AN INEQUALITY DUE TO H. HORNICH

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H. Hornich[1] proved a theorem on the average risk of the sum of equal insurance policies. It seems of interest to note that when translated from its actuarial formulation into the terminology of the calculus of probabilities this theorem becomes an inequality for mean deviations of random variables, and to present it with a concise proof in non-actuarial language.

Let x be a random variable with a symmetrical probability distribution, $D_1 = E(|x|)$ its mean deviation, $x_1, x_2, \cdots, x_n$ independent repetitions of x, and $D_n = E(|x_1 + x_2 + \cdots + x_n|)$ the mean deviation of $x_1 + x_2 + \cdots + x_n$. Then D_n fulfills the inequality

$$D_n \geqq \frac{D_1 n}{2^{n-1}} \binom{n-1}{\left[\frac{n}{2}\right]} \tag{1}$$

where $\left[\frac{n}{2}\right]$ denotes the greatest integer $\leqq \frac{n}{2}$. If the distribution of x is not symmetrical but $E(x) = 0$, the inequality becomes

$$D_n \geqq \frac{D_1 n}{2^{n}} \binom{n-1}{\left[\frac{n}{2}\right]}. \tag{2}$$

The proof will be given for a continuous random variable but it clearly holds quite generally. If $f(x)$ is the probability density of x, and $E(x) = 0$, then one has

$$D_1 = \int_{-\infty}^{+\infty} |x| f(x)\, dx = 2\int_0^{\infty} x f(x)\, dx. \tag{3}$$

In the expression for D_n, the integration over the entire n-space $(x_1, x_2, \cdots, x_n)$ may be performed by integrating separately over each of the 2^n "octants" which correspond to the different combinations of signs of the coordinates, and thus one obtains, for a symmetrical distribution, the estimates

$$D_n = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left|\sum_{i=1}^{n} x_i\right|\prod_{i=1}^{n} f(x_i)\prod_{i=1}^{n} dx_i$$

$$= \sum_{\epsilon_1=\pm1,\, \epsilon_2=\pm1,\, \cdots,\, \epsilon_n=\pm1} \int_{\operatorname{sgn} x_1=\epsilon_1}\int_{\operatorname{sgn} x_2=\epsilon_2}\cdots\int_{\operatorname{sgn} x_n=\epsilon_n}$$

$$= \sum_{s=0}^{n}\binom{n}{s}\underset{\substack{\operatorname{sgn} x_1=\operatorname{sgn} x_2=\cdots=\operatorname{sgn} x_s=-1\\ \operatorname{sgn} x_{s+1}=\operatorname{sgn} x_{s+2}=\cdots=\operatorname{sgn} x_n=+1}}{\int\int\cdots\int} \geqq 2\sum_{s=0}^{[(n-1)/2]}\binom{n}{s}\underset{\substack{\operatorname{sgn} x_1=\operatorname{sgn} x_2=\cdots=\operatorname{sgn} x_s=-1\\ \operatorname{sgn} x_{s+1}=\operatorname{sgn} x_{s+2}=\cdots=\operatorname{sgn} x_n=+1}}{\int\int\cdots\int}$$

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[1] Hans Hornich, "Zur theorie des Risikos," *Monatsh. Math. Phys.*, Vol. 50 (1941), pp. 142–150.

$$\geqq 2 \sum_{s=0}^{[(n-1)/2]} \binom{n}{s} \underset{\substack{\operatorname{sgn} x_1 = \operatorname{sgn} x_2 = \cdots = \operatorname{sgn} x_s = -1 \\ \operatorname{sgn} x_{s+1} = \operatorname{sgn} x_{s+2} = \cdots = \operatorname{sgn} x_n = +1}}{\int \int \cdots \int} \left(\sum_{i=s+1}^{n} x_i - \sum_{i=1}^{s} x_i \right) \prod_{i=1}^{n} f(x_i) \prod_{i=1}^{n} dx_i$$

$$= 2 \sum_{s=0}^{[(n-1)/2]} \binom{n}{s} (n - 2s) \frac{D_1}{2} \cdot \frac{1}{2^{n-1}} = \frac{D_1}{2^{n-1}} \left\{ n \sum_{s=0}^{[(n-1)/2]} \binom{n}{s} - 2 \sum_{s=0}^{[(n-1)/2]} s \binom{n}{s} \right\}$$

$$= \frac{D_1 n}{2^{n-1}} \left\{ \sum_{s=0}^{[(n-1)/2]} \binom{n}{s} - 2 \sum_{s=0}^{[(n-1)/2-1]} \binom{n-1}{s} \right\}$$

$$= \frac{D_1 n}{2^{n-1}} \left\{ \binom{n}{\left[\frac{n-1}{2}\right]} + \sum_{s=0}^{[(n-1)/2-1]} \left(\binom{n}{s} - \binom{n-1}{s} \right) - \sum_{s=0}^{[(n-1)/2-1]} \binom{n-1}{s} \right\}$$

$$= \frac{D_1 n}{2^{n-1}} \left\{ \binom{n}{\left[\frac{n-1}{2}\right]} + \sum_{s=1}^{[(n-1)/2-1]} \binom{n-1}{s-1} - \sum_{s=0}^{[(n-1)/2-1]} \binom{n-1}{s} \right\}$$

$$= \frac{D_1 n}{2^{n-1}} \left\{ \binom{n}{\left[\frac{n-1}{2}\right]} - \binom{n-1}{\left[\frac{n-1}{2} - 1\right]} \right\}$$

$$= \frac{D_1 n}{2^{n-1}} \binom{n-1}{\left[\frac{n-1}{2}\right]} = \frac{D_1 n}{2^{n-1}} \binom{n-1}{\left[\frac{n}{2}\right]}$$

If x is not symmetrical but $E(x) = 0$, we consider the random variable x' with the probability density $g(x') = f(-x')$, and the random variable $y = x + x'$. In view of (3) we find

$$E(|y|) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x + x'| f(x) g(x') \, dx \, dx' \geqq 2 \int_0^{\infty} \int_0^{\infty} (x + x') f(x) g(x') \, dx \, dx'$$

$$= 2 \int_0^{\infty} g(x') \int_0^{\infty} x f(x) \, dx \, dx' + 2 \int_0^{\infty} f(x) \int_0^{\infty} x' g(x') \, dx' \, dx$$

$$= E(x) \left\{ -\int_0^{\infty} g(-x) \, dx + \int_0^{\infty} f(x) \, dx \right\} = E(x).$$

Let $x_1, x_2, \cdots, x_n$ and $x_1', x_2', \cdots, x_n'$ be independent repetitions of x and x', respectively, and $y_i = x_i + x_i'$. Since y has a symmetrical distribution, an application of (1) gives

$$E(|x_1 + x_2 + \cdots x_n|) = \tfrac{1}{2}\{E(|x_1 + \cdots + x_n|) + E(|x_1' + \cdots + x_n'|)\}$$

$$\geqq \tfrac{1}{2} E(|x_1 + x_1' + \cdots + x_n + x_n'|) = \tfrac{1}{2} E(|y_1 + \cdots + y_n|)$$

$$\geqq \frac{1}{2} \cdot \frac{E(|y|) n}{2^{n-1}} \binom{n-1}{\left[\frac{n}{2}\right]} \geqq \frac{E(|x|) n}{2^n} \binom{n-1}{\left[\frac{n}{2}\right]}$$

An application of Stirling's formula shows that the right hand sides in (1) and (2) are of the order of magnitude of $\sqrt{n}$.