The upper end point of the confidence interval is the root in λ of the equation

$$\frac{N-p}{r}\frac{Q(\lambda)}{Q_a} = F_1 \tag{2.19}$$

and the lower end point is the root in λ of the equation

$$\frac{N-p}{r}\frac{Q(\lambda)}{Q_a} = F_2. \tag{2.20}$$

If equation (2.20) has no root, the lower end point of the confidence interval is put equal to zero.

REFERENCE

[1] A. Wald, "On the analysis of variance in case of multiple classifications with unequal class frequencies", *Annals. of Math. Stat.*, Vol. 12 (1941).

ON THE SHAPE OF THE ANGULAR CASE OF CAUCHY'S DISTRIBUTION CURVES

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1. Let ξ be a *linear* random variable, that is, a random variable capable of values x represented by points of a line $-\infty < x < \infty$, and suppose, for simplicity, that ξ has a density of probability, $f(x)$. Then, subject to provisos of convergence, the series

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

represents a periodic function, of period 1, having the following significance: $F(x)$ is the density of probability of the *angular* random variable, say Ξ, which is obtained if all the states

$$\cdots, \quad \xi-2, \quad \xi-1, \quad \xi, \quad \xi+1, \quad \xi+2, \cdots$$

of the linear random variable are identified.

In other words, if a circle of unit circumference rolls from $-\infty$ to ∞ on the ξ-line, then every point of the circumference collects the various densities of probability attached to congruent points of the ξ-line, and a state of Ξ represents a point of the circumference. For a detailed study of the mapping $\xi \to \Xi$ or $f \to F$, cf. [2].

According to Poisson's summation formula, the Fourier constants of the periodic function $F(x)$ can be obtained by restricting u in $g(u)$ to an equidistant sequence of discrete values, where $g(u)$ denotes the Fourier transform of $f(x)$; cf., e.g., [5], p. 78 or [9], pp. 477–478.

2. Consider, in particular, the case in which $f(x)$ is the density of a symmetric distribution which is stable in Cauchy's sense. The determination of the totality of these linear densities of probability is due to Lévy [6]. It was shown in [8] that every such $f(x) = f(-x)$ is a decreasing function of $| x |$. As explained in [8], p. 70, this fact makes superfluous one of the axioms occurring in Gauss' postulational approach to "errors of observation."

The purpose of the present note is the deduction of the angular analogue of the fact just quoted. The analogue states that, if $f(x)$ is symmetric and stable, then the corresponding periodic $F(x)$ is decreasing for $0 \leqq x \leqq \frac{1}{2}$ (and so, for reasons of symmetry, is increasing for $\frac{1}{2} \leqq x \leqq 1$). This is contained in the italicized statement of §4 below.

In view of Poisson's rule, quoted above, the periodic densities in question can be defined by certain Fourier series representing generalizations of elliptic theta-series. From this point of view, not even the existence (i.e., the *positivity*) of the periodic densities is obvious, if arbitrary values of the "precision constant" (denoted below by q) are allowed. The difficulties involved are explained in §3.

3. If q and λ are positive constants the first of which is less than 1, then the (even, periodic) function

$$\theta_\lambda(x;q) = 1 + 2\sum_{n=1}^{\infty} q^{n^\lambda} \cos nx, \tag{1}$$

where $q^{n^\lambda} > 0$, has derivatives of arbitrarily high order at every real x. It is regular-analytic at every real x if and only if $\lambda > 0$ is replaced by $\lambda \geqq 1$, where the sign of equality holds if and only if the analytic continuation (from the x-axis) is not an entire function. In fact, it is known that a Fourier series $\Sigma(a_n \cos nx + b_n \sin nx)$ is that of a function which is regular-analytic at every real x, and has the period 2π, if and only if $| a_n | + | b_n |$ is majorized by a constant multiple of the nth power of a positive constant which is less than 1; and that the latter constant can be chosen arbitrarily small if and only if the analytic continuation does not lead to any singularity (at a $z \neq \infty$).

Since the function (1) tends to 1 uniformly in x as $q \to +0$, if λ is fixed, there belongs to every $\lambda > 0$ a positive $q^* = q^*(\lambda)$ having the property that

$$\theta_\lambda(x;q) > 0 \text{ for } 0 \leqq x < 2\pi \tag{2}$$

if $0 < q < q^*(\lambda)$. It is less obvious that, if q is sufficiently small with reference to λ, say if $0 < q < q^{**}(\lambda)$, then

$$\theta_\lambda(x;q) \text{ is decreasing for } 0 \leqq x \leqq \pi \tag{3}$$

(hence, increasing for $\pi \leqq x < 2\pi$). The existence of such a $q^{**}(\lambda) < \infty$ for every $\lambda > 0$ can be assured as follows:

If $s_n(x)$ denotes the nth partial sum of the Fourier series $\Sigma(\sin nx)/n$, then $s_n(x)$ is positive for $0 < x < \pi$ (Gronwall, Jackson; for a short proof, cf. [4]).

Hence, a partial summation shows that the sum of a sine series, $\Sigma b_n \sin nx$, must be positive for $0 < x < \pi$ if

$$nb_n - (n + 1)b_{n+1} > 0 \text{ and } nb_n \to 0.$$

Since the first derivative of (1) (with respect to x) results by choosing $b_n = -2nq^{n^\lambda}$, it follows that (3) must be true if

$$n^2 q^{n^\lambda} - (n + 1)^2 q^{(n+1)^\lambda} > 0$$

holds for $n = 1, 2, \cdots$. But the last inequality is readily seen to be satisfied from $n = 1$ onward if, while λ is fixed, q tends to 0. This proves that $q^{**}(\lambda)$ exists for every $\lambda > 0$.

4. From these deductions alone, it is quite unexpected that (the best values of) both $q^*(\lambda)$ and $q^{**}(\lambda)$ turn out to be independent of λ when

$$0 < \lambda \leqq 2, \tag{4}$$

i.e., that (1) *satisfies both* (2) *and* (3) *for* $0 < q < 1$, *if* (4) *is assumed.* This fact is of statistical significance, since, on the one hand, it is precisely the restriction (4) which is necessary and sufficient for the existence of Cauchy's (symmetric) "stable" distributions (cf. [6], pp. 254–263) and, on the other hand, the reduction (mod 2π) of the densities of these *linear* distributions leads to the functions (1) as angular densities (cf. [9], pp. 477–478); the numerical value of $q(< 1)$ being determined by the "precision" or "dispersion" of the resulting *angular* distributions.

Under the necessary restriction (4), the *linear* analogue of $q^*(\lambda) = 1$ and of $q^{**}(\lambda) = 1$ was proved in [6], pp. 258–263 and in [8], pp. 71–77, respectively. It will remain undecided whether the restriction (4) is necessary in either of the *angular* cases.

5. Suppose that λ has a fixed value in the range (4). Then there exists a monotone function of t, say $\alpha_\lambda(t)$, for which

$$\exp(-u^\lambda) = \int_0^\infty \exp(-u^2 t)\, d\alpha_\lambda(t)$$

is an identity in u, where $0 < u < \infty$ (cf. [1], p. 769, where further references will be found). Hence, a change of variables shows that

$$q^{n^\lambda} = \int_0^\infty q^{tn^2}\, d\alpha_\lambda(t \mid \log q \mid^{1-2\lambda})$$

is an identity in q and n, where $0 < q < 1$ and $n = 0, 1, 2, \cdots$ (the integration variable is t). Consequently from (1),

$$\theta_\lambda(x; q) = \int_0^\infty \theta_2(x; q^t)\, d\alpha_\lambda(t \mid \log q \mid^{1-2/\lambda}),$$

where $0 < q < 1$ and $-\infty < x < \infty$. In fact, the legitimacy of the term-by-term integration is obvious from $0 < q < 1$ and $d\alpha_\lambda \geqq 0$ (even though the integrals are improper).

6. Since α_λ is a non-decreasing function, it is clear from the last formula line that both (2) and (3) will be proved for $0 < q < 1$ and for every λ (satisfying (4)), if it is ascertained that both (2) and (3) hold for $0 < q < 1$ when $\lambda = 2$. But the case $\lambda = 2$ of (1) is an elliptic theta-function, for which both properties in question (cf. the diagram in [3], p. 44) are known; a simple proof can be concluded from what, in Hecke's terminology, is the Eulerian factorization of $\theta_2(x\,;q)$, as follows:

According to Jacobi, the factorization of the case $\lambda = 2$ of (1) is

$$\theta_2(x\,;q) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1}\cos x + q^{4n-2})$$

(cf. [7], pp. 64–65). Thus

$$\theta_2(x\,;q) = c_q \prod_{n=1}^{\infty} P(x + \pi\,; q^{2n-1}),$$

where

$$c_q = \prod_{n=1}^{\infty} (1 - q^{2n})$$

and

$$(5) \qquad P(x\,;r) = 1 - 2r\cos x + r^2, \qquad (0 < r < 1),$$

hence

$$P(x\,;r) > 0 \qquad (0 < r < 1).$$

Since $0 < q < 1$, this proves the case $\lambda = 2$ of (2). Furthermore, logarithmic differentiation of the product representation of $\theta_2(x\,;q)$ gives

$$\theta_2'(x\,;q) = \theta_2(x\,;q) \sum_{n=1}^{\infty} P'(x + \pi\,; q^{2n-1})/P(x + \pi\,; q^{2n-1}),$$

where $f' = df/dx$; so that, by (5),

$$P'(x + \pi\,; r) = -2r\sin x.$$

Since $0 < q < 1$, the last three formula lines and the case $\lambda = 2$ of (2) imply that

$$\theta_2'(x\,;q) < 0 \text{ if } 0 < x < \pi,$$

as claimed by the case $\lambda = 2$ of (3).

This completes the proof of the italicized assertion.

REFERENCES

[1] P. Hartman and A. Wintner, "On the spherical approach to the normal distribution law," *Amer. Jour. of Math.*, Vol. 62 (1940), pp. 759–779.

[2] E. K. Haviland, "On the distribution functions of the reciprocal of a function and of a function reduced mod 1," *Amer. Jour. of Math.*, Vol. 68 (1941), pp. 408–414.

[3] E. Jahnke and F. Emde, *Tables of Functions with Formulae and Curves*, 3d ed. (1938), reprinted by G. E. Stechert and Co., New York.

[4] E. Landau, "Ueber eine trigonometrische Ungleichung," *Math. Zeit.*, Vol. 37 (1933), p. 36.

[5] H. Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1905.

[6] P. Lévy, *Calcul des probabilités*, Paris, 1925.

[7] H. Weber, *Elliptische Funktionen und algebraische Zahlen*, Braunschweig, 1891.

[8] A. Wintner, "On a class of Fourier transforms," *Amer. Jour. of Math.*, Vol. 58 (1936), pp. 45–90 and p. 425.

[9] F. Zernike, "Wahrscheinlichkeitsrechnung und mathematische Statistik," *Handbuch der Physik*, Vol. 3 (1928), pp. 419–492.

A NOTE ON THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

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1. Introduction. Let $\{z_i\}$, $(i = 1, 2, 3, \cdots)$, be a sequence of real valued random variables identically distributed according to the cumulative distribution function $F(z)$. Define the sums $Z_N = z_1 + z_2 + \cdots + z_N$ for every positive integer N. Choose two positive constants a and b and define the random variable n as the smallest integer N for which one of the inequalities $Z_N \geqq a$ or $Z_N \leqq -b$ holds. The notations $P(u \mid F)$ and $E(u \mid F)$ will denote the probability of u and its expectation respectively assuming that F is the distribution of the z_i.

Wald [1] has established the results contained in the following lemmas.

Lemma 1. *If the variance of $F(z)$ is positive, $P(n < \infty \mid F)$ equals one.*

Lemma 2. *If there exists a positive number δ such that $P(e^z < 1 - \delta \mid F) > 0$ and $P(e^z > 1 + \delta \mid F) > 0$ and if the moment generating function $\varphi(t) = E(e^{zt} \mid F)$ exists for all real values of t, then $\varphi(t)$ has one and only one minimum at some finite value $t = t_0$. Moreover, $\varphi''(t) > 0$ for all real values of t.*

It is the purpose of this note to establish the following extension of the validity of certain results given by Wald [1], [2].

Theorem.[1] *Under the conditions of Lemma 2 the identity*

$$E\{e^{Z_n t}[\varphi(t)]^{-n} \mid F\} = 1 \tag{1}$$

[1] Wald's results show (1) to be valid for all complex t in the domain over which $|\varphi(t)| \geqq 1$ and the validity of the differentiation clause for all real t in that domain. The importance of the present extension arises from the fact that, if $E(x \mid F) \neq 0$, then $0 < \varphi(t) < 1$ on a certain interval of the real axis.