It is interesting to note from Table I that in small samples the average length of confidence interval for the G test compares favourably with that for the t-test. The test has the advantage that it is easier for computation.

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A PROPERTY OF SOME TYPE A REGIONS

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1. Summary. In a test of an hypothesis one may regard a sample in the critical region as evidence that the hypothesis is false. Let us assume that for some reason it is desired to increase the critical size of the test, i.e., to make rejection of the hypothesis more probable. Then one may expect that an observation which led to rejection in the first test should still lead to rejection in the new test. In other words, one should expect $W_\alpha \supset W_{\alpha'}$ if $\alpha > \alpha'$, where W_α is the critical region for the test of size α. An example is given where regions of type A[1] are uniquely specified except for sets of measure zero, but fail to have this property.

2. Example. We shall consider type A regions for the hypothesis $\theta = 0$ where our sample consists of one observation with density

$$p(x, \theta) = (2\pi)^{-\frac{1}{2}}(1 + \theta)^{\frac{1}{2}} \exp [- (x - \theta)^2(1 + \theta)/2] \qquad \text{for } \theta > -1.$$

[1] Regions of type A were introduced by Neyman and Pearson (see [1]).

Since $p(x, \theta)$ permits differentiation under the integral sign, i.e., $\int_W \frac{\partial^i p(x, \theta)}{\partial \theta^i} dx = \frac{\partial^i}{\partial \theta^i} \int_W p(x, \theta)\, dx$ for $i = 1, 2$, then W is a region of type A of critical size α for the hypothesis $\theta = 0$ if

$$\int_W p(x, 0)\, dx = \alpha, \tag{1}$$

$$\int_W \frac{\partial p(x, 0)}{\partial \theta}\, dx = 0, \tag{2}$$

and if for every region W' satisfying equations (1) and (2)

$$\int_W \frac{\partial^2 p(x, 0)}{\partial \theta^2}\, dx \geq \int_{W'} \frac{\partial^2 p(x, 0)}{\partial \theta^2}\, dx. \tag{3}$$

As a consequence of the results of Dantzig and Wald which made use of the Neyman-Pearson Fundamental Lemma (see [1] and [2]), it follows that there is a region of type A for any α, $0 \leqq \alpha \leqq 1$, and that a necessary and sufficient condition that W be a region of type A of size α is that equations (1) and (2) hold and that there exist real numbers k_1 and k_2 so that, with the possible exception of a set of measure zero,

$$\frac{\partial^2 p(x, 0)}{\partial \theta^2} \geq k_1 p(x, 0) + k_2 \frac{\partial p(x, 0)}{\partial \theta} \qquad \text{for } x \text{ in } W \tag{4}$$

and the opposite inequality holds for x not in W.

We shall show that for α close to zero, a critical region of type A consists of the union of three intervals $(-\infty, a)$, (b, c), and (d, ∞), where $a < 1 < b < c < d$ (except possibly for a set of measure zero). Furthermore, as $\alpha \to 0$, $a \to -\infty$, $d \to \infty$, and b and $c \to 1$. From this last remark it follows that there are critical sizes α and α', $\alpha > \alpha'$, so that the corresponding critical regions of type A, W_α and $W_{\alpha'}$, are uniquely specified except for sets of measure zero, but a subset of $W_{\alpha'}$ of positive measure is not in W_α.

3. Proof.

Part 1. First we note that inequality (4) reduces to

$$(2x - 3/2) + (\tfrac{1}{2} + x - x^2/2)^2 \geq k_1 + k_2(\tfrac{1}{2} + x - x^2/2) \quad \text{for } x \text{ in } W. \tag{5a}$$

This inequality in turn reduces to

$$z^4 + c_1 z^2 + z + c_2 \geq 0 \qquad \text{for } x \text{ in } W, \tag{5b}$$

where $z = (x - 1)/2$ and c_1 and c_2 are real numbers depending on k_1 and k_2. Hence, for $0 < \alpha < 1$, W is (except possibly for a set of measure zero) either the union of two intervals $(-\infty, a)$ and (b, ∞), where $a < b$, or the union of three intervals $(-\infty, a)$, (b, c), and (d, ∞), where $a \leq b \leq c \leq d$, depending on the number of real roots of $z^4 + c_1 z^2 + z + c_2 = 0$.

Let us define $\varphi(t) = (2\pi)^{-\frac{1}{2}} \exp[-t^2/2]$, $A(t) = \int_{-\infty}^{t} \varphi(x)\,dx$, $B(t) = \int_{-\infty}^{t} x\varphi(x)\,dx = -\varphi(t)$, $C(t) = \int_{-\infty}^{t} x^2\varphi(x)\,dx = -t\varphi(t) + A(t)$, and $\psi(t) = (t-2)\varphi(t)$. Then we see that in the case where W is the union of two intervals, equations (1) and (2) reduce to

$$A(b) - A(a) = 1 - \alpha, \tag{6}$$

$$\psi(b) = \psi(a), \tag{7}$$

and in the case of three intervals equations (1) and (2) reduce to

$$A(d) - A(c) + A(b) - A(a) = 1 - \alpha, \tag{8}$$

$$\psi(d) - \psi(c) + \psi(b) - \psi(a) = 0. \tag{9}$$

Since $\psi(t) > 0$ for $t > 2$ and $\psi(t) < 0$ for $t < 2$, the largest value of $A(b) - A(a)$ consistent with equation (7) is $\int_{-\infty}^{2} \varphi(t)\,dt < .98$. Thus for α sufficiently small, a region of type A must consist of the union of three intervals except possibly for a set of measure zero.

Part 2. That $a \to -\infty$ and $d \to \infty$ as $\alpha \to 0$ is obvious. Equation (9) prevents b and c from both being less than $1 - \sqrt{2}$ (the minimizing value of $\psi(t)$) or from both exceeding $1 + \sqrt{2}$ (the maximizing value of $\psi(t)$). Since the interval (b, c) must then have some points in common with the interval $(1 - \sqrt{2}, 1 + \sqrt{2})$ it follows that b and c are bounded as $\alpha \to 0$.

The roots of equation (5b) are $a^* = (a-1)/2$, $b^* = (b-1)/2$, $c^* = (c-1)/2$, and $d^* = (d-1)/2$. Since the coefficients of z^3 and z are zero and one, respectively, we have $(a^* + d^*) + (b^* + c^*) = 0$ and $(a^* + d^*)b^*c^* + a^*d^*(b^* + c^*) = -1$. Setting $a^* + d^* = -2L$, we have $a^*d^* = b^*c^* - 1/(2L)$. Since $a^*d^* \to -\infty$, $L \to 0+$. Then a^* and d^* are both of the order of magnitude of $L^{-\frac{1}{2}}$. Let $b^* = L - \epsilon$. Then $c^* = L + \epsilon$ and equation (8) gives us $\epsilon = O(\alpha)$.[2] Applying equation (9) and noting that for $z = 0$, $x = 1$ and $\psi'(1) > 0$, we see that $\epsilon \neq 0$, and $\epsilon = O[\psi(a) - \psi(d)] = o(L)$. The two results $L \to 0+$ and $\epsilon = o(L)$ imply that for α small enough b and $c \to 1$ but b and c exceed one. This is the desired result.

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[2] We write $a = o(b)$ if $a/b \to 0$ as $\alpha \to 0$ and $a = O(b)$ if there is a constant k so that $|a| \leq k\,|b|$ for α small enough.