NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A NOTE ON THE THEORY OF MOMENT GENERATING FUNCTIONS

By J. H. Curtiss

Cornell University

Let X be a one-dimensional variate and let $F(x)$ be its distribution function.[1] The function

$$G(\alpha) = E(e^{\alpha X}) = \int_{-\infty}^{+\infty} e^{\alpha x}\, dF(x), \qquad \alpha \text{ real},$$

in which the integral is assumed to converge for α in some neighborhood of the origin, is called the moment generating function of X. In dealing with certain distribution problems, this function has been widely used by statisticians, and especially by the English writers, in place of the closely-related characteristic function $f(t) = E(e^{itX})$. It is known that a characteristic function uniquely determines the corresponding distribution, and that if a sequence of characteristic functions approaches a limit, the corresponding sequence of distribution functions does likewise. (These results are more accurately stated below.) The appropriate analogues for the moment generating function of these theorems are apparently not too readily accessible in the literature, if they have been treated at all, and it seems worthwhile to record them in this note.

Henceforth we abbreviate distribution function to d.f., moment generating function to m.g.f., and characteristic function to c.f. The variables α and t will always be real, in contradistinction to the complex variable s, to be introduced in the next paragraph.

The uniqueness property of the c.f. may be stated as follows: If $F_1(x)$ and $f_1(t)$ are the d.f. and c.f. of one variate, and $F_2(x)$ and $f_2(t)$ are those of another, and if $f_1(t) \equiv f_2(t)$ for all[2] t, then $F_1(x) \equiv F_2(x)$ for all x [1, p. 28]. To study the corresponding situation for the m.g.f., we first observe that

$$\varphi(s) = E(e^{sX}) = \int_{-\infty}^{+\infty} e^{sx}\, dF(x), \qquad s \text{ complex},$$

[1] Or cumulative frequency function; our notation and terminology are uniform with that of [1] except for the use of the term "variate" instead of "random variable."

[2] It is possible for two non-identical distributions to have c.f.'s which are identical throughout an interval of values of t containing the origin; an example is given in [4], p. 190. The author is obliged to Professor Wintner and Professor Feller for pointing out the existence of this particular example.

is a bilateral Laplace-Stieltjes transform. If such a transform exists for real values of s in an interval $-\alpha_1 < s < \alpha_1$, $\alpha_1 > 0$, it must exist for all complex values of s in the strip $-\alpha_1 < \Re s < \alpha_1$, and represent there an analytic function of s [5, p. 238]. Evidently $\varphi(\alpha) = G(\alpha)$, $\varphi(it) = f(t)$. Suppose now that $F_1(x)$, $G_1(\alpha)$, $f_1(t)$, are the d.f., m.g.f., and c.f. of a variate X_1, and $F_2(x)$, $G_2(\alpha)$, $f_2(t)$, are those of X_2. Let $\varphi_1(s) = E(e^{sX_1})$, $\varphi_2(s) = E(e^{sX_2})$, s complex. If $G_1(\alpha) \equiv G_2(\alpha)$ for all α in some interval, however small, containing the origin, then by a familiar property of analytic functions [2, p. 116], $\varphi_1(s) \equiv \varphi_2(s)$ throughout the corresponding strip of analyticity, and so on the axis of imaginaries. This means that $f_1(t) \equiv f_2(t)$, all t, and therefore $F_1(x) \equiv F_2(x)$. We have:

THEOREM 1. *A m.g.f. existing in some neighborhood of* $\alpha = 0$ *uniquely determines the corresponding distribution.*

We turn now to distributions of variable form. Because certain of the versions to be found in the literature are incomplete, it seems worth while to give here a full statement of the basic limit theorem for sequences of c.f.'s, due to P. Lévy and sometimes called Lévy's Continuity Theorem [4, pp. 48–50].

THEOREM 2. *Let the distribution of a variate* X_n *depend on a parameter* n, *and let* $F_n(x)$ *and* $f_n(t)$ *be the d.f. and c.f. of* X_n.

(a) *If there exists a variate* X *with d.f.* $F(x)$ *such that* $\lim_{n\to\infty} F_n(x) = F(x)$ *at every continuity point of* $F(x)$, *then* $\lim_{n\to\infty} f_n(t) = f(t)$ *uniformly in each finite interval on the t-axis, where* $f(t)$ *is the c.f. of* X.

(b) *If there exists a function* $f(t)$ *such that* $\lim_{n\to\infty} f_n(t) = f(t)$, *all* t,[3] *and uniformly*[4] *in some open interval containing the origin, then there exists a variate* X *with d.f.* $F(x)$ *such that* $\lim_{n\to\infty} F_n(x) = F(x)$ *at each continuity point and uniformly in any finite or infinite interval of continuity of* $F(x)$. *The c.f. of* X *is* $f(t)$, *and* $\lim_{n\to\infty} f_n(t) = f(t)$ *uniformly in each finite interval.*

We now develop the corresponding theorem for the m.g.f. In the first place, it is not difficult to see that part (a) will have no direct analogue, even if we add to the hypothesis the conditions that the m.g.f. of X_n exists in some fixed interval for all n and that the m.g.f. of X also exists in some interval. For example, the d.f.

$$F_n(x) = \begin{cases} 0, \; x < -n \\ \frac{1}{2} + k_n \arctan nx, & -n \leqq x < n \\ 1, \; x \geqq n \end{cases}$$

[3] The condition that $\lim_{n\to\infty} f_n(t)$ exist on at least an everywhere dense set of points on the t-axis is essential to the proof as given in Cramer's book [1, pp. 29–30], but is omitted in his statement of the theorem, and is not stated clearly in certain other treatments by other authors.

[4] For a discussion of this uniformity condition, and possible alternatives, see [1, p. 29 (footnote)]. The condition may, for instance, be replaced by the assumption that $f(t)$ is continuous at $t = 0$.

where $k_n = 1/(2 \text{ arc tan } n^2)$, clearly tends as $n \to \infty$ to the d.f.

$$F(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqq 0 \end{cases}$$

at all points of continuity of the latter d.f. The m.g.f. corresponding to $F_n(x)$ is

$$G_n(\alpha) = \int_{-n}^{n} k_n e^{\alpha x} \frac{n}{1 + n^2x^2} dx,$$

which for each n exists for all α, and the m.g.f. corresponding to $F(x)$ is simply the constant 1. Clearly

$$G_n(\alpha) > \int_0^n k_n \frac{|\alpha|^3 x^3}{3!} \cdot \frac{n}{1 + n^2x^2} dx,$$

and from this it can easily be verified that $\lim_{n\to\infty} G_n(\alpha) = \infty$ if $\alpha \neq 0$. In short, mere convergence of a sequence of d.f.'s tells little about the behavior of the corresponding sequence of m.g.f.'s.

Part (b) assumes the following form:

THEOREM 3. *Let $F_n(x)$ and $G_n(\alpha)$ be respectively the d.f. and m.g.f. of a variate X_n. If $G_n(\alpha)$ exists for $|\alpha| < \alpha_1$ and for all $n \geqq n_0$, and if there exists a finite-valued function $G(\alpha)$ defined for $|\alpha| \leqq \alpha_2 < \alpha_1$, $\alpha_2 > 0$, such that $\lim_{n\to\infty} G_n(\alpha) = G(\alpha)$, $|\alpha| \leqq \alpha_2$, then there exists a variate X with d.f. $F(x)$ such that $\lim_{n\to\infty} F_n(x) = F(x)$ at each continuity point and uniformly in each finite or infinite interval of continuity of $F(x)$. The m.g.f. of X exists for $|\alpha| \leqq \alpha_2$ and is equal to $G(\alpha)$ in that interval.*

To prove the theorem, we introduce the Laplace transform $\varphi_n(s) = E(e^{sX_n})$ and observe that $|\varphi_n(s)| \leqq \varphi_n(\alpha) = G_n(\alpha)$, $s = \alpha + it$, $n \geqq n_0$, for any s in the strip $-\alpha_1 < \mathfrak{R}s < \alpha_1$. By applying Leibniz's rule for differentiation under an integral sign (extended to Stieltjes integrals), we find [5, p. 240] that

$$G_n''(\alpha) = \int_{-\infty}^{+\infty} x^2 e^{\alpha x} dF_n(x), \qquad |\alpha| < \alpha_1,$$

from which it appears that $G_n''(\alpha) > 0$, $|\alpha| < \alpha_1$. This means that the function $G_n(\alpha)$ assumes its maximum value in the interval $|\alpha| \leqq \alpha_2$ at either or both endpoints of the interval. But of course $G_n(\alpha_2)$ and $G_n(-\alpha_2)$ both approach finite limits as n becomes infinite, so it follows that the sequence $\{G_n(\alpha)\}$, $n \geqq n_0$, is uniformly bounded in the interval $|\alpha| \leqq \alpha_2$. Thus the sequence $\{|\varphi_n(s)|\}$, $n \geqq n_0$, is uniformly bounded in the strip $-\alpha_2 \leqq \mathfrak{R}s \leqq \alpha_2$, and moreover has a limit at each point of an infinite set possessing a limit point in the strip (i.e., at each point of the interval $-\alpha_2 \leqq s \leqq \alpha_2$). So by Vitali's Theorem [3, pp. 156–160, 240], there exists an analytic function $\varphi^*(s)$ such that $\lim_{n\to\infty} \varphi_n(s) = \varphi^*(s)$ uniformly in each bounded closed subregion of the strip $-\alpha_2 < \mathfrak{R}s < \alpha_2$. Since $\varphi_n(it)$ is the c.f. of X_n, the existence of the limiting distribution follows from Theorem 2(b).

Of course, $\varphi^*(\alpha) = G(\alpha)$, $-\alpha_2 < \alpha < \alpha_2$. It remains to show that $\varphi^*(\alpha)$ is the m.g.f. of X. Theorem 2(b) states that $\varphi^*(it)$ is the c.f. of X. If we can show that the function $\varphi(s) = E(e^{sX})$ exists at least in the strip $-\alpha_2 < \Re s < \alpha_2$, then since $\varphi(s) \equiv \varphi^*(s)$ on the axis of imaginaries, the equality must be valid in the entire strip, and so in particular on the interval of the real axis inside the strip.

It will suffice for this purpose to show that $\varphi(\alpha)$ exists for $-\alpha_2 \leqq \alpha \leqq \alpha_2$. Suppose indeed that $\varphi(\alpha)$ does not exist at some point $\alpha = \alpha_3$ in this interval. That means that if

$$M = [\text{l.u.b. } G_n(\alpha_3), n \geqq n_0],$$

we can find a real number A such that

$$\int_{-A}^{A} e^{\alpha_3 x}\, dF(x) > M. \tag{1}$$

But

$$\int_{-A}^{A} e^{\alpha_3 x}\, dF(x) = \int_{-A}^{A} e^{\alpha_3 x}\, dF_n(x) + \left[\int_{-A}^{A} e^{\alpha_3 x}\, dF(x) - \int_{-A}^{A} e^{\alpha_3 x}\, dF_n(x)\right].$$

Since $\lim_{n\to\infty} F_n(x) = F(x)$ at all continuity points of $F(x)$, and so on an everywhere dense set of points, the Helly-Bray Theorem [5, p. 31] states that the expression in brackets in (2) approaches zero as n becomes infinite. Meanwhile

$$\int_{-A}^{A} e^{\alpha_3 x}\, dF_n(x) \leqq \int_{-\infty}^{+\infty} e^{\alpha_3 x}\, dF_n(x) \leqq M, \qquad n \geqq n_0.$$

Thus we arrive at the conclusion that the left member of (2) must be less than or equal to M, which contradicts (1).

To be sure, we have only proved that the m.g.f. of X is equal to $\varphi^*(\alpha)$ or $G(\alpha)$ in the open interval $-\alpha_2 < \alpha < \alpha_2$, and not in the corresponding closed interval, as promised. But because of the absolute (and therefore uniform) convergence of the integrals defining $G_n(\alpha)$ and $\varphi(\alpha)$, these functions must be continuous in the closed interval $-\alpha_2 \leqq \alpha \leqq \alpha_2$. Since $\lim_{n\to\infty} G_n(\alpha) = G(\alpha)$ uniformly in this interval, $G(\alpha)$ must also be continuous there. This implies that $\varphi(\alpha)$, the m.g.f. of X, is identically equal to $G(\alpha)$ in the closed interval, and the proof is complete.

It is perhaps worth while to point out explicitly that in the course of the foregoing argument we have proved this proposition:

THEOREM 4. *If a sequence of m.g.f.'s converges in an open interval containing* $\alpha = 0$, *then it must converge uniformly in every closed subinterval of the open interval, and the limit function is itself a m.g.f.*

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