Thanks are due to Dr. J. C. P. Miller, Technical Director, Scientific Computing Service, Limited, London, England, for helpful suggestions in the preparation of this paper.

REFERENCES

[1] T. A. Bancroft, "Some extensions of the incomplete beta function tables." (in preparation)

[2] Karl Pearson, *Tables of the Incomplete Beta-Function*, Cambridge University Press, 1934.

[3] Wilhelm Magnus und Fritz Oberhettinger, *Formeln and Sätze für die Speziellen Funktionen der Mathematischen Physik*, Julius Springer, Berlin, 1943.

[4] T. A. Bancroft, "On biases in estimation due to the use of preliminary tests of significance, *Annals of Math. Stat.*," Vol. 15 (1944).

ON A THEOREM BY WALD AND WOLFOWITZ

By Gottfried E. Noether

New York University

Let $\mathfrak{H}_n = (h_1, \cdots, h_n)$, $(n = 1, 2, \cdots)$, be sequences of real numbers and for all n denote by $H_{e_1 \cdots e_m}$ the symmetrical function generated by $h_1^{e_1} \cdots h_m^{e_m}$, i.e., $H_{e_1 \cdots e_m} = \Sigma\, h_{i_1}^{e_1} \cdots h_{i_m}^{e_m}$ where the summation is extended over the $n(n-1) \cdots (n-m+1)$ possible arrangements of the m integers $i_1, \cdots, i_m$, such that $1 \leq i_j \leq n$ and $i_j \neq i_k$, $(j, k = 1, \cdots, m)$. According to Wald and Wolfowitz [1] the sequences $\mathfrak{H}_n$ are said to satisfy condition W, if for all integral $r > 2$

$$\frac{\frac{1}{n}\sum_{i=1}^{n}(h_i - \bar{h})^r}{\left[\frac{1}{n}\sum_{i=1}^{n}(h_i - \bar{h})^2\right]^{r/2}} = O(1),^1$$

where $\bar{h} = 1/n\, \Sigma_{i=1}^{n} h_i$.

Given sequences $\mathfrak{A}_n = (a_1, \cdots, a_n)$ and $\mathfrak{D}_n = (d_1, \cdots, d_n)$, consider the chance variable

$$L_n = d_1 x_1 + \cdots + d_n x_n,$$

where the domain of $(x_1, \cdots, x_n)$ consists of the $n!$ equally likely permutations of the elements of $\mathfrak{A}_n$. Then it is shown in [1] that if the sequences $\mathfrak{A}_n$ and $\mathfrak{D}_n$ satisfy condition W, the distribution of $L_n^0 = (L_n - EL_n)/\sigma(L_n)$ approaches the normal distribution with mean 0 and variance 1 as $n \to \infty$. These conditions

[1] The symbol O, as well as the symbols o and $\sim$ to be used later, have their usual meaning. See e. g. Cramér [2, p. 122].

for asymptotic normality can be weakened. It will be shown that the following theorem holds:

THEOREM. *L_n^0 is asymptotically normal with mean* 0 *and variance* 1 *provided the sequences $\mathfrak{D}_n$ satisfy condition W while for the sequences $\mathfrak{A}_n$*

$$\frac{\sum_{i=1}^{n} (a_i - \bar{a})^r}{\left[\sum_{i=1}^{n} (a_i - \bar{a})^2\right]^{r/2}} = o(1), \qquad (r = 3, 4, \cdots). \tag{1}$$

We note that L_n^0 is not changed if a_i is replaced by $[1/n\Sigma_{i=1}^{n} (a_i - \bar{a})^2]^{-1/2}(a_i - \bar{a})$ and d_i by $[1/n \Sigma_{i=1}^{n} (d_i - \bar{d})^2]^{-1/2}(d_i - \bar{d})$. Therefore it is sufficient to prove asymptotic normality provided

$$D_1 = 0, \quad D_2 = n, \quad D_r = O(n), \qquad (r = 3, 4, \cdots); \tag{2}$$

$$A_1 = 0, \quad A_2 = n, \quad A_r = o(n^{r/2}), \qquad (r = 3, 4, \cdots). \tag{3}$$

Then

$$EL_n = D_1 E x_1 = 0,$$

$$\text{var } L_n = EL_n^2 = D_2 E x_1^2 + D_{11} E x_1 x_2$$

$$= \frac{1}{n} A_2 D_2 + \frac{1}{n(n-1)} (A_1^2 - A_2)(D_1^2 - D_2) \sim n,$$

and it is sufficient to show that $n^{-r/2} EL_n^r$ tends to the rth moment of a normal distribution with mean 0 and variance 1.

Now we can write

$$\mu_r = n^{-r/2} EL_n^r = n^{r/2} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} E d_{i_1} x_{i_1} \cdots d_{i_r} x_{i_r}$$

$$= n^{-r/2} [D_r E x_1^r + \cdots + c(r, e_1, \cdots, e_m) D_{e_1 \cdots e_m} E x_1^{e_1} \cdots x_m^{e_m} + \cdots + D_{1 \cdots 1} E x_1 \cdots x_r] \tag{4}$$

where $e_1 + \cdots + e_m = r$ with e_k, $(k = 1, \cdots, m)$, positive integral and the coefficient $c(r, e_1, \cdots, e_m)$ stands for the number of ways in which the r indices $i_1, \cdots, i_r$ can be tied in m groups of size $e_1, \cdots, e_m$, respectively, so as to produce the terms of $D_{e_1 \cdots e_m} E x_1^{e_1} \cdots x_m^{e_m}$.

Since $E x_1^{e_1} \cdots x_m^{e_m} \sim n^{-m} A_{e_1 \cdots e_m}$ we have

$$n^{-r/2} D_{e_1 \cdots e_m} E x_1^{e_1} \cdots x_m^{e_m} \sim n^{-(r/2+m)} D_{e_1 \cdots e_m} A_{e_1 \cdots e_m} = B(r, e_1, \cdots, e_m), \text{ say.} \tag{5}$$

LEMMA. *$B(r, e_1, \cdots, e_m) \sim 0$ unless*

$$m = r/2, \qquad e_1 = \cdots = e_{r/2} = 2. \tag{6}$$

In that case $B(r, 2, \cdots, 2) \sim 1$.

Before proving this lemma we shall show that our theorem follows immediately. By (4) μ_r is the sum of a finite number of expressions $B(r, e_1, \cdots, e_m)$.

Therefore if $r = 2s + 1$, $(s = 1, 2, \cdots)$, $\mu_{2s+1} \sim 0$, since at least one of the e_k, $(k = 1, \cdots, m)$, in all the $B(2s + 1, e_1, \cdots, e_m)$ adding up to μ_{2s+1} must be odd. If $r = 2s$, $\mu_{2s} \sim c(2s, 2, \cdots, 2)$. Since the first index in (4) can be tied with any one of the other $2s - 1$ indices, the next free index with any one of the remaining $2s - 3$ indices, etc., it is seen that $\mu_{2s} \sim (2s - 1)(2s - 3) \cdots 3$. However these are the moments of a normal distribution with mean 0 and variance 1. This proves the theorem.

PROOF OF LEMMA. Define $A(j_1, \cdots, j_h) = A_{j_1} \cdots A_{j_h}$. Then $A_{e_1 \cdots e_m}$ is the sum of a finite number of expressions $A(j_1, \cdots, j_h)$, where the j_g, $(g = 1, \cdots, h)$, are obtained from $e_1, \cdots, e_m$ by addition in such a way that

$$j_1 + \cdots + j_h = e_1 + \cdots + e_m = r. \tag{7}$$

Since by (3) $A_1 = 0$, we need only consider those $A(j_1, \cdots, j_h)$ for which $j_g \geq 2$, $(g = 1, \cdots, h)$. If some $j_g > 2$ by (3) and (7)

$$A(j_1, \cdots, j_h) = o(n^{r/2}). \tag{8}$$

If $j_g \equiv 2$,

$$A(2, \cdots, 2) = A_2^{r/2} = n^{r/2}. \tag{9}$$

This last case can only happen if r is even and e_k, $(k = 1, \cdots, m)$, equals either 1 or 2. Therefore, unless (6) is true

$$m > r/2. \tag{10}$$

Similarly, writing $D_{e_1 \cdots e_m}$ as a sum of products of the kind $D_{j_1} \cdots D_{j_h}$ it is seen that by (2)

$$D_{e_1 \cdots e_m} = \begin{cases} O(n^m) & \text{if } m < r/2 \\ O(n^{r/2}) & \text{if } m \geq r/2. \end{cases} \tag{11}$$

Thus by (8)–(11)

$$A_{e_1 \cdots e_m} D_{e_1 \cdots e_m} = o(n^{r/2+m}), \tag{12}$$

unless (6) is true. In that case

$$A_{2 \cdots 2} \sim A_2^{r/2} = n^{r/2}, \tag{13}$$

$$D_{2 \cdots 2} \sim D_2^{r/2} = n^{r/2}. \tag{14}$$

(12)–(14) together with (5) prove the lemma.

Let $a_1, a_2, \cdots$ be independent observations on the same chance variable Y. We may ask what conditions have to be imposed on the distribution of Y to insure—at least with probability 1—that condition (1) is satisfied. Wald and Wolfowitz state in Corollary 2 of [1] that provided Y has positive variance and finite moments of all orders the $a_1, a_2, \cdots$ satisfy condition W with probability 1 and therefore insure asymptotic normality of L_n provided the sequences $\mathfrak{D}_n$ satisfy condition W. On the other hand, it can be shown that the $a_1, a_2, \cdots$

satisfy condition (1) with probability 1, provided Y has positive variance and a finite absolute moment of order 3. Thus condition (1) constitutes a considerable improvement over condition W.

REFERENCES

[1] A. Wald and J. Wolfowitz, "Statistical tests based on permutations of the observations," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 358–372.
[2] H. Cramér, *Mathematical Methods of Statistics*, Princeton, 1946.

ON SUMS OF SYMMETRICALLY TRUNCATED NORMAL RANDOM VARIABLES

By Z. W. Birnbaum and F. C. Andrews[1]
University of Washington, Seattle

1. Introduction. Let X_a be the random variable with the probability density

$$f_a(x) = \begin{cases} Ce^{-x^2/2} & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a, \end{cases} \tag{1.1}$$

obtained from the normal probability density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ by symmetrical truncation at the "terminus" $|x| = a$, and let $S_a^{(m)}$ be the sum of m independent sample-values of X_a. We consider the following *problem*: An integer $m \geq 2$ and the real numbers $A > 0$, $\epsilon > 0$ are given; how does one have to choose the terminus a so that the probability of $|S_a^{(m)}| \geq A$ is equal to ϵ,

$$P(|S_a^{(m)}| \geq A) = \epsilon? \tag{1.2}$$

This problem arises for example when single components of a product are manufactured under statistical quality control, so that each component has the length $Z = k + X$ where X has the probability density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and the final product consists of m components so that its total length S is the sum of the lengths of the components. We wish to have probability $1 - \epsilon$ that S differs from mk by not more than a given A. To achieve this we decide to reject each single component for which $|Z - k| = |X| > a$; how do we determine a?

The exact solution of this problem would require laborious computations.[2] In the present paper methods are given for obtaining approximate values of a which are "safe", that is such that

$$P(|S_a^{(m)}| \geq A) \leq \epsilon. \tag{1.3}$$

[1] Research done under the sponsorship of the Office of Naval Research.

[2] A similar problem has been studied by V. J. Francis [2] for one-sided truncation; he actually had the exact probabilities for the solution of his problem computed and tabulated for $m = 2, 4$.