \not\supset T(x) \end{array}\right\} \qquad (5.3) \\ &= \left\{x \,\middle|\, \begin{array}{l} \exists h(x) \ni f_\mu(x) = h(x)f_\mu(x') \text{ for all } \mu, \text{ and } g_\mu(x) = h(x)g_\mu(x') \\ \text{for all } \mu \text{ for which } S_\mu \not\supset T(x) \end{array}\right\}. \qquad (5.4) \end{aligned}$$

(5.4) follows from (5.3) since it was shown above that $T(x) = T(x')$ is a consequence of $T'(x) = T'(x')$.

$$\begin{aligned} E_{x'} &= \left\{x \,\middle|\, \begin{array}{l} \exists h(x) \ni f_\mu(x) = h(x)f_\mu(x') \text{ for all } \mu, \text{ and } g_\mu(x) = \\ h(x)g_\mu(x') \text{ for all } \mu \text{ for which } S_\mu \not\supset T(x) \end{array}\right\} \\ &= \{x \mid \exists h(x) \ni f_\mu(x) = h(x)f_\mu(x') \text{ for all } \mu\}, \end{aligned}$$

if $g(x)$, which so far is only restricted to positive values, is given the following functional form:

$$g_\mu(x) = c_\mu(T) f_{v_\mu(T)}(x)$$

where $v_\mu(T(x))$ can be any element of $\mathfrak{M}$ for which $S_v \supset T(x)$. Thus E is a set of the partition corresponding to T'. This completes the proof of Theorem 1.

As remarked in the proof, $g_\mu(x)$ can be chosen independent of μ, as for example

$$g(x) = C(T) f_{v(T)}(x)$$

where $v(T(x))$ can be any element of $\mathfrak{M}$ for which $S_v \supset T(x)$.

In a recent paper [3] R. C. Davis has investigated sufficient statistics for density functions on R^1 satisfying certain regularity conditions and having the upper and lower extremities of the range of the distribution depend on the parameter. The proofs of necessity in Theorems 1 and 2 in [3] follow immediately from Lemma 4 and the theorem in this section by noting that in each case the statistic considered is equivalent to the statistic of selection.

6. Parameter of selection. In the special case in which μ acts as a parameter of selection for $\{f_\mu(x)\}$, the following theorem gives explicitly an f-sufficient statistic.

THEOREM 2. *If μ acts as a parameter of selection for $\{f_\mu(x)\}$ then the 'statistic of selection' is the minimal f-sufficient statistic for $\{f_\mu(x)\}$.*

PROOF. Since μ is a parameter of selection, $f_\mu(x)$ has the form

$$\begin{aligned} f_\mu(x) &= \phi_{S_\mu}(x) f(x) g(\mu) \\ &= \psi(T, S_\mu) f(x) g(\mu). \end{aligned}$$

Hence T is f-sufficient. By Theorem 1, T is minimal f-sufficient for $\{f_\mu(x)\}$.

7. Real-valued parameter. We now consider f-sufficiency when the parameter is real-valued and is a parameter of selection.

THEOREM 3. *If μ is a parameter of selection for $\{f_\mu(x)\}$, $\{S_\mu\}$ are Borel sets and ordered by $\subset$, and S_μ can be put in monotone 1-1 correspondence with a subset of R^1, then there exists a real-valued statistic which is sufficient and minimal f-sufficient for $\{f_\mu(x)\}$.*

PROOF. By Theorem 2 the statistic of selection is minimal f-sufficient, hence we want a real valued statistic T^* which can be put into 1-1 correspondence with the statistic of selection. First we define a real valued parameter essentially equivalent to μ.

By hypothesis we can choose $\theta(\mu) \in R^1$ such that $S_{\mu'} \subset S_{\mu''} \leftrightarrow \theta(\mu') \leqq \theta(\mu'')$ and $S_{\mu'} = S_{\mu''} \leftrightarrow \theta(\mu') = \theta(\mu'')$. Since μ is a parameter of selection, an equivalent parameter is S_μ and as is seen above θ is equivalent to S_μ. Hence we use the parameter θ to index the densities.

We define $T^*(x) \in R^1$ as follows:

$$T^*(x) = \inf \theta \text{ for all } \theta \text{ having } x \in S_\theta .$$

Therefore

$$T^*(x) = \inf \theta \text{ for all } \theta \text{ having } T_x \subset S_\theta .$$

Thus T^* is a function of T. If T^* is also f-sufficient then it can be put into 1-1 correspondence with T and is minimal f-sufficient. Define $\psi^*(T^*, \theta)$ by

$$\psi^*(T^*, \theta) = 1 \quad \text{if } T^* \leqq \theta,$$
$$= 0 \quad \text{if } T^* > \theta.$$

We now prove $\psi^*(T^*(x), \theta) \equiv \psi(T(x), S_\theta)$.

$$\psi^*(T^*(x), \theta) = 1$$

is equivalent to

$$T^*(x) \leqq \theta,$$

which is equivalent to

$$T(x) \subset S_\theta ,$$

that is,

$$\psi(T(x), S_\theta) = 1.$$

Hence

$$\psi^*(T^*(x), \theta) \equiv \psi(T(x), S_\theta).$$

The densities $f_\mu(x)$ have the following form:

$$f_\mu(x) = \phi_{S_\theta}(x) \cdot f(x) \cdot g(\theta)$$
$$= \psi(T(x), S_\theta) \cdot f(x) \cdot g(\theta)$$
$$= \psi^*(T^*(x), \theta) \cdot f(x) \cdot g(\theta).$$

Therefore $T^*(x)$ is f-sufficient and hence by previous argument is minimal f-sufficient. A more direct proof of this statement could have been made by establishing the 1-1 correspondence, but the above functional form for $f_\mu(x)$ is desirable for the remainder of the proof.

To prove that $T^*(x)$ is sufficient we need only verify that the conditions of Lemma 4 are fulfilled for the set of densities $\{f_\mu(x)\}$. Since the sets S_θ are Borel sets, $T^*(x)$ is a measurable function. Also $\psi^*(T^*, \theta)$ is a measurable function of T^*, and $f_\mu(x)$ is integrable. We now verify that $f(x)$ or a modification of $f(x)$ is integrable over $\bigcup S_\theta$. Take a monotone sequence $\{\theta_i\}$ such that $S_{\theta_i} \to \bigcup S_\theta$. Let

$$\int_{\theta_1} f(x)\, d\lambda(x) = e_1.$$

Choose $r(\theta_1) > 0$ such that

$$\int_{S_{\theta_2} - S_{\theta_1}} \frac{f(x)}{r(\theta_1)}\, d\lambda(x) = \tfrac{1}{2} e_1,$$

and similarly

$$\int_{S_{\theta_i}-S_{\theta_{i-1}}} \frac{f(x)}{r(\theta_{i-1})}\, d\lambda(x) = e_1/2^{i-1}.$$

Define

$$\begin{aligned} r(T^*) &= 1 && \text{if} \quad T^* \leqq \theta_1, \\ &= r(\theta_1) && \text{if} \quad \theta_1 < T^* \leqq \theta_2, \\ &= r(\theta_i) && \text{if} \quad \theta_i < T^* \leqq \theta_{i+1}. \end{aligned}$$

Then

$$\int_{S_{\theta_i}} \frac{f(x)}{r(T^*)}\, d\lambda(x) = 2e_1[1 - (\tfrac{1}{2})^i].$$

Therefore

$$\int_{US_\theta} \frac{f(x)}{r(T^*)}\, d\lambda(x) = 2e_1.$$

Therefore the modification of $f(x)$, that is, $f(x)/r(T^*)$, is integrable over US_θ. The theorem follows by noting that the modification does not affect the factorization needed for sufficiency:

$$f_u(x) = \psi^*(T^*(x), \theta) \cdot g(\theta) \cdot r(T^*(x)) \cdot \frac{f(x)}{r(T^*(x))}.$$

8. Examples.

EXAMPLE 1. Consider the class of all uniform distributions on R^1 having range 1. Let the parameter θ be the midpoint of the range. For a sample of n from a distribution of this family, we look for a sufficient statistic.

The parameter θ is a parameter of selection. Hence the statistic of selection T is f-sufficient. T is the nth power of the interval obtained from the intersection of all the unit intervals which contain the n points. It is equivalent to the closed interval $[x_L, x_G]$ where x_L and x_G are respectively the least and greatest values in the sample. Thus x_L and x_G are jointly f-sufficient for θ. It is easy to verify that the conditions of Lemma 4 are fulfilled and hence x_L and x_G are jointly sufficient for θ.

EXAMPLE 2. To a Poisson variable with fixed mean is added an "error" uniformly distributed between $-\mu$ and $+\mu$ where $\mu < \frac{1}{2}$. Let μ be the unknown parameter. For a sample of n from this distribution, we look for a sufficient statistic.

This class of distributions satisfies the hypothesis of Theorem 3. Therefore a real valued sufficient statistic T exists. T is equal to the largest deviation for the n sample values: the deviation for x_i is defined to be the absolute value of the distance from x_i to the nearest integer.

If we let (x_i) be the integer nearest the sample element x_i, then

$$T = \sup_{i=1,\cdots,n} | x_i - (x_i) | .$$

EXAMPLE 3. To a Poisson variable with parameter m is added an "error" uniformly distributed between $-\mu$ and $+\mu$, where $\mu < \frac{1}{2}$. Let (m, μ) be the unknown parameter. For a sample of n, we find the minimal f-sufficient statistic and then check for sufficiency.

The statistic of selection T is as calculated in Example 2. (We only consider points in R^1 for which the coordinates are all $> -\frac{1}{2}$). The carrier of the distribution S_μ can be defined by

$$S_\mu = [\{x \mid | x - (x)| < \mu\}]^n.$$

Therefore

$$f_{\mu,m}(\bar{x}) = \phi_{S_u}(\bar{x}) \cdot n(\mu) \prod_{i=1}^{n} \frac{m^{(x_i)}}{(x_i)!} e^{-m}.$$

We wish to define $\bar{f}_{\mu,m}(\bar{x})$ to facilitate the calculation of the minimal f-sufficient statistic and yet consistent with the functional form prescribed in Theorem 1. The obvious definition fulfills the necessary conditions:

$$\bar{f}_{\mu,m}(\bar{x}) = \prod_{i=1}^{n} \frac{m^{(x_i)}}{(x_i)!} e^{-m}.$$

The minimal f-sufficient statistic for $\{\bar{f}_{\mu,m}(x)\}$ is $\sum(x_i)$.

By Theorem 1 the minimal f-sufficient statistic for $\{f_{\mu,m}(x)\}$ is

$$\left[\sup_{i=1,\cdots,n} | x_i - (x_i) |, \quad \sum_{i=1}^{n} (x_i) \right].$$

Since the conditions of Lemma 4 are easily seen to be fulfilled, the above statistic is also sufficient.

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