$$\left\{A^{ij} - \frac{1}{N}\sum_{\alpha} (y^i_\alpha - x^i_\mu a^\mu)(y^j_\alpha - x^j_\nu a^\nu)\right\} \delta_{ij} = 0, \tag{7}$$

$$\left\{A^{ij} - \frac{1}{N}\sum_{\alpha} (y^i_\alpha - x^i_\mu a^\mu)(y^j_\alpha - x^j_\nu a^\nu)\right\} x^i_\sigma x^j_\tau = 0,$$

$$A^{ij} = \sigma^2 \delta^{ij} + x^i_\mu A^{\mu\nu} x^j_\nu,$$

for determining the maximum likelihood estimates. The first of equations (7) is already solved for the a^μ, and the solution of the simultaneous equations for the remaining essential parameters yields the estimates

$$\hat{\sigma}^2 = \frac{1}{N(p - t)} \sum_{\alpha, i} (y^i_\alpha - x^i_\mu \hat{y}^\mu_\alpha)^2 \tag{8}$$

$$\hat{A}^{\mu\nu} = \frac{1}{N}\sum_{\alpha} (\hat{y}^\mu_\alpha - \hat{a}^\mu)(\hat{y}^\nu_\alpha - \hat{a}^\nu) - v^{\mu\nu}\hat{\sigma}^2. \tag{9}$$

A considerable amount of algebraic manipulation is required to put the solutions in the form given above; but since the results are about what one would expect in view of (5), we omit the details. As is often the case, some bias remains in the "optimum" estimates (9). However, this can be eliminated by writing $N - 1$ in place of N. The estimate (8) of σ^2 is unbiased as it stands.

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[3] Helen deSales Forrest, "Correlations between the constants in the curve of learning," *Studies in Psychology and Psychiatry*, the Catholic University of America, Vol. 5, no. 1 (1941).

CONFIDENCE LIMITS FOR AN UNKNOWN DISTRIBUTION FUNCTION

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Let $x_1, x_2, \cdots, x_n$ be mutually independent random variables following the same distribution law

$$P\{x_i \leq \xi\} = F(\xi). \tag{1}$$

A recent paper by A. Wald and J. Wolfowitz[1] deals with the problem of using

[1] A. Wald and J. Wolfowitz, "Confidence limits for distribution functions," *Annals of Math. Stat.*, Vol. 10 (1939), pp. 105–118.

the observable values of the x's to estimate the function $F(\xi)$. In this connection it may be useful to recall the following results published by me in 1933.[2]

Put

$$F_n(\xi) = \frac{N(\xi)}{n} \tag{2}$$

where $N(\xi)$ denotes the number of those x's whose observed values do not exceed ξ.

THEOREM 1: *If the function $F(\xi)$ is continuous then the distribution law of the quantities*

$$D_n = \sup | F(\xi) - F_n(\xi) | \sqrt{n} \tag{3}$$

does not depend on $F(\xi)$.

Denote by $\Phi_n(\lambda)$ the value of the probability $P\{D_n \leq \lambda\}$ which is common to all continuous distribution functions $F(\xi)$.

THEOREM 2: *For n tending to infinity, the distribution function $\Phi_n(\lambda)$ tends to*

$$\Phi(\lambda) = \sum_{k=-\infty}^{+\infty} (-1)^k e^{-2k^2\lambda^2} \tag{4}$$

uniformly with respect to λ.

A more elementary proof of Theorem 2 was given by N. Smirnoff in 1939.[3] Another paper by the same author[4] gives a table of the function $\Phi(\lambda)$.

Without the assumption that $F(\xi)$ is continuous, we easily obtain

THEOREM 3: *Whatever be the distribution function $F(\xi)$,*

$$P\{D_n \leq \lambda\} \geq \Phi_n(\lambda). \tag{5}$$

Theorems 1 and 3 giving the exact lower bound of the probability that $F_n(\xi)$ will satisfy the inequality

$$| F(\xi) - F_n(\xi) | \leq \frac{\lambda}{\sqrt{n}} \tag{6}$$

for all values of ξ, can be used to establish confidence limits for $F(\xi)$ corresponding to the confidence coefficient

$$\alpha = \Phi_n(\lambda). \tag{7}$$

These confidence limits will be free from any restriction concerning the nature of the function $F(\xi)$.

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[2] A. Kolmogoroff, "Sulla determinatione empirica di una legge di distributione," *Giornale dell'Istituto Italiano degli Attuari*, Vol. 4 (1933), pp. 83–91.

[3] N. Smirnoff, "Sur les écarts de la courbe de distribution empirique," *Recueil Math. de Moscou*, Vol. 6 (1939), pp. 3–26.

[4] N. Smirnoff, "On the estimation of the discrepancy between empirical curves of distribution for two independent samples," *Bulletin de l'Université de Moscou, Série internationale (Mathématiques)*, Vol. 2, fasc. 2 (1939).

For sufficiently large values of n we can use the limiting distribution (4) and write

$$\alpha = \Phi(\lambda). \tag{8}$$

The following short table, based on that of Smirnoff,[4] gives the values of λ corresponding to a few chosen confidence coefficients α.

TABLE OF λ

α	λ
.95	1.35
.98	1.52
.99	1.63
.995	1.73
.998	1.86
.999	1.95

Smirnoff's paper[4] contains still another application of the function $\Phi(\lambda)$. Denote by $x_1', x_2', \cdots, x_{n_1}'$ and $x_1'', x_2'', \cdots, x_{n_2}''$ two sequences of mutually independent random variables following the same probability law $F(\xi)$. Let further $F_{n_1}(\xi)$ and $F_{n_2}(\xi)$ be two random step functions corresponding to these series, defined as in (2). Smirnoff proves then the following

THEOREM 4: *If the probability law $F(\xi)$ is continuous, then the probability*

$$P\left\{\sup \mid F_{n_1}(\xi) - F_{n_2}(\xi) \mid \leq \lambda \sqrt{\frac{n_1 + n_2}{n_1 n_2}}\right\} = \Phi_{n_1, n_2}(\lambda) \tag{9}$$

is independent of the function $F(\xi)$. If n_1 and n_2 are indefinitely increased subject to the restriction that the ratio n_1/n_2 remains between two fixed numbers a_1 and a_2

$$0 < a_1 \leq \frac{n_1}{n_2} \leq a_2 < +\infty \tag{10}$$

then

$$\Phi_{n_1, n_2}(\lambda) \rightarrow \Phi(\lambda). \tag{11}$$

In the general case, where the probability law $F(\xi)$ is absolutely arbitrary we have

$$P\left\{\sup \mid F_{n_1}(\xi) - F_{n_2}(\xi) \mid \leq \lambda \sqrt{\frac{n_1 + n_2}{n_1 n_2}}\right\} \leq \Phi_{n_1, n_2}(\lambda). \tag{12}$$

Owing to the above results the quantity

$$D_{n_1, n_2} = \sup \mid F_{n_1}(\xi) - F_{n_2}(\xi) \mid \sqrt{\frac{n_1 n_2}{n_1 + n_2}} \tag{13}$$

could be used as a criterion to test the hypothesis that the probability laws of the two series of observable variables are actually the same.