NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

NOTE ON THE INDEPENDENCE OF CERTAIN QUADRATIC FORMS

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Various approaches to the problem of the independence of quadratic forms in normally and independently distributed variables have been made by R. A. Fisher, Cochran, Madow and others. It is the purpose of this note to point out a few simple propositions which, in so far as the writer is aware, have not had specific mention in the literature.

1. Independence of certain quadratic forms. Theorem 1: *A necessary and sufficient condition that two real symmetric quadratic forms, in* n *normally and independently distributed variables, be independent in the probability sense is that the product of the matrices of the forms be zero.*

Let the chance variable x be normally distributed with mean zero and unit variance. Let $x_1, x_2, \cdots, x_n$ be n independent values of x and let A and B be two real symmetric matrices, each of order n. Write $Q_1 = \Sigma\Sigma a_{ij}x_ix_j$ and $Q_2 = \Sigma\Sigma b_{ij}x_ix_j$ where $\| a_{ij} \| = A$ and $\| b_{ij} \| = B$. It is well known that the generating function of the moments of the joint distribution of Q_1 and Q_2 can be written

$$G(\lambda, \lambda') = | I - \lambda A - \lambda' B |^{-\frac{1}{2}},$$

so that

$$| I - \lambda A - \lambda' B | = | I - \lambda A || I - \lambda' B |, \tag{1}$$

for all real values of λ and λ', is necessary and sufficient for the independence of Q_1 and Q_2.

If Q_1 and Q_2 are independent, then (1), being true for all real values of λ and λ', is in particular true for $\lambda = \lambda'$. Thus

$$| I - \lambda(A + B) | \equiv | I - \lambda A || I - \lambda B |. \tag{2}$$

Denote by r_1, r_2 and $r \leq r_1 + r_2$ respectively the ranks of A, B and $A + B$. Then $r = r_1 + r_2$ since (2) expresses the identity of two polynomials in λ of degrees r and $r_1 + r_2$.

Further, if we write

$$| I - \lambda A | = (1 - \lambda p_1) \cdots (1 - \lambda p_{r_1}),$$

$$| I - \lambda B | = (1 - \lambda q_1) \cdots (1 - \lambda q_{r_2}),$$

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and $| I - \lambda(A + B) | = (1 - \lambda s_1) \cdots (1 - \lambda s_{r_1+r_2})$, then, because the factorization of polynomials is unique, each s_j can be paired with one of the numbers $p_1, \cdots, p_{r_1}, q_1, \cdots, q_{r_2}$. Thus, if Q_1 and Q_2 are independent, the rank of $A + B$ is the sum of the ranks of A and B, and the non-zero roots of the characteristic equation of $A + B$ are those of the characteristic equation of A together with those of the characteristic equation of B. There exists an appropriately chosen orthogonal matrix L of order n such that $L'(A + B)L$, L' being the conjugate of L, is a matrix with the reciprocals of the numbers $p_1, \cdots, p_{r_1}, q_1, \cdots, q_{r_2}$ on the principal diagonal and zeros elsewhere. Then $L'AL$ and $L'BL$ have no overlapping non-zero elements and $L'ALL'BL = 0$. But $L' = L^{-1}$, the inverse of L. Hence, upon multiplying both members of the preceding equation on the right by L' and on the left by L, we have $AB = 0$. Since $A = A'$ and $B = B'$, likewise $BA = 0$.

Conversely, suppose $AB = 0$. Then the matrix $(I - \lambda A)(I - \lambda' B) = I - \lambda A - \lambda' B$. These matrices being equal, their determinants are equal and the condition (1) for the independence of Q_1 and Q_2 is satisfied.

The theorem is readily extended to the case of the mutual independence of any finite number of such quadratic forms.

The product of a non-singular matrix and a matrix of rank R is a matrix of rank R. Hence, every non-singular quadratic form of the kind here discussed is correlated with every non-identically vanishing quadratic form in the same variables.

2. Conditions for independent Chi-Square distributions. The preceding theorem enables one to determine, by multiplication of matrices, whether real symmetric quadratic forms in normally and independently distributed variables are themselves independent in the probability sense. The following theorem affords a simple test as to whether the distributions are of the Chi-Square type.

THEOREM 2: *Necessary and sufficient conditions that each of two real symmetric quadratic forms, in n normally and independently distributed variables with mean zero and unit variance, be independently distributed as is Chi-Square, are that the product of the matrices of the forms be zero and that each matrix equal its own square.*

If Q_1 and Q_2 are independently distributed as is Chi-Square, then $AB = 0$ and each of the non-zero roots of the characteristic equations of A and B is $+1$. For an appropriately chosen orthogonal matrix L, of order n, $L'AL$ is a matrix with r_1 elements on the principal diagonal $+1$, all other elements being zero. For such a matrix it is seen that $(L'AL)(L'AL) = L'A^2L = L'AL$ and $A^2 = A$. A similar argument shows that $B^2 = B$.

Conversely, if $AB = 0$, then Q_1 and Q_2 are independent. Further, if $A^2 = A$ and $B^2 = B$, each of the non-zero roots of the characteristic equations of A and B is $+1$. This follows from the fact that the roots of the characteristic equation of the square of any matrix are themselves the squares of the roots of the characteristic equation of that matrix. Since A and B are real and symmetric, the roots under consideration are real. Thus Q_1 and Q_2 have independent Chi-Square distributions with r_1 and r_2 degrees of freedom respectively.

This theorem can likewise be extended to any finite number of these quadratic forms.

Of special interest is the case of, say k, quadratic forms for which the sum of the k matrices is the identity matrix. Thus $A_1 + A_2 + \cdots + A_k = I$. By Theorem 1, it is both necessary and sufficient for the mutual independence of the k forms that $A_u A_v = 0$, $u \neq v$.

Now

$$A_i = I - A_1 - \cdots - A_{i-1} - A_{i+1} - \cdots - A_j - \cdots - A_k$$

and

$$A_i A_j = A_j - A_1 A_j - \cdots - A_{i-1} A_j - A_{i+1} A_j - \cdots - A_j^2 - \cdots - A_k A_j,$$

so that $A_j = A_j^2$. In this particular case it is to be seen that the mutual independence of the forms implies that their several distributions are of the Chi-Square type.

A CHARACTERIZATION OF THE NORMAL DISTRIBUTION

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In 1925 R. A. Fisher gave a geometric derivation of the joint distribution of mean and variance in samples from a normal population (*Metron*, Vol. 5, pp. 90–104). On examining the argument however, we find that an (apparently) more general result is actually established: if $f(x_1) \cdots f(x_n)$ is a function $g(m, s)$ of the sample mean m and standard deviation s, then the probability density of m and s in samples of n from the population $f(x)$ is $g(m, s)s^{n-2}$. This condition on $f(x)$ is of course satisfied if $f(x)$ is normal; in this note we shall conversely show that for $n \geqq 3$ it characterizes the normal distribution. In the proof it will be assumed that $g(m, s)$ possesses partial derivatives of the first order, although a weaker assumption would probably suffice.

Let us for the moment restrict the variables x_i to values such that $f(x_i) > 0$. After a change of notation we have

$$\phi(x_1) + \cdots + \phi(x_n) = h(u, v),$$

where $\phi = \log f$, $u = x_1 + \cdots + x_n$, $v = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$. A differentiation yields

$$\phi'(x_i) = h_u + h_v x_i .$$