LOW MOMENTS FOR SMALL SAMPLES: A COMPARATIVE STUDY OF ORDER STATISTICS

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1. Summary. The means, variances, and covariances for samples of size ≤ 10 from the normal distribution, a selected long-tailed distribution, and the uniform distribution are tabled and compared with the usual asymptotic approximations. The methods of computation used and the accuracy expected are discussed. Use is made of the representation of an arbitrarily distributed variate as a monotone function of a uniformly (rectangularly) distributed variate. It is hoped that these tables will encourage experimentation with new statistical procedures.

2. Introduction. Two sorts of statistical procedures have been widely exploited in theoretical statistics—first the use of linear and quadratic combinations of the unordered observations and, second, the use of ranked (ordered) observations. Statistics based on ordered observations have recently been dubbed *systematic statistics* [2, Mosteller, 1946]. Analytic processes and a few necessary numerical tables have advanced the study of the first procedure greatly, at least for the special case of the normal distribution; but analytic procedures have not done much to exhibit the behavior of systematic statistics and the necessary tables have been lacking.

It would be very helpful to have (1) at least the first two moments (including product moments) of the order statistics, and (2) tables of the percentage points of their distributions, for samples of sizes from 1 to some moderately large value such as 100 and for a large representative family of distributions. This is a large order and will require much computation.

The first step in this direction was taken by Fisher and Yates [1] by tabulating the means, to two decimal places, of all order statistics from normal samples of size ≤ 50. The present paper continues the process by supplying all means, variances, and covariances for samples of size ≤ 10 from (*a*) the normal distribution, (*b*) the uniform (rectangular) distribution, (*c*) a special distribution with long tails. For purposes of comparison, we also supply approximate means, variances, and covariances for the uniform and the special distribution computed from suitable asymptotic formulas.

The special distribution has the representing function

$$r(u) = (1 - u)^{-1/10} - u^{-1/10}, \tag{1}$$

where u has the uniform distribution on the interval [0, 1], and $x = r(u)$ is the variable whose order statistics interest us. This special distribution was especially constructed 1) to have high tails and 2) to provide moments of order statistics in closed form which could be evaluated with a reasonable amount of labor. The normal distribution is rather unreasonable in this latter respect—there being no known expression except in terms of single and double quadratures of some considerable numerical difficulty.

We have restricted ourselves to samples of size ≤ 10, and to only three distributions, all of these symmetrical, because of limited man-power rather than limited interest. Additional tables of a similar nature will surely prove helpful.

In order to obtain even as much information as provided in this paper, it has been necessary to make a joint effort, dividing the labor. The various parts of the work have been carried out more or less separately by the various authors—the means and variances for the normal by Mosteller, the covariances for the normal (which, with their double quadratures, required far more time than all the other thought and computation combined) by Hastings with some assistance from Mosteller, the choice of the special distribution by Tukey, and the computation for it by Winsor.

3. Results. In this section we provide the various tables that have been computed.

Table I gives the mean and standard deviation of the ith order statistic $x(i \mid n)$, [or $x_{i|n}$, we use whichever notation seems less likely to confuse and agree that $x(1 \mid n) \geq x(2 \mid n) \geq \cdots \geq x(n \mid n)$] from a sample of size n drawn from a uniform (U), normal (N), and a special distribution (S). All three distributions have been adjusted to have zero mean and unit variance. In addition Table I gives approximations for the mean and standard deviation as computed from asymptotic formulas for the normal (AN) and the special (AS).

If $f(x)$ is the density function, the asymptotic approximation for the mean $m(i \mid n)$ of the ith order statistic from a sample of size n is obtained by solving the equation

$$\int_{m(i|n)}^{\infty} f(x)\, dx = i/(n+1)$$

for $m(i \mid n)$. Similarly the formula used for the asymptotic variance of $x(i \mid n)$ is

$$\frac{i(n-i+1)}{n(n+1)^2\{f[m(i \mid n)]\}^2}.$$

Values are given for $n = 1, 2, \cdots, 10$ and $i = 1, \cdots, \left[\frac{n}{2}\right]$. If $m(i \mid n)$ is an entry in the table for means, a missing entry $m(n-i+1 \mid n) = -m(i \mid n)$; if $w(i \mid n)$ is an entry in the table of standard deviations, a missing entry

$$w(n-i+1 \mid n) = w(i \mid n).$$

Table II gives the variances and covariances of the order statistics for the normal distribution (N) and the same quantities as approximated by the asymp-

TABLE I

Means and standard deviations of order statistics $x(i|n)$ for uniform distribution (U), normal (N), special (S), asymptotic normal (AN), asymptotic special (AS)

		Mean					Standard Deviation				
n	i	U	N	S	AN	AS	U	N	S	AN	AS
1	1	0	0	0	0	0	1.00000	1.00000	1.00000	1.2533	.9804
2	1	.57735	.56419	.53493	.4307	.3418	.81650	.82565	.84490	.9168	.7486
3	1	.86603	.84628	.80240	.6745	.5466	.67082	.74798	.82783	.7867	.6823
	2	0	0	0	0	0	.77460	.66983	.58457	.7236	.5660
4	1	1.03923	1.02938	.98473	.8416	.6954	.56569	.70122	.82982	.7144	.6542
	2	.34641	.29701	.25540	.2533	.1992	.69282	.60038	.52582	.6340	.5035
5	1	1.15470	1.16296	1.12449	.9674	.8136	.48795	.66898	.83642	.6670	.6415
	2	.57735	.49502	.42567	.4307	.3418	.61721	.55814	.50390	.5798	.4730
	3	0	0	0	0	0	.65465	.53557	.44903	.5605	.4384
6	1	1.23718	1.26721	1.23847	1.0676	.9114	.42857	.64492	.84423	.6331	.6330
	2	.74231	.64176	.55458	.5659	.4539	.55328	.52874	.49425	.5426	.4567
	3	.24744	.20155	.16785	.1800	.1412	.60609	.49620	.41648	.5147	.4057

TABLE I (*Continued*)

		Mean					Standard Deviation				
n	i	U	AN	N	AS	S	U	AN	N	AS	S
7	1	1.29904	1.1504	1.35218	.9957	1.33506	.38188	.6072	.62603	.6141	.85217
	2	.86603	.6745	.75737	.5462	.65892	.50000	.5150	.50670	.4359	.48992
	3	.43301	.3186	.35271	.2512	.29375	.55902	.4826	.46875	.3772	.39963
	4	0	0	0	0	0	.57735	.4737	.45874	.3617	.37747
8	1	1.34715	1.2207	1.42360	1.0697	1.41892	.34427	.5867	.61066	.6276	.85988
	2	.96225	.7647	.85222	.6259	.74690	.45542	.4936	.48930	.4402	.48823
	3	.57735	.4307	.47282	.3418	.39498	.51640	.4584	.44807	.3743	.38998
	4	.19245	.1397	.15251	.1094	.12502	.54433	.4447	.43264	.3494	.35616
9	1	1.38564	1.2816	1.48501	1.1358	1.49358	.31334	.5691	.59780	.6268	.86725
	2	1.03923	.8416	.93230	.6954	.82317	.41779	.4763	.47508	.4361	.48800
	3	.69282	.5244	.57197	.4191	.47995	.47863	.4393	.43171	.3722	.38414
	4	.34641	.2533	.27453	.1992	.22504	.51168	.4227	.41303	.3356	.34321
	5	0	0	0	0	0	.52223	.4178	.40751	.3268	.33173

TABLE I (*Concluded*)

		Mean					Standard Deviation				
n	i	U	AN	N	AS	S	U	AN	N	AS	S
10	1	1.41713	1.3352	1.53875	1.1956	1.56057	.28748	.5557	.58681	.6275	.87423
	2	1.10222	.9085	1.00135	.7574	.89062	.38569	.4619	.46318	.4334	.48859
	3	.78730	.6046	.65608	.4866	.55336	.44536	.4238	.41826	.3604	.38054
	4	.47238	.3488	.37572	.2754	.30866	.48105	.4052	.39756	.3261	.33477
	5	.15746	.1142	.12274	.0894	.09961	.49793	.3973	.38857	.3117	.31190

totic formulas (AN). The asymptotic covariance between $x(i \mid n)$ and $x(j \mid n)$ is given by

$$\frac{j(n - i + 1)}{n(n + 1)^2 f[m(i \mid n)] f[m(j \mid n)]}, \quad j \leq i.$$

Symmetry relations exist for supplying the missing entries,

$$\text{cov}\,[x(i \mid n), x(j \mid n)] = \text{cov}\,[x(n - i + 1 \mid n), x(n - j + 1 \mid n)].$$

It might seem more natural to use the factor $n + 2$ rather than n in the denominator of the asymptotic variances and covariances so that the formulas would more nearly agree with those for the uniform distribution. However the use of n gives much better approximations for the normal and the special distribution.

Table III gives the variances and covariances of the order statistics for the uniform distribution (U), and Table IV gives the corresponding results for the special distribution (S). Table V gives the asymptotic variances and covariances for the special distribution (AS).

Table VI compares the correlation coefficients between the order statistics $x(i \mid n)$ and $x(j \mid n)$ for the uniform (U), the normal (N), and the special distribution (S).

It seems worthwhile to call attention to the following:

(1). Even for $n = 10$, the asymptotic formulas do not give satisfactory mean values for the order statistics.

(2). For $n \geq 8$, the asymptotic standard deviations for the normal are close enough to be very useful. For the special distribution we must except the two order statistics on each end from this statement.

TABLE II

Variances and covariances of the order statistics $x(i|n)$ for the normal (N) and the asymptotic normal (AN)

n	i \ j	1		2		3		4		5		6		7		8		9		10	
		N	AN	N	AN	N	AN	N	AN	N	AN	N	AN	N	AN	N	AN	N	AN	N	AN
2	1	.68	.84	.32	.42																
3	1	.56	.62	.28	.33	.17	.21														
	2			.45	.52																
4	1	.49	.51	.24	.28	.16	.18	.11	.13												
	2			.36	.40	.24	.27														
5	1	.45	.44	.22	.24	.15	.17	.11	.12	.07	.09										
	2			.31	.34	.21	.23	.15	.17												
	3					.29	.31														
6	1	.42	.40	.21	.22	.13	.15	.11	.12	.07	.09	.06	.07								
	2			.28	.29	.19	.20	.14	.15	.10	.12										
	3					.25	.26	.18	.20												
7	1	.39	.37	.20	.20	.13	.14	.10	.11	.08	.09	.06	.07	.05	.05						
	2			.26	.27	.17	.19	.13	.14	.10	.11	.08	.09								
	3					.22	.23	.17	.18	.13	.14										
	4							.21	.22												
8	1	.37	.34	.19	.19	.13	.13	.09	.10	.08	.08	.06	.07	.04	.05	.04	.04				
	2			.24	.24	.17	.17	.12	.13	.10	.10	.08	.09	.07	.07						
	3					.20	.21	.15	.16	.12	.13	.09	.11								
	4							.19	.20	.15	.16										
9	1	.36	.32	.18	.18	.12	.13	.09	.10	.07	.08	.06	.07	.05	.05	.04	.05	.04	.04		
	2			.23	.23	.16	.16	.11	.12	.09	.10	.08	.08	.06	.07	.05	.06				
	3					.19	.19	.14	.15	.11	.12	.10	.10	.08	.08						
	4							.17	.18	.14	.14	.12	.12								
	5									.17	.17										
10	1	.34	.31	.17	.17	.12	.12	.09	.09	.07	.08	.06	.06	.05	.05	.04	.05	.03	.04	.03	.03
	2			.21	.21	.14	.15	.11	.12	.09	.09	.07	.08	.06	.07	.05	.06	.04	.05		
	3					.17	.18	.13	.14	.11	.11	.09	.09	.08	.08	:06	.07				
	4							.16	.16	.12	.13	.11	.11	.09	.09						
	5									.15	.16	.13	.13								

(3). For $n \geq 8$, the asymptotic variances and covariances of the normal are close enough for many, if not most purposes.

(4). For the special distribution, only the variances and covariances of moderately central order statistics are adequately given by the asymptotic formulas.

TABLE III

Variances and covariances for the uniform distribution (U)

n	i \ j	1	2	3	4	5	6	7	8	9	10
2	1	.66667	.33333								
3	1	.45000	.30000	.15000							
	2		.60000								
4	1	.32000	.24000	.16000	.08000						
	2		.48000	.32000							
5	1	.23810	.19047	.14286	.09522	.04762					
	2		.38095	.28571	.19047						
	3			.42857							
6	1	.18367	.15306	.12245	.09184	.06122	.03061				
	2		.30612	.24490	.18367	.12245					
	3			.36735	.27551						
7	1	.14583	.12500	.10417	.08333	.06250	.04167	.02083			
	2		.25000	.20833	.16667	.12500	.08333				
	3			.31250	.25000	.18750					
	4				.33333						
8	1	.11852	.10370	.08889	.07407	.05925	.04444	.02963	.01481		
	2		.20741	.17778	.14815	.11852	.08889	.05925			
	3			.26667	.22222	.17778	.13333				
	4				.29630	.23704					
9	1	.09818	.08727	.07636	.06545	.05455	.04363	.03273	.02182	.01091	
	2		.17455	.15273	.13091	.10909	.08727	.06545	.04363		
	3			.22909	.19636	.16364	.13091	.09818			
	4				.26182	.21818	.17455				
	5					.27273					
10	1	.08264	.07438	.06611	.05785	.04959	.04132	.03306	.02479	.01653	.00826
	2		.14876	.13223	.11570	.09917	.08264	.06611	.04959	.03306	
	3			.19835	.17355	.14876	.12397	.09917	.07438		
	4				.23140	.19835	.16529	.13223			
	5					.24793	.20661				

(5). The correlation coefficients change rather little from distribution to distribution, the poorest approximation being for end order statistics.

TABLE IV

Variances and covariances for the special distribution (S)

n	i \ j	1	2	3	4	5	6	7	8	9	10
2	1	.71385	.28615								
3	1	.68530	.24214	.15957							
	2		.34172								
4	1	.68860	.23277	.14141	.11123						
	2		.27649	.17532							
5	1	.69960	.23154	.13655	.10004	.08614					
	2		.25391	.15418	.11490						
	3			.20163							
6	1	.71272	.23310	.13544	.09667	.07786	.07080				
	2		.24429	.14506	.10486	.08514					
	3			.17345	.12762						
7	1	.72619	.23577	.13582	.09565	.07517	.06409	.06042			
	2		.24002	.14065	.10004	.07913	.06776				
	3			.15970	.11509	.09184					
	4				.14249						
8	1	.73940	.23890	.13687	.09562	.07420	.06179	.05471	.05291		
	2		.23837	.13850	.09754	.07608	.06359	.05645			
	3			.15208	.10822	.08499	.07138				
	4				.12685	.10053					
9	1	.75211	.24219	.13825	.09608	.07398	.06085	.05266	.04789	.04721	
	2		.23814	.13756	.09625	.07443	.06141	.05327	.04852		
	3			.14756	.10413	.08097	.06707	.05835			
	4				.11780	.09225	.07680				
	5					.11004					
10	1	.76428	.24550	.13978	.09680	.07414	.06053	.05176	.04604	.04271	.04272
	2		.23872	.13732	.09565	.07354	.06018	.05156	.04594	.04266	
	3			.14481	.10158	.07846	.06444	.05533	.04940		
	4				.11207	.08707	.07180	.06186			
	5					.10016	.08300				

4. Methods of calculation and accuracy for the normal distribution. The means and variances of the order statistics for the normal distribution were obtained from direct quadrature of forms like

$$\int_{-\infty}^{\infty} x^k [F(x)]^j [1 - F(x)]^{n-j-1} f(x)\, dx, \qquad k = 1, 2,$$

where

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt \quad \text{and} \quad f(x) = F'(x).$$

It is believed that the means are correct to within one unit in the fifth decimal and that the standard deviations are correct to within 2 or 3 units in the fifth decimal.

TABLE V

Variances and covariances of the special distribution as computed from asymptotic formulas

n	i \ j	1	2	3	4	5	6	7	8	9	10
2	1	.56044	.28022								
3	1	.46550	.22297	.15517							
	2		.32038								
4	1	.42792	.20168	.13444	.10698						
	2		.25347	.16898							
5	1	.41156	.19167	.12579	.09605	.08231					
	2		.22368	.14679	.11208						
	3			.19221							
6	1	.40072	.18667	.12105	.09080	.07464	.06679				
	2		.20861	.13529	.10147	.08341					
	3			.16457	.12343						
7	1	.37715	.17527	.11304	.08394	.06782	.05842	.05388			
	2		.19004	.12258	.09103	.07354	.06335				
	3			.14232	.10569	.08538					
	4				.13731						
8	1	.39389	.18276	.11746	.08669	.06935	.05873	.05221	.04924		
	2		.19382	.12458	.09194	.07355	.06229	.05538			
	3			.14011	.10341	.08272	.07005				
	4				.12211	.09769					
9	1	.39286	.18226	.11881	.08591	.06829	.05727	.05092	.04556	.04367	
	2		.19019	.12398	.08965	.07126	.05977	.05313	.04754		
	3			.13855	.10019	.07963	.06678	.05938			
	4				.11265	.08958	.07512				
	5					.10680					
10	1	.39373	.18242	.11677	.08560	.06775	.05646	.04891	.04379	.04054	.03937
	2		.18784	.12024	.08813	.06977	.05814	.05036	.04508	.04174	
	3			.12988	.09520	.07536	.06280	.05440	.04871		
	4				.10633	.08417	.07014	.06076			
	5					.09716	.08098				

The evaluation of the covariances was much more troublesome, requiring the evaluation of iterated integrals of the form

$$\int_{-\infty}^{\infty} xf(x)F^j(x) \int_{-\infty}^{x} tf(t)[1 - F(t)]^i \, dt \, dx.$$

Necessary linear combinations of such forms give rise to considerable loss of accuracy. The covariances are believed to be correct to within 1 unit in the second decimal (except for one or two values which may be off by two units).

TABLE VI

Correlation coefficients × 10^2 between order statistics $x(i \mid n)$, $x(j \mid n)$ for the uniform (U), normal (N), and special distribution (S)

n \ j	i	2			3			4			5			6			7			8			9			10		
		U	N	S	U	N	S	U	N	S	U	N	S	U	N	S	U	N	S	U	N	S	U	N	S	U	N	S
2	1	50	47	40																								
3	1	58	55	50	33	29	23																					
4	1	61	58	53	41	38	32	25	21	16																		
	2				67	65	63																					
5	1	63	60	55	45	41	36	32	28	24	20	16	12															
	2				71	70	68	50	48	42																		
6	1	65	61	56	47	44	39	35	32	28	26	23	19	17	13	10												
	2				73	72	70	55	53	51	40	38	35															
	3							75	74	74																		
7	1	66	62	56	49	45	40	38	34	30	29	26	22	22	19	15	14	11	08									
	2				74	73	72	58	56	54	45	43	40	33	31	28												
	3							78	77	76	60	59	58															
8	1	66	62	57	50	46	41	40	36	32	32	28	24	25	22	18	19	16	13	12	10	07						
	2				76	74	73	60	58	56	48	46	44	38	36	33	29	26	24									
	3							79	79	78	63	62	61	50	49	47												
	4										80	80	79															
9	1	67	63	57	51	47	42	41	37	32	33	30	26	27	24	20	22	19	16	17	14	11	11	09	06			
	2				76	75	73	61	60	57	50	48	46	41	39	37	33	31	28	25	23	20						
	3							80	80	79	66	65	64	54	52	51	43	41	40									
	4										82	81	81	67	66	65												
10	1	67	63	57	52	47	42	42	38	33	35	31	27	29	25	22	24	21	18	19	17	14	15	13	10	10	08	06
	2				77	76	74	62	61	58	52	50	48	43	42	40	36	34	32	29	27	25	22	20	18			
	3							81	80	80	67	66	66	56	55	54	46	45	43	38	35	34						
	4										83	82	82	69	69	68	57	56	55									
	5													83	83	83												

Better tables of these covariances are badly needed, and it is hoped that someone will provide them.

The asymptotic values are correct to the two decimals given.

5. Computation in terms of the representing function. It will prove convenient in working with the special distribution, as indeed it does in many statistical procedures, to introduce the *representing function* $r(u)$, which is a monotone function such that

$$Pr\,\{r(u_1) \leq x \leq r(u_2)\} = u_2 - u_1\,, u_2 \geq u_1\,.$$

Thus if u has a uniform (= rectangular on [0, 1]) distribution then $x = r(u)$ defines a variate with the given distribution.

The ith order statistic of n from the uniform distribution, $u_{i|n}$, is distributed according to

$$i\binom{n}{i}u^{n-i}(1 - u)^{i-1}\,du,\ 0 \leq u \leq 1,$$

where it is important to remember that $u_{1|n}$ is the largest and not the smallest order statistic; and the joint distribution of $u = u_{i|n}$ and $v = u_{j|n}$, $(j > i)$, is given by

$$i(j - i)\begin{bmatrix} n \\ i, j - i, n - j\end{bmatrix} v^{n-j}(u - v)^{j-i-1}(1 - u)^{i-1}\,du\,dv, \qquad 0 \leq v \leq u \leq 1,$$

where $\begin{bmatrix} n \\ i, j - i, n - j\end{bmatrix}$ is a multinomial coefficient.

The means, variances, and covariances which we desire can be written as follows (it is immaterial whether we think of expectations over x's or over u's):

$$E(x_{i|n}) = E(r(u_{i|n})) = i\binom{n}{i}\int_0^1 r(u)u^{n-i}(1 - u)^{i-1}\,du,$$

$$\begin{aligned}\text{var}\,(x_{i|n}) &= E(x_{i|n})^2 - (E(x_{i|n}))^2 = E(r^2(u_{i|n})) - E(x_{i|n})^2\\ &= i\binom{n}{i}\int_0^1 r^2(u)u^{n-i}(1 - u)^{i-1}\,du - (E(x_{i|n}))^2,\end{aligned}$$

$$\begin{aligned}\text{cov}\,(x_{i|n}\,, x_{j|n}) &= E(x_{i|n}x_{j|n}) - E(x_{i|n})\cdot E(x_{j|n})\\ &= E(r(u_{i|n})r(u_{j|n})) - E(x_{i|n})E(x_{j|n})\\ &= i(j - i)\begin{bmatrix} n \\ i, j - i, n - j\end{bmatrix}\int_0^1\int_v^1 r(u)r(v)v^{n-j}(u - v)^{j-i-1}(1 - u)^{i-1}\,du\,dv\\ &\qquad - E(x_{i|n})E(x_{j|n})\end{aligned}$$

Introducing $E_{s,t}$ by

$$E_{s,t} = \int_0^1\int_v^1 r(u)r(v)u^s v^t\,du\,dv,$$

we have

$$E(x_{i|n}x_{j|n}) = i(j - 1)\begin{bmatrix} n \\ i, j - i, n - j\end{bmatrix} \cdot \sum_{k,m}(-1)^{k+m}\binom{j - i - 1}{k}\binom{i - 1}{m}E_{k+m,n-i-1-k},$$

and, in particular,

$$E(x_{1|2}x_{2|2}) = 2E_{0,0} ,$$

$$E(x_{1|5}x_{4|5}) = 60E_{2,1} - 120E_{1,8} + 60E_{0,3} .$$

Introducing $E_{s,s}$ by

$$E_{s,s} = \int_0^1 r^2(u)u^s \, du,$$

we have

$$E(x^2_{i|n}) = i\binom{n}{i}\sum_k (-1)^k \binom{i-1}{k} E_{n-i+k,n-i+k}$$

and, in particular,

$$E(x^2_{2|5}) = 20E_{3,3} - 20E_{4,4} .$$

Introducing E_s by

$$E_s = \int_0^1 r(u)u^s \, du,$$

we have

$$E(x_{i|n}) = i\binom{n}{i}\sum_k (-1)^k \binom{i-1}{k} E_{n-i+k}$$

and, in particular,

$$E(x_{3|5}) = 30E_2 - 60E_3 + 30E_4 .$$

Thus the computation of the desired means, variances, and covariances is reduced to the computation of the integrals E_s , $E_{s,s}$, and $E_{s,t}$.

We shall also want to calculate the asymptotic approximations to the means, variances, and covariances of the order statistics. For the uniform distribution, it is well known that

$$\text{mean } (u_{i|n}) = \frac{n-i+1}{n+1},$$

$$\text{var } (u_{i|n}) = \frac{i(n-i+1)}{(n+1)^2(n+2)},$$

$$\text{cov } (u_{i|n}\, u_{j|n}) = \frac{i(n-j+1)}{(n+1)^2(n+2)}, \qquad (i < j).$$

These asymptotic formulas are transformed from u to x by the relations $x = r(u)$ and $dx = r'(u)\, du$, giving

$$\text{approx mean } (x_{i|n}) = r\left(\frac{n-i+1}{n+1}\right),$$

$$\text{approx var } (x_{i|n}) = \left(r'\left(\frac{n-i+1}{n+1}\right)\right)^2 \frac{i(n-i+1)}{(n+1)^2(n+2)},$$

$$\text{approx cov } (x_{i|n}x_{j|n}) = r'\left(\frac{n-j+1}{n+1}\right) \cdot r'\left(\frac{n-i+1}{n+1}\right)\frac{i(n-j+1}{(n+1)^2(n+2)}, \qquad (i \leq j),$$

as noted above, in our calculations we have replaced $n + 2$ by n in the denominator.

6. Reduction of integrals for the special case. When the reoresenting function is

$$x = r(u) = \frac{1}{(1-u)^\lambda} - \frac{1}{u^\lambda}, \qquad (\lambda > 0),$$

we obtain a symmetrical distribution with long tails. (For the normal distribution $r(u) = o(\ln u)$ as $u \to 0$). The integrals we want are

$$E_s = \int_0^1 \{(1-u)^{-\lambda} - u^{-\lambda}\}u^s\,du,$$

$$E_{s,s} = \int_0^1 \{(1-u)^{-\lambda} - u^{-\lambda}\}^2 u^s\,du,$$

$$E_{s,t} = \int_0^1 \int_v^1 \{(1-u)^{-\lambda} - u^{-\lambda}\}\{(1-v)^{-\lambda} - v^{-\lambda}\}u^s v^t\,du\,dv,$$

which can be expressed as

$$E_s = A_s(\lambda) - B_s(\lambda),$$

$$E_{s,s} = A_{s,s}(\lambda) - 2B_{s,s}(\lambda) + C_{s,s}(\lambda),$$

$$E_{s,t} = A_{s,t}(\lambda) - B_{s,t}(\lambda) - C_{s,t}(\lambda) + D_{s,t}(\lambda),$$

where

$$A_s(\lambda) = \int_0^1 (1-u)^{-\lambda}u^s\,du = b(-\lambda, s),$$

$$B_s(\lambda) = \int_0^1 u^{-\lambda}u^s\,du = \frac{1}{s+1-\lambda},$$

$$A_{s,s}(\lambda) = \int_0^1 (1-u)^{-2\lambda}u^s\,du = b(-2\lambda, s),$$

$$B_{s,s}(\lambda) = \int_0^1 (1-u)^{-\lambda}u^{-\lambda}u^s\,du = b(-\lambda, s-\lambda),$$

$$C_{s,s}(\lambda) = \int_0^1 u^{-2\lambda}u^s\,du = \frac{1}{s+1-2\lambda},$$

$$A_{s,t}(\lambda) = \int_0^1 \int_v^1 (1-u)^{-\lambda}(1-v)^{-\lambda}u^s v^t\,du\,dv$$

$$= \sum_{i=0}^{s} \binom{s}{i} (-)^i \frac{b(i + 1 - 2\lambda, t)}{i + 1 - \lambda},$$

$$B_{s,t}(\lambda) = \int_0^1 \int_v^1 (1 - u)^{-\lambda} v^{-\lambda} u^s v^t \, du \, dv$$

$$= \sum_{i=0}^{s} \binom{s}{i} (-)^i \frac{b(i + 1 - \lambda, t - \lambda)}{i + 1 - \lambda} = \frac{b(s + t + 1 - \lambda, -\lambda)}{t + 1 - \lambda},$$

$$C_{s,t}(\lambda) = \int_0^1 \int_0^1 u^{-\lambda}(1 - v)^{-\lambda} u^s v^t \, du \, dv$$

$$= \frac{1}{s + 1 - \lambda} \{b(-\lambda, t) - b(s + t + 1, -\lambda, -\lambda)\},$$

$$D_{s,t}(\lambda) = \int_0^1 \int_v^1 u^{-\lambda} v^{-\lambda} u^s v^t \, du \, dv$$

$$= \frac{1}{(t + 1 - \lambda)(s + t + 2 - 2\lambda)},$$

where throughout

$$b(p, q) = \frac{p!q!}{(p + q + 1)!} = \frac{\Gamma(p + 1)\Gamma(q + 1)}{\Gamma(p + q + 2)} = \mathrm{B}(p + 1, q + 1).$$

7. Calculations for the special distribution. The computations for the special distribution were made from the formulas in the preceding section. The quantities A, B, C, D were computed from $r = s = 0$ to $r + s = 8$, whence the values of E_s, E_{ss}, E_{st} were calculated. The values of the means, variances, and covariances were then obtained from the formulas of section 3.

The means, variances, and covariances are believed to be accurate to the five decimal places given.

8. Formulas and accuracy for the uniform. The means, variances, and covariances of the uniform are given near the end of section 5. Since $r(u) \equiv u$, they are also the values given by the asymptotic approximation, when $n + 2$ is used.

The tabulated values were computed to six decimal places and rounded to the four or five decimals given.

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