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The slice spectral sequence for singular schemes and applications

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We examine the slice spectral sequence for the cohomology of singular schemes with respect to various motivic T-spectra, especially the motivic cobordism spectrum. When the base field k admits resolution of singularities and X is a scheme of finite type over k, we show that Voevodsky's slice filtration leads to a spectral sequence for MGL_X whose terms are the motivic cohomology groups of X defined using the cdh-hypercohomology. As a consequence, we establish an isomorphism between certain geometric parts of the motivic cobordism and motivic cohomology of X.

A similar spectral sequence for the connective K-theory leads to a cycle class map from the motivic cohomology to the homotopy invariant K-theory of X. We show that this cycle class map is injective for a large class of projective schemes. We also deduce applications to the torsion in the motivic cohomology of singular schemes.

1. Introduction

The motivic homotopy theory of schemes was put on a firm foundation by Voevodsky and his coauthors beginning with the work of Morel and Voevodsky [1999] and

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its stable counterpart [Voevodsky 1998]. It was observed by Voevodsky [2002b] that the motivic T-spectra in the stable homotopy category $\mathcal{SH}_X$ over a noetherian scheme X of finite Krull dimension can be understood via their slice filtration. This slice filtration leads to spectral sequences, which then become a very powerful tool in computing various cohomology theories for smooth schemes over X.

The main problem in the study of the slice filtration for a given motivic T-spectrum is twofold: the identification of its slices and the analysis of the convergence properties for the corresponding slice spectral sequence. When k is a field which admits resolution of singularities, the slices for many of these motivic T-spectra in $\mathcal{SH}_k$ are now known. In particular, we can compute these generalized cohomology groups of smooth schemes over k using the slice spectral sequence.

In this paper, we study a descent property of the motivic T-spectra in $\mathcal{SH}_X$ when X is a possibly singular scheme of finite type over k. This descent property tells us that the cohomology groups of a scheme $Y \in \mathbf{Sm}_X$, associated to an absolute motivic T-spectra in $\mathcal{SH}_X$ [Déglise 2014, §1.2], can be computed using only $\mathcal{SH}_k$.

Even though our methods apply to any of these absolute T-spectra, we restrict our study to the motivic cobordism spectrum MGL_X. We show using the above descent property of motivic spectra that MGL_X can be computed using the motivic cohomology groups of X. Recall from [Friedlander and Voevodsky 2000, Definitions 4.3 and 9.2] that the motivic cohomology groups of X are defined to be the cdh-hypercohomology groups $H^p(X, \mathbb{Z}(q)) = \mathbb{H}^{p-2q}_{\mathrm{cdh}}(X, C_* z_{\mathrm{equi}}(\mathbb{A}^q_k, 0)_{\mathrm{cdh}})$. Using these motivic cohomology groups, we show the following:

Theorem 1.1. *Let k be a field which admits resolution of singularities and let X be a separated scheme of finite type over k. Then for any integer $n \in \mathbb{Z}$, there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(n-q)) \otimes_{\mathbb{Z}} \mathbb{L}^q \Rightarrow \mathrm{MGL}^{p+q,n}(X), \tag{1.2}$$

and the differentials of this spectral sequence are given by $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$. Furthermore, this spectral sequence degenerates with rational coefficients.

If k is a perfect field of positive characteristic p, we obtain a similar spectral sequence after inverting p, except that we can not guarantee strong convergence unless X is smooth over k (see Remark 4.25).

As a consequence of Theorem 1.1 and its positive characteristic version, we get the following relation between the motivic cobordism and cohomology of singular schemes.

Theorem 1.3. *Let k be a field which admits resolution of singularities* (*resp. a perfect field of positive characteristic p*). *Then for any separated* (*resp. smooth*) *scheme X of finite type over k and dimension d and every $i \geq 0$, the edge map in* *the spectral sequence* (1.2)

$$\nu_X : \mathrm{MGL}^{2d+i,d+i}(X) \to H^{2d+i}(X, \mathbb{Z}(d+i))$$
$$\big(\textit{resp.}\quad \nu_X : \mathrm{MGL}^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]\big)$$

is an isomorphism.

We apply our descent result to obtain a similar spectral sequence for the connective KH-theory, KGL^0 (see Section 5). We use this spectral sequence and the canonical map $CKH(-) \to KH(-)$ from the connective KH-theory to obtain the following cycle class map from the motivic cohomology of a singular scheme to its homotopy invariant K-theory.

Theorem 1.4. *Let k be a field of exponential characteristic p and let X be a separated scheme of dimension d which is of finite type over k. Then the map $\mathrm{KGL}^0_X \to s_0 \, \mathrm{KGL}_X \cong H\mathbb{Z}$ induces, for every integer $i \geq 0$, an isomorphism*

$$CKH^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \xrightarrow{\cong} H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big].$$

In particular, there is a natural cycle class map

$$\mathrm{cyc}_i : H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to KH_i(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big].$$

We use this cycle class map and the Chern class maps from the homotopy invariant K-theory to the Deligne cohomology of schemes over $\mathbb{C}$ to construct intermediate Jacobians and Abel–Jacobi maps for the motivic cohomology of singular schemes over $\mathbb{C}$. More precisely, we prove the following. This generalizes intermediate Jacobians and Abel–Jacobi maps of Griffiths and the torsion theorem of Roitman for smooth schemes.

Theorem 1.5. *Let X be a projective scheme over $\mathbb{C}$ of dimension d. Assume that either $d \leq 2$ or X is regular in codimension one. Then there is a semiabelian variety $J^d(X)$ and an Abel–Jacobi map $\mathrm{AJ}^d_X : H^{2d}(X, \mathbb{Z}(d))_{\deg 0} \to J^d(X)$ which is surjective and whose restriction to the torsion subgroups is an isomorphism.*

In a related work, Kohrita [2017, Theorem 6.5] has constructed an Abel–Jacobi map for the Lichtenbaum motivic cohomology $H^{2d}_L(X, \mathbb{Z}(d))$ of singular schemes over $\mathbb{C}$ using a different technique. He has also proven a version of the Roitman torsion theorem for the Lichtenbaum motivic cohomology. The natural map $H^{2d}(X, \mathbb{Z}(d)) \to H^{2d}_L(X, \mathbb{Z}(d))$ is not an isomorphism in general if $d \geq 3$. Note also that the Roitman torsion theorem for $H^{2d}(X, \mathbb{Z}(d))$ is a priori a finer statement than that for the analogous Lichtenbaum cohomology.

Using Theorem 1.5, we prove the following property of the cycle class map of Theorem 1.4, which is our final result. The analogous result for smooth projective schemes was proven by Marc Levine [1987, Theorem 3.2]. More generally, Levine shows that a relative Chow group of 0-cycles on a normal projective scheme over $\mathbb{C}$ injects inside $K_0(X)$.

Theorem 1.6. *Let X be a projective scheme of dimension d over $\mathbb{C}$. Assume that either $d \leq 2$ or X is regular in codimension one. Then the cycle class map* $\mathrm{cyc}_0 : H^{2d}(X, \mathbb{Z}(d)) \to KH_0(X)$ *is injective.*

We end this section with the comment that our motivation behind this work was to exploit powerful tools of the motivic homotopy theory to study several questions about the motivic cohomology and K-theory of singular schemes which were previously known only for smooth schemes. We hope that the methods and techniques of our proofs can be advanced further to answer many other cohomological questions about singular schemes. We refer to [Krishna and Pelaez 2018] for more results based on the techniques of this text.

2. A descent theorem for motivic spectra

In this section, we set up our notation, discuss various model structures used in our proofs and show the Quillen adjunction property of many functors among these model structures. The main objective of this section is to prove a cdh-descent property of the motivic T-spectra; see Theorem 2.14.

2.1. ***Notations and preliminary results.*** Let k be a perfect field of exponential characteristic p; in some instances we require that the field k admits resolution of singularities [Voevodsky 2010, Definition 4.1]. We write $\mathbf{Sch}_k$ for the category of separated schemes of finite type over k and $\mathbf{Sm}_k$ for the full subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes over k. If $X \in \mathbf{Sch}_k$, let $\mathbf{Sm}_X$ denote the full subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes over X. We write $(\mathbf{Sm}_k)_{\mathrm{Nis}}$ (resp. $(\mathbf{Sm}_X)_{\mathrm{Nis}}$, $(\mathbf{Sch}_k)_{\mathrm{cdh}}$, $(\mathbf{Sch}_k)_{\mathrm{Nis}}$) for $\mathbf{Sm}_k$ equipped with the Nisnevich topology (resp. $\mathbf{Sm}_X$ equipped with the Nisnevich topology, $\mathbf{Sch}_k$ equipped with the cdh-topology, $\mathbf{Sch}_k$ equipped with the Nisnevich topology). The product $X \times_{\mathrm{Spec}\, k} Y$ is denoted by $X \times Y$.

Let $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$) be the category of pointed simplicial presheaves on $\mathbf{Sm}_k$ (resp. $\mathbf{Sm}_X$, $\mathbf{Sch}_k$) equipped with the motivic model structure described in [Isaksen 2005] considering the Nisnevich topology on $\mathbf{Sm}_k$ (resp. Nisnevich topology on $\mathbf{Sm}_X$, cdh-topology on $\mathbf{Sch}_k$) and the affine line $\mathbb{A}^1_k$ as an interval. A simplicial presheaf is often called a *motivic space*.

We define T in $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$) as the pointed simplicial presheaf represented by $S^1_s \wedge S^1_t$, where S^1_t is $\mathbb{A}^1_k \setminus \{0\}$ (resp. $\mathbb{A}^1_X \setminus \{0\}$, $\mathbb{A}^1_k \setminus \{0\}$) pointed by 1, and S^1_s denotes the simplicial circle. Given an arbitrary integer $r \geq 1$, let S^r_s denote the iterated smash product $S^1_s \wedge \cdots \wedge S^1_s$ of S^1_s with r factors, and S^r_t the iterated smash product $S^1_t \wedge \cdots \wedge S^1_t$ of S^1_t with r factors; $S^0_s = S^0_t$ is by definition equal to the pointed simplicial presheaf represented by the base scheme $\mathrm{Spec}\, k$ (resp. X, $\mathrm{Spec}\, k$).

Since T is cofibrant in $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\text{cdh}}$) we can apply freely the results in [Hovey 2001, §8]. Let $\text{Spt}(\mathcal{M})$ (resp. $\text{Spt}(\mathcal{M}_X)$, $\text{Spt}(\mathcal{M}_{\text{cdh}})$) denote the category of symmetric T-spectra on $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\text{cdh}}$) equipped with the motivic model structure defined in [Hovey 2001, Definition 8.7]. We write $\mathcal{SH}$ (resp. $\mathcal{SH}_X$, $\mathcal{SH}_{\text{cdh}}$) for the homotopy category of $\text{Spt}(\mathcal{M})$ (resp. $\text{Spt}(\mathcal{M}_X)$, $\text{Spt}(\mathcal{M}_{\text{cdh}})$), which is a tensor triangulated category. For any two integers $m, n \in \mathbb{Z}$, let $\Sigma^{m,n}$ denote the automorphism $\Sigma_s^{m-n} \circ \Sigma_t^n : \mathcal{SH} \to \mathcal{SH}$ (this also makes sense in $\mathcal{SH}_X$ and $\mathcal{SH}_{\text{cdh}}$). We write Σ_T^n for $\Sigma^{2n,n}$, and $E \wedge F$ for the smash product of $E, F \in \mathcal{SH}$ (resp. $\mathcal{SH}_X$, $\mathcal{SH}_{\text{cdh}}$).

Given a simplicial presheaf A, we write A_+ for the pointed simplicial presheaf obtained by adding a disjoint base point (isomorphic to the base scheme) to A. For any $B \in \mathcal{M}$, let $\Sigma_T^\infty(B)$ denote the object $(B, T \wedge B, \dots) \in \text{Spt}(\mathcal{M})$. This functor makes sense for objects in $\mathcal{M}_{\text{cdh}}$ and $\mathcal{M}_X$ as well.

If $F : \mathcal{A} \to \mathcal{B}$ is a functor with right adjoint $G : \mathcal{B} \to \mathcal{A}$, we say that $(F, G) : \mathcal{A} \to \mathcal{B}$ is an adjunction. We freely use the language of model and triangulated categories. We write Σ^1 for the suspension functor in a triangulated category, and Σ^n is the suspension (or desuspension in case $n < 0$) functor iterated n (or $-n$) times.

We use the following notation in all the categories under consideration: $*$ denotes the terminal object, and $\cong$ denotes that a map is an isomorphism or that a functor is an equivalence of categories.

2.2. *Change of site.* Let $X \in \mathbf{Sch}_k$ and let $v : X \to \operatorname{Spec} k$ denote the structure map. We write Pre_X and $\underline{\text{Pre}}_k$ for the categories of pointed simplicial presheaves on $\mathbf{Sm}_X$ and $\mathbf{Sch}_k$, respectively. If $X = \operatorname{Spec} k$, where k is the base field, we write Pre_k instead of Pre_X. These categories are equipped with the objectwise flasque model structure [Isaksen 2005, §3]. To recall this model structure, we consider a finite set I of monomorphisms $\{V_i \to U\}_{i \in I}$ for any $U \in \mathbf{Sm}_X$. The categorical union $\bigcup_{i \in I} V_i$ is the coequalizer of the diagram

$$\coprod_{i,j \in I} V_i \times_U V_j \rightrightarrows \coprod_{i \in I} V_i$$

formed in Pre_X. We denote by i_I the induced monomorphism $\bigcup_{i \in I} V_i \to U$. Note that $\varnothing \to U$ arises in this way. The pushout product of maps of i_I and a map between simplicial sets exists in Pre_X. In particular, we may form the sets

$$I_{\text{clo}}^{\text{sch}}(\mathbf{Sm}_X) = \{i_I \,\square\, (\partial \Delta^n \subset \Delta^n)_+\}_{I, n \geq 0},$$
$$J_{\text{clo}}^{\text{sch}}(\mathbf{Sm}_X) = \{i_I \,\square\, (\Lambda_i^n \subset \Delta^n)_+\}_{I, n \geq 0, 0 \leq i \leq n},$$

where I is a finite set of monomorphisms $\{V_i \to U\}_{i \in I}$ with $U \in \mathbf{Sm}_X$, and $i_I : \bigcup_{i \in I} V_i \to U$ is the induced monomorphism defined above.

A map between simplicial presheaves is called a closed objectwise fibration if it has the right lifting property with respect to $J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)$. A map $u : E \to F$ between simplicial presheaves is called a weak equivalence if $E(U) \to F(U)$ is a weak equivalence of simplicial sets for each $U \in \mathbf{Sm}_X$. A closed objectwise cofibration is a map having the left lifting property with respect to every trivial closed objectwise fibration. Note that this notion of weak equivalence, cofibrations and fibrations makes sense for simplicial presheaves in any category with finite products (e.g., $\mathbf{Sm}_k$, $\mathbf{Sch}_k$). It follows from [Isaksen 2005, Theorem 3.7] that the above notion of weak equivalence, cofibrations and fibrations forms a proper, simplicial and cellular model category structure on Pre_k, Pre_X and $\underline{\mathrm{Pre}}_k$. We call this the *objectwise flasque model structure*. Our reason for choosing this model structure is the following result.

Lemma 2.3 [Isaksen 2005, Lemma 6.2]. *If $V \to U$ is a monomorphism in $\mathbf{Sm}_k$ (resp. $\mathbf{Sm}_X$, $\mathbf{Sch}_k$), then U_+/V_+ is cofibrant in the flasque model structure on Pre_k (resp. Pre_X, $\underline{\mathrm{Pre}}_k$). In particular, $T^n \wedge U_+$ is cofibrant for any $n \geq 0$.*

It is clear that Pre_X and $\underline{\mathrm{Pre}}_k$ are cofibrantly generated model categories with generating cofibrations $I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)$ and $I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k)$ and generating trivial cofibrations $J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)$ and $J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k)$, respectively.

Let $\pi : (\mathbf{Sch}_k)_{\mathrm{cdh}} \to (\mathbf{Sm}_k)_{\mathrm{Nis}}$ be the continuous map of sites considered in [Voevodsky 2010, §4]. We write $(\pi^*, \pi_*) : \mathrm{Pre}_k \to \underline{\mathrm{Pre}}_k$ and $(v^*, v_*) : \mathrm{Pre}_k \to \mathrm{Pre}_X$ for the adjunctions induced by π and v, respectively.

We also consider the morphism of sites $\pi_X : (\mathbf{Sch}_k)_{\mathrm{cdh}} \to (\mathbf{Sm}_X)_{\mathrm{Nis}}$ and the corresponding adjunction $(\pi_X^*, \pi_{X*}) : \mathrm{Pre}_X \to \underline{\mathrm{Pre}}_k$. These adjunctions are related by the following lemma.

Lemma 2.4. *The following diagram commutes*:

$$\begin{tikzcd} \mathrm{Pre}_k \arrow[r, "\pi^*"] \arrow[dr, "v^*"'] & \underline{\mathrm{Pre}}_k \arrow[d, "\pi_{X*}"] \\ & \mathrm{Pre}_X \end{tikzcd}$$

Proof. We first notice that for every simplicial set K, $Y \in \mathbf{Sm}_k$ and $Z \in \mathbf{Sm}_X$, one has

$$\begin{aligned} \pi^*(K \otimes Y_+) &= K \otimes Y_+ \in \underline{\mathrm{Pre}}_k, \\ v^*(K \otimes Y_+) &= K \otimes (Y \times X)_+ \in \mathrm{Pre}_X, \end{aligned} \tag{2.5}$$

and

$$\pi_X^*(K \otimes Z_+) = K \otimes Z_+ \in \underline{\mathrm{Pre}}_k.$$

We observe that π^* and v^* commute with colimits since they are left adjoint, and that π_{X*} also commutes with colimits since it is a restriction functor. Hence, it suffices to show that for every simplicial set K and every $Y \in \mathbf{Sm}_k$, we have

$\pi_{X*}(\pi^*(K \otimes Y_+)) = v^*(K \otimes Y_+)$. Finally, a direct computation shows that

$$\pi_{X*}(K \otimes Y_+) = K \otimes (Y \times X)_+ \in \mathrm{Pre}_X$$

and we conclude by (2.5). □

Lemma 2.6. *The adjunctions* $(\pi^*, \pi_*) : \mathrm{Pre}_k \to \underline{\mathrm{Pre}}_k$, $(v^*, v_*) : \mathrm{Pre}_k \to \mathrm{Pre}_X$ *and* $(\pi_X^*, \pi_{X*}) : \mathrm{Pre}_X \to \underline{\mathrm{Pre}}_k$ *are all Quillen adjunctions. Moreover*, π_{X*} *and* π_* *preserve weak equivalences.*

Proof. We have seen above that all the three model categories (with the objectwise flasque model structure) are cofibrantly generated. Moreover, it follows from (2.5) that

$$\begin{aligned} \pi^*(I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_k)) &\subseteq I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k), & \pi^*(J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_k)) &\subseteq J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k), \\ v^*(I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_k)) &\subseteq I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X), & v^*(J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_k)) &\subseteq J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X), \\ \pi_X^*(I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)) &\subseteq I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k), & \pi_X^*(J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)) &\subseteq J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k). \end{aligned}$$

Hence, it follows from [Hovey 1999, Lemma 2.1.20] that (π^*, π_*), (v^*, v_*) and (π_X^*, π_{X*}) are Quillen adjunctions. The second part of the lemma is an immediate consequence of the fact that π_{X*} and π_* are restriction functors and the weak equivalences in the objectwise flasque model structure are defined schemewise. □

To show that the Quillen adjunction of Lemma 2.6 extends to the level of motivic model structures, we consider a distinguished square α [Voevodsky 2010, §2]

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd} \tag{2.7}$$

in $(\mathbf{Sm}_k)_{\mathrm{Nis}}$, $(\mathbf{Sm}_X)_{\mathrm{Nis}}$ or $(\mathbf{Sch}_k)_{\mathrm{cdh}}$, and write $P(\alpha)$ for the pushout of $Z \leftarrow Z' \to Y'$ in Pre_k, Pre_X or $\underline{\mathrm{Pre}}_k$, respectively.

The motivic model category $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$, $\mathcal{M}_{\mathrm{ft}}$) is the left Bousfield localization of Pre_k (resp. Pre_X, $\underline{\mathrm{Pre}}_k$, $\underline{\mathrm{Pre}}_k$) with respect to the following two sets of maps:

- $P(\alpha) \to Y$ indexed by the distinguished squares in $(\mathbf{Sm}_k)_{\mathrm{Nis}}$ (resp. $(\mathbf{Sm}_X)_{\mathrm{Nis}}$, $(\mathbf{Sch}_k)_{\mathrm{cdh}}$, $(\mathbf{Sch}_k)_{\mathrm{Nis}}$),
- $p_Y : Y \times \mathbb{A}^1_k \to Y$ for $Y \in \mathbf{Sm}_k$ (resp. $Y \in \mathbf{Sm}_X$, $Y \in \mathbf{Sch}_k$, $Y \in \mathbf{Sch}_k$).

Notice that as we are working with the flasque model structures, by [Isaksen 2005, Theorems 4.8–4.9] it is possible to consider maps from the ordinary pushout $P(\alpha)$ instead of maps from the homotopy pushout of the diagram $Z \leftarrow Z' \to Y'$ in (2.7).

Remark 2.8. We also consider the Nisnevich (resp. cdh) local model structure, i.e., the left Bousfield localization of Pre_k (resp. $\underline{\mathrm{Pre}}_k$) with respect to the set of maps $P(\alpha) \to Y$ indexed by the distinguished squares in $(\mathbf{Sm}_k)_{\mathrm{Nis}}$ (resp. $(\mathbf{Sch}_k)_{\mathrm{cdh}}$).

We abuse notation and write $(\pi^*, \pi_*) : \mathcal{M} \to \mathcal{M}_{\mathrm{cdh}}$, $(v^*, v_*) : \mathcal{M} \to \mathcal{M}_X$ and $(\pi_X^*, \pi_{X*}) : \mathcal{M}_X \to \mathcal{M}_{\mathrm{cdh}}$ for the adjunctions induced by π, v and π_X, respectively.

Proposition 2.9. *The adjunctions* $(\pi^*, \pi_*) : \mathcal{M} \to \mathcal{M}_{\mathrm{cdh}}$, $(v^*, v_*) : \mathcal{M} \to \mathcal{M}_X$ *and* $(\pi_X^*, \pi_{X*}) : \mathcal{M}_X \to \mathcal{M}_{\mathrm{cdh}}$ *are Quillen adjunctions.*

Proof. We give the argument for (π^*, π_*), since the other cases are parallel. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Pre}_k \arrow[r, "\pi^*"] \arrow[d, "\mathrm{id}"'] & \underline{\mathrm{Pre}}_k \arrow[d, "\mathrm{id}"] \\ \mathcal{M} \arrow[r, dashed, "\pi^*"] & \mathcal{M}_{\mathrm{cdh}} \end{tikzcd}$$

where the solid arrows are left Quillen functors by [Hirschhorn 2003, Lemma 3.3.4(1)] and Lemma 2.6. Thus, it follows from [Hirschhorn 2003, Definition 3.1.1(1)(b), Theorem 3.3.19] that it suffices to check that $\pi^*(P(\alpha) \to Y)$ and $\pi^*(Y \times \mathbb{A}_k^1 \to Y)$ are weak equivalences in $\mathcal{M}_{\mathrm{cdh}}$.

On the one hand, it is immediate that $\pi^*(Y \times \mathbb{A}_k^1 \to Y) = (Y \times \mathbb{A}_k^1 \to Y) \in \mathcal{M}_{\mathrm{cdh}}$, and is hence a weak equivalence in $\mathcal{M}_{\mathrm{cdh}}$. On the other hand, π^* commutes with pushouts since it is a left adjoint functor. It thus follows from (2.5) that

$$\pi^*(P(\alpha) \to Y) = (P(\alpha) \to Y) \in \mathcal{M}_{\mathrm{cdh}},$$

and is hence a weak equivalence in $\mathcal{M}_{\mathrm{cdh}}$. $\square$

We write $\mathcal{H}$ (resp. $\mathcal{H}_X$, $\mathcal{H}_{\mathrm{cdh}}$) for the homotopy category of $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$) and $(\boldsymbol{L}\pi^*, \boldsymbol{R}\pi_*) : \mathcal{H} \to \mathcal{H}_{\mathrm{cdh}}$, $(\boldsymbol{L}v^*, \boldsymbol{R}v_*) : \mathcal{H} \to \mathcal{H}_X$, $(\boldsymbol{L}\pi_X^*, \boldsymbol{R}\pi_{X*}) : \mathcal{H}_X \to \mathcal{H}_{\mathrm{cdh}}$ for the derived adjunctions of the Quillen adjunctions in Proposition 2.9; see [Hirschhorn 2003, Theorem 3.3.20].

2.10. ***A* cdh*-descent for motivic spectra.*** It follows from (2.5) that the adjunctions between the categories of motivic spaces induce levelwise adjunctions

$$(\pi^*, \pi_*) : \mathrm{Spt}(\mathcal{M}) \to \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}}),$$
$$(v^*, v_*) : \mathrm{Spt}(\mathcal{M}) \to \mathrm{Spt}(\mathcal{M}_X),$$
$$(\pi_X^*, \pi_{X*}) : \mathrm{Spt}(\mathcal{M}_X) \to \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$$

between the corresponding categories of symmetric T-spectra such that the following diagram commutes (see Lemma 2.4):

$$\begin{tikzcd} \mathrm{Spt}(\mathcal{M}) \arrow[r, "\pi^*"] \arrow[dr, "v^*"'] & \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}}) \arrow[d, "\pi_{X*}"] \\ & \mathrm{Spt}(\mathcal{M}_X) \end{tikzcd} \tag{2.11}$$

We further conclude from Proposition 2.9 and [Hovey 2001, Theorem 9.3] the following:

Proposition 2.12. *The pairs*

(1) $(\pi^*, \pi_*) : \mathrm{Spt}(\mathcal{M}) \to \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$,

(2) $(v^*, v_*) : \mathrm{Spt}(\mathcal{M}) \to \mathrm{Spt}(\mathcal{M}_X)$ *and*

(3) $(\pi_X^*, \pi_{X*}) : \mathrm{Spt}(\mathcal{M}_X) \to \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$

are Quillen adjunctions between stable model categories.

We deduce from Proposition 2.12 that there are pairs of adjoint functors

$$(\boldsymbol{L}\pi^*, \boldsymbol{R}\pi_*) : \mathcal{SH} \to \mathcal{SH}_{\mathrm{cdh}},$$
$$(\boldsymbol{L}v^*, \boldsymbol{R}v_*) : \mathcal{SH} \to \mathcal{SH}_X,$$
$$(\boldsymbol{L}\pi_X^*, \boldsymbol{R}\pi_{X*}) : \mathcal{SH}_X \to \mathcal{SH}_{\mathrm{cdh}}$$

between the various stable homotopy categories of motivic T-spectra. We observe that for $a \geq b \geq 0$, the suspension functor $\Sigma^{a,b}$ in $\mathcal{SH}$ (resp. $\mathcal{SH}_X$, $\mathcal{SH}_{\mathrm{cdh}}$) is the derived functor of the left Quillen functor $E \mapsto S_s^{a-b} \wedge S_t^b \wedge E$ in $\mathrm{Spt}(\mathcal{M})$ (resp. $\mathrm{Spt}(\mathcal{M}_X)$, $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$). Since the functors π^*, v^*, π_X^* are simplicial and symmetric monoidal, we deduce that they commute with the suspension functors $\Sigma^{m,n}$, i.e., for every $m, n \in \mathbb{Z}$,

$$\boldsymbol{L}\pi^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \boldsymbol{L}\pi^*(-),$$
$$\boldsymbol{L}v^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \boldsymbol{L}v^*(-),$$
$$\boldsymbol{L}\pi_X^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \boldsymbol{L}\pi_X^*(-).$$

Recall that $\mathcal{M}_{\mathrm{ft}}$ is the motivic category for the Nisnevich topology in $\mathbf{Sch}_k$. We write $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ for the category of symmetric T-spectra on $\mathcal{M}_{\mathrm{ft}}$ equipped with the stable model structure considered in [Hovey 2001, Definition 8.7].

It is well known [Jardine 2003, p. 198] that $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ and $\mathrm{Spt}(\mathcal{M}_X)$ (for $X \in \mathbf{Sch}_k$) are simplicial model categories [Hirschhorn 2003, Definition 9.1.6]. For E, E' in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ or $\mathrm{Spt}(\mathcal{M}_X)$, we write $\mathrm{Map}(E, E')$ and $\mathrm{Map}_X(E, E')$ for the simplicial set of maps from E to E', i.e., the simplicial set with n-simplices of the form $\mathrm{Hom}_{\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})}(E \otimes \Delta^n, E')$ or $\mathrm{Hom}_{\mathrm{Spt}(\mathcal{M}_X)}(E \otimes \Delta^n, E')$, respectively.

For $f : X \to X'$, note that the Quillen adjunction $(f^*, f_*) : \mathrm{Spt}(\mathcal{M}_{X'}) \to \mathrm{Spt}(\mathcal{M}_X)$ [Ayoub 2007b, Théorème 4.5.14] is enriched on simplicial sets, i.e., we have $\mathrm{Map}_X(f^*E', E) \cong \mathrm{Map}_{X'}(E', f_*E)$ for $E \in \mathrm{Spt}(\mathcal{M}_X)$, $E' \in \mathrm{Spt}(\mathcal{M}_{X'})$.

The following result is a direct consequence of the proper base change theorem in motivic homotopy theory [Ayoub 2007a, Corollaire 1.7.18; Cisinski and Déglise 2012, Proposition 2.3.11(2); Cisinski 2013, Proposition 3.7].

Proposition 2.13. *$\boldsymbol{L}v^*$ is naturally equivalent to the composition $\boldsymbol{R}\pi_{X*} \circ \boldsymbol{L}\pi^*$.*

Proof. We observe that the following diagram of left Quillen functors commutes:

$$\begin{tikzcd} \mathrm{Spt}(\mathcal{M}) \arrow[r, "\pi^*"] \arrow[dr, "\pi_{\mathrm{ft}}^*"'] & \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}}) \\ & \mathrm{Spt}(\mathcal{M}_{\mathrm{ft}}) \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

Let E be a motivic T-spectrum in $\mathrm{Spt}(\mathcal{M})$. Without any loss of generality, we can assume that E is cofibrant in $\mathrm{Spt}(\mathcal{M})$. Let $\nu : \pi_{\mathrm{ft}}^* E \to E'$ be a functorial fibrant replacement of $\pi_{\mathrm{ft}}^* E$ in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$.

The argument in [Jardine 2003, pp. 198–199] shows that the restriction functor π_{X*} maps weak equivalences in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ into weak equivalences in $\mathrm{Spt}(\mathcal{M}_X)$. Combining this with (2.11), we deduce that

$$\pi_{X*}(\nu) : \pi_{X*}(\pi_{\mathrm{ft}}^* E) = \pi_{X*}(\pi^* E) = v^* E \to \pi_{X*} E'$$

is a weak equivalence in $\mathrm{Spt}(\mathcal{M}_X)$. Since E is cofibrant in $\mathrm{Spt}(\mathcal{M})$, $\boldsymbol{L}v^* E \cong v^* E$. Hence, to conclude it suffices to show that E' is fibrant in $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$.

For the rest of the proof, for $Y \in \mathbf{Sch}_k$ we write $v_Y : Y \to \mathrm{Spec}\,(k)$ for the structure map. Notice that we have proved that $\boldsymbol{L}v_Y^* E \cong v_Y^* E \cong \pi_{Y*} E'$ in $\mathcal{SH}_Y$. Consider a distinguished abstract blow-up square in $\mathbf{Sch}_k$, i.e., a distinguished square in the lower cd-structure defined in [Voevodsky 2010, §2]:

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"'] & Y' \arrow[d, "f"] \\ Z \arrow[r, "i"'] & Y \end{tikzcd}$$

Let $j = i \circ f'$. Then

$$\boldsymbol{R} f_* \boldsymbol{L} f^* (\boldsymbol{L} v_Y^* E) \cong \boldsymbol{R} f_* \boldsymbol{L} (v_Y \circ f)^* E \cong \boldsymbol{R} f_* \pi_{Y'*} E' \cong f_* \pi_{Y'*} E'$$

in $\mathcal{SH}_Y$. In particular, the last isomorphism above follows from the fact that $\pi_{Y'*} E'$ is fibrant in $\mathrm{Spt}(\mathcal{M}_{Y'})$, since E' is fibrant in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ and the restriction functor $\pi_{Y'} : \mathrm{Spt}(\mathcal{M}_{\mathrm{ft}}) \to \mathrm{Spt}(\mathcal{M}_{Y'})$ is a right Quillen functor (using the same argument as in Proposition 2.12). Similarly, we conclude that $\boldsymbol{R} i_* \boldsymbol{L} i^* (\boldsymbol{L} v_Y^* E) \cong i_* \pi_{Z*} E'$ and $\boldsymbol{R} j_* \boldsymbol{L} j^* (\boldsymbol{L} v_Y^* E) \cong j_* \pi_{Z'*} E'$ in $\mathcal{SH}_Y$.

Thus, by [Cisinski 2013, Proposition 3.7] we conclude that the commutative diagram

$$\begin{tikzcd} \pi_{Y*} E' \arrow[r] \arrow[d] & f_* \pi_{Y'*} E' \arrow[d] \\ i_* \pi_{Z*} E' \arrow[r] & j_* \pi_{Z'*} E' \end{tikzcd}$$

is a homotopy cofiber square in $\mathrm{Spt}(\mathcal{M}_Y)$ [Hirschhorn 2003, Definition 13.5.8], and thus also a homotopy fiber square since $\mathrm{Spt}(\mathcal{M}_Y)$ is a stable model category, i.e., its homotopy category is triangulated. Since $\Sigma_T^\infty Y_+$ is cofibrant in $\mathrm{Spt}(\mathcal{M}_Y)$ and $\pi_{Y*}E'$, $f_*\pi_{Y'*}E'$, $i_*\pi_{Z*}E'$ and $j_*\pi_{Z'*}E'$ are fibrant, combining [Hirschhorn 2003, Definition 9.1.6(M7)] and [Hirschhorn 2003, Corollary 9.7.5(1)] we conclude that the induced commutative diagram is a homotopy fiber square of simplicial sets:

$$\begin{tikzcd}
\mathrm{Map}_Y(\Sigma_T^\infty Y_+, \pi_{Y*}E') \arrow[r] \arrow[d] & \mathrm{Map}_Y(\Sigma_T^\infty Y_+, f_*\pi_{Y'*}E') \arrow[d] \\
\mathrm{Map}_Y(\Sigma_T^\infty Y_+, i_*\pi_{Z*}E') \arrow[r] & \mathrm{Map}_Y(\Sigma_T^\infty Y_+, j_*\pi_{Z'*}E')
\end{tikzcd}$$

Since the adjunction (f^*, f_*) is enriched in simplicial sets, we conclude that

$$\mathrm{Map}_Y(\Sigma_T^\infty Y_+, f_*\pi_{Y'*}E') \cong \mathrm{Map}_{Y'}(f^*\Sigma_T^\infty Y_+, \pi_{Y'*}E') \cong \mathrm{Map}_{Y'}(\Sigma_T^\infty Y'_+, \pi_{Y'*}E')$$

and by definition $\mathrm{Map}_{Y'}(\Sigma_T^\infty Y'_+, \pi_{Y'*}E') \cong \mathrm{Map}(\Sigma_T^\infty Y'_+, E')$. Similarly, we conclude that

$$\mathrm{Map}_Y(\Sigma_T^\infty Y_+, \pi_{Y*}E') \cong \mathrm{Map}(\Sigma_T^\infty Y_+, E'),$$
$$\mathrm{Map}_Y(\Sigma_T^\infty Y_+, i_*\pi_{Z*}E') \cong \mathrm{Map}(\Sigma_T^\infty Z_+, E'),$$
$$\mathrm{Map}_Y(\Sigma_T^\infty Y_+, j_*\pi_{Z'*}E') \cong \mathrm{Map}(\Sigma_T^\infty Z'_+, E').$$

Therefore, the following is a homotopy fiber square of simplicial sets:

$$\begin{tikzcd}
\mathrm{Map}(\Sigma_T^\infty Y_+, E') \arrow[r] \arrow[d] & \mathrm{Map}(\Sigma_T^\infty Y'_+, E') \arrow[d] \\
\mathrm{Map}(\Sigma_T^\infty Z_+, E') \arrow[r] & \mathrm{Map}(\Sigma_T^\infty Z'_+, E')
\end{tikzcd}$$

Since $\Sigma_T^\infty Z'_+ \to \Sigma_T^\infty Y'_+$ is a cofibration in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ and E' is fibrant in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$, we deduce that $\mathrm{Map}(\Sigma_T^\infty Y'_+, E') \to \mathrm{Map}(\Sigma_T^\infty Z'_+, E')$ is a fibration of simplicial sets; see [Hirschhorn 2003, Definition 9.1.6(M7)]. We observe that the functor $\mathrm{Map}(-, E')$ maps pushout squares in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ into pullback squares of simplicial sets [Hirschhorn 2003, Proposition 9.1.8]; thus, by [Hirschhorn 2003, Corollary 13.3.8] we conclude that the map

$$\mathrm{Map}(\Sigma_T^\infty Y_+, E') \to \mathrm{Map}(\Sigma_T^\infty P(\alpha), E')$$

induced by $P(\alpha) \to Y$ is a weak equivalence of simplicial sets, where $P(\alpha)$ is the pushout of $Z \leftarrow Z' \to Y'$ in $\underline{\mathrm{Pre}}_k$. Finally, by [Hirschhorn 2003, Theorem 4.1.1(2)] we conclude that E' is fibrant in $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$, since by construction $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$ is the left Bousfield localization of $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ with respect to the maps of the form $\Sigma_T^\infty(P(\alpha) \to Y_+)$ indexed by the abstract blow-up squares in $\mathbf{Sch}_k$. □

The following result should be compared with [Cisinski 2013, Proposition 3.7].

Theorem 2.14. *Let $v : X \to \mathrm{Spec}\,(k)$ be in $\mathbf{Sch}_k$. Given a motivic T-spectrum $E \in \mathcal{SH}$, $Y \in \mathbf{Sm}_X$ and integers $m, n \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}v^* E) \cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}\pi^* E).$$

Proof. By Proposition 2.13, $\boldsymbol{L}v^*(-) \cong (\boldsymbol{R}\pi_{X*} \circ \boldsymbol{L}\pi^*)(-)$ in $\mathcal{SH}_X$. Thus, by adjointness,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}v^* E) &\cong \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \boldsymbol{L}v^*(\Sigma^{m,n} E)) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\boldsymbol{L}\pi_X^* \Sigma_T^\infty Y_+, \boldsymbol{L}\pi^*(\Sigma^{m,n} E)) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\boldsymbol{L}\pi_X^* \Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}\pi^* E).\end{aligned}$$

Finally, it follows from Lemma 2.3 that $\Sigma_T^\infty Y_+$ is cofibrant in the levelwise flasque model structure and hence in any of its localizations. In particular, it is cofibrant in the stable model structure of motivic T-spectra. We conclude that

$$\boldsymbol{L}\pi_X^* \Sigma_T^\infty Y_+ \cong \pi_X^* \Sigma_T^\infty Y_+ \cong \Sigma_T^\infty Y_+.$$

The corollary now follows. □

Remark 2.15. The above result could be called a cdh-descent theorem because it implies cdh-descent for many motivic spectra; see [Cisinski 2013, Proposition 3.7]. In particular, it implies cdh-descent for absolute motivic spectra (for example, KGL and MGL). Recall from [Déglise 2014, §1.2] that an absolute motivic spectrum E is a section of a 2-functor from $\mathbf{Sch}_k$ to triangulated categories such that for any $f : X' \to X$ in $\mathbf{Sch}_k$, the canonical map $f^* E_X \to E_{X'}$ is an isomorphism.

Lemma 2.16. *Let $f : Y \to X$ be a smooth morphism in $\mathbf{Sch}_k$. Let $v : X \to \mathrm{Spec}\,(k)$ be the structure map and $u = v \circ f$. Given any $E \in \mathcal{SH}$, the map*

$$\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \boldsymbol{L}v^* E) \to \mathrm{Hom}_{\mathcal{SH}_Y}(\Sigma_T^\infty Y_+, \boldsymbol{L}u^* E)$$

is an isomorphism.

Proof. The functor $\boldsymbol{L}f^* : \mathcal{SH}_X \to \mathcal{SH}_Y$ admits a left adjoint $\boldsymbol{L}f_\sharp : \mathcal{SH}_Y \to \mathcal{SH}_X$ by [Ayoub 2007b, Proposition 4.5.19]; see also [Ayoub 2007a, Scholium 1.4.2]. Since $f : Y \to X$ is smooth, we have $\boldsymbol{L}f_\sharp(\Sigma_T^\infty Y_+) = \Sigma_T^\infty Y_+$ by [Morel and Voevodsky 1999, Proposition 3.1.23(1)] and we get

$$\begin{aligned}\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \boldsymbol{L}v^* E) &\cong \mathrm{Hom}_{\mathcal{SH}_X}(\boldsymbol{L}f_\sharp(\Sigma_T^\infty Y_+), \boldsymbol{L}v^* E) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_Y}(\Sigma_T^\infty Y_+, \boldsymbol{L}f^* \circ \boldsymbol{L}v^* E) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_Y}(\Sigma_T^\infty Y_+, \boldsymbol{L}u^* E),\end{aligned}$$

and the lemma follows. □

A combination of Lemma 2.16 and Theorem 2.14 yields the following corollary:

Corollary 2.17. *Under the same hypotheses and notation of Theorem 2.14, assume in addition that $X \in \mathbf{Sm}_k$. Then there are natural isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{SH}}(\Sigma_T^\infty Y_+, \Sigma^{m,n} E) &\cong \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}v^* E) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}\pi^* E).\end{aligned}$$

3. Motivic cohomology of singular schemes

We continue to assume that k is a perfect field of exponential characteristic p. In this section, we show that the motivic cohomology of a scheme $X \in \mathbf{Sch}_k$, defined in terms of a cdh-hypercohomology (see Definition 3.1), is representable in the stable homotopy category $\mathcal{SH}_{\mathrm{cdh}}$.

Recall from [Mazza et al. 2006, Lecture 16] that given $T \in \mathbf{Sch}_k$ and an integer $r \geq 0$, the presheaf $z_{\mathrm{equi}}(T, r)$ on $\mathbf{Sm}_k$ is defined by letting $z_{\mathrm{equi}}(T, r)(U)$ be the free abelian group generated by the closed and irreducible subschemes $Z \subsetneq U \times T$ which are dominant and equidimensional of relative dimension r (any fiber is either empty or all its components have dimension r) over a component of U. It is known that $z_{\mathrm{equi}}(T, r)$ is a sheaf on the big étale site of $\mathbf{Sm}_k$.

Let $\underline{C}_* z_{\mathrm{equi}}(T, r)$ denote the chain complex of presheaves of abelian groups associated via the Dold–Kan correspondence to the simplicial presheaf on $\mathbf{Sm}_k$ given by $\underline{C}_n z_{\mathrm{equi}}(T, r)(U) = z_{\mathrm{equi}}(T, r)(U \times \Delta_k^n)$. The simplicial structure on $\underline{C}_* z_{\mathrm{equi}}(T, r)$ is induced by the cosimplicial scheme $\Delta_k^\bullet$. Recall the following definition of motivic cohomology of singular schemes from [Friedlander and Voevodsky 2000, Definition 9.2].

Definition 3.1. The motivic cohomology groups of $X \in \mathbf{Sch}_k$ are defined as the hypercohomology

$$H^m(X, \mathbb{Z}(n)) = \mathbb{H}_{\mathrm{cdh}}^{m-2n}(X, \underline{C}_* z_{\mathrm{equi}}(\mathbb{A}_k^n, 0)_{\mathrm{cdh}}) = A_{0,2n-m}(X, \mathbb{A}^n).$$

We also need to consider $\mathbb{Z}[1/p]$-coefficients. In this case, we write

$$H^m\big(X, \mathbb{Z}\big[\tfrac{1}{p}\big](n)\big) = \mathbb{H}_{\mathrm{cdh}}^{m-2n}\big(X, \underline{C}_* z_{\mathrm{equi}}(\mathbb{A}_k^n, 0)\big[\tfrac{1}{p}\big]\big).$$

For $n < 0$, we set $H^m(X, \mathbb{Z}(n)) = H^m(X, \mathbb{Z}[1/p](n)) = 0$.

3.2. ***The motivic cohomology spectrum.*** In order to represent the motivic cohomology of a singular scheme X in $\mathcal{SH}_X$, let us recall the Eilenberg–MacLane spectrum

$$H\mathbb{Z} = (K(0, 0), K(1, 2), \dots, K(n, 2n), \dots)$$

in $\mathrm{Spt}(\mathcal{M})$, where $K(n, 2n)$ is the presheaf of simplicial abelian groups on $\mathbf{Sm}_k$ associated to the presheaf of chain complexes $\underline{C}_* z_{\mathrm{equi}}(\mathbb{A}_k^n, 0)$ via the Dold–Kan correspondence. The external product of cycles induces product maps

$$K(m, 2m) \wedge K(n, 2n) \to K(m+n, 2(m+n)).$$

Notice $K(1, 2) \cong \underline{C}_*(z_{\mathrm{equi}}(\mathbb{P}^1_k, 0)/z_{\mathrm{equi}}(\mathbb{P}^0_k, 0))$ [Mazza et al. 2006, Theorem 16.8], so composing the product maps with the canonical map

$$g : T \cong \mathbb{P}^1_k/\mathbb{P}^0_k \to \underline{C}_*(z_{\mathrm{equi}}(\mathbb{P}^1_k, 0)/z_{\mathrm{equi}}(\mathbb{P}^0_k, 0)) \cong K(1, 2)$$

(where the first map assigns to any morphism $U \to \mathbb{P}^1_k$ its graph in $U \times \mathbb{P}^1_k$), we obtain the bonding maps. $H\mathbb{Z}$ is a symmetric spectrum whose symmetric structure is obtained by permuting the coordinates in $\mathbb{A}^n_k$. We shall not distinguish between a simplicial abelian group and the associated chain complex of abelian groups from now on in this text and will use them interchangeably.

3.3. ***Motivic cohomology via $\mathcal{SH}_{\mathrm{cdh}}$.*** Let $\mathbf{1} = \Sigma^\infty_T(S^0_s)$ be the sphere spectrum in $\mathcal{SH}$, and let $\mathbf{1}[1/p] \in \mathcal{SH}$ be the homotopy colimit [Neeman 2001, Definition 1.6.4] of the filtering diagram in $\mathcal{SH}$:

$$\mathbf{1} \xrightarrow{p} \mathbf{1} \xrightarrow{p} \mathbf{1} \xrightarrow{p} \cdots$$

where $\mathbf{1} \xrightarrow{r} \mathbf{1}$ is the composition of the sum map with the diagonal $\mathbf{1} \xrightarrow{\Delta} \bigoplus_{i=1}^r \mathbf{1} \xrightarrow{\Sigma} \mathbf{1}$. For $E \in \mathcal{SH}$, we define $E[1/p] \in \mathcal{SH}$ to be $E \wedge \mathbf{1}[1/p]$. This also makes sense in $\mathcal{SH}_X$ and $\mathcal{SH}_{\mathrm{cdh}}$.

The following is a reformulation of the main result in [Friedlander and Voevodsky 2000] when k admits resolution of singularities, and the main result in [Kelly 2012] when k has positive characteristic.

Theorem 3.4 [Cisinski and Déglise 2015]. *Let k be a perfect field of exponential characteristic p, and let $v : X \to \operatorname{Spec}(k)$ be a separated scheme of finite type. Then for any $m, n \in \mathbb{Z}$, there is a natural isomorphism*

$$\theta_X : H^m\big(X, \mathbb{Z}\big[\tfrac{1}{p}\big](n)\big) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{SH}_X}\big(\Sigma^\infty_T X_+, \Sigma^{m,n} \boldsymbol{L}v^* H\mathbb{Z}\big[\tfrac{1}{p}\big]\big). \tag{3.5}$$

Proof. Recall that $H^m(X, \mathbb{Z}[1/p](n)) = A_{0,2n-m}(X, \mathbb{A}^n)$ (Definition 3.1). We observe that $\underline{C}_* z_{\mathrm{equi}}(\mathbb{A}^n_k, 0)$ is the motive with compact supports $M^c(\mathbb{A}^n_k)$ of $\mathbb{A}^n_k$ [Voevodsky 2000, §4.1; Mazza et al. 2006, Definition 16.13]. Combining [Voevodsky 2000, Corollary 4.1.8] (or [Mazza et al. 2006, Theorem 16.7, Example 16.14]) with [Cisinski and Déglise 2015, 4.2, Proposition 4.3, Theorem 5.1 and Corollary 8.6], we conclude that

$$H^m\big(X, \mathbb{Z}\big[\tfrac{1}{p}\big](n)\big) \cong \operatorname{Hom}_{\mathcal{SH}_X}\big(\Sigma^{2n-m,0}(\Sigma^\infty_T X_+), \Sigma^{2n,n} \boldsymbol{L}v^* H\mathbb{Z}\big[\tfrac{1}{p}\big]\big),$$

which finishes the proof. □

As a combination of Theorem 2.14 and Theorem 3.4, we get a corollary:

Corollary 3.6. *Under the hypothesis and with the notation of Theorem 3.4, there are natural isomorphisms*

$$\begin{aligned} H^m\big(X, \mathbb{Z}[\tfrac{1}{p}](n)\big) &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma_T^\infty X_+, \Sigma^{m,n} \boldsymbol{L}\pi^* H\mathbb{Z}[\tfrac{1}{p}]\big) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_X}\big(\Sigma_T^\infty X_+, \Sigma^{m,n} \boldsymbol{L}v^* H\mathbb{Z}[\tfrac{1}{p}]\big). \end{aligned}$$

4. Slice spectral sequence for singular schemes

Let k be a perfect field of exponential characteristic p. Given $X \in \mathbf{Sch}_k$, recall that Voevodsky's slice filtration of $\mathcal{SH}_X$ is given as follows. For an integer $q \in \mathbb{Z}$, let $\Sigma_T^q \mathcal{SH}_X^{\mathrm{eff}}$ denote the smallest full triangulated subcategory of $\mathcal{SH}_X$ which contains C_{eff}^q and is closed under arbitrary coproducts, where

$$C_{\mathrm{eff}}^q = \{\Sigma^{m,n}\Sigma_T^\infty Y_+ : m, n \in \mathbb{Z},\ n \geq q,\ Y \in \mathbf{Sm}_X\}. \tag{4.1}$$

In particular, $\mathcal{SH}_X^{\mathrm{eff}}$ is the smallest full triangulated subcategory of $\mathcal{SH}_X$ which is closed under infinite direct sums and contains all spectra of the type $\Sigma_T^\infty Y_+$ with $Y \in \mathbf{Sm}_X$. The slice filtration of $\mathcal{SH}_X$ [Voevodsky 2002b] is the sequence of full triangulated subcategories

$$\cdots \subseteq \Sigma_T^{q+1}\mathcal{SH}_X^{\mathrm{eff}} \subseteq \Sigma_T^{q}\mathcal{SH}_X^{\mathrm{eff}} \subseteq \Sigma_T^{q-1}\mathcal{SH}_X^{\mathrm{eff}} \subseteq \cdots.$$

It follows from [Neeman 1996; 2001] that the inclusion $i_q : \Sigma_T^q \mathcal{SH}_X^{\mathrm{eff}} \to \mathcal{SH}_X$ admits a right adjoint $r_q : \mathcal{SH}_X \to \Sigma_T^q \mathcal{SH}_X^{\mathrm{eff}}$ and the functors $f_q, s_{<q}, s_q : \mathcal{SH}_X \to \mathcal{SH}_X$ are triangulated, where $r_q \circ i_q$ is the identity, $f_q = i_q \circ r_q$ and $s_{<q}, s_q$ are characterized by the existence of the distinguished triangles

$$\begin{aligned} f_q E &\longrightarrow E \longrightarrow s_{<q} E, \\ f_{q+1} E &\longrightarrow f_q E \longrightarrow s_q E \end{aligned} \tag{4.2}$$

in $\mathcal{SH}_X$ for every $E \in \mathcal{SH}_X$.

Definition 4.3. Let $a, b, n \in \mathbb{Z}$ and $Y \in \mathbf{Sm}_X$. Let $F^n E^{a,b}(Y)$ be the image of the map induced by $f_n E \to E$ in (4.2):

$$\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b} f_n E) \to \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b} E).$$

This determines a decreasing filtration $F^\bullet$ on $E^{a,b}(Y) = \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b} E)$, and we write $\mathrm{gr}^n F^\bullet$ for the associated graded $F^n E^{a,b}(Y)/F^{n+1}E^{a,b}(Y)$.

The following result is well known; see [Voevodsky 2002b, §2].

Proposition 4.4. *The filtration $F^\bullet$ on $E^{a,b}(Y)$ is exhaustive* (in the sense of [Boardman 1999, Definition 2.1]).

Proof. Recall that $\mathcal{SH}_X$ is a compactly generated triangulated category in the sense of [Neeman 1996, Definition 1.7], with set of compact generators [Ayoub 2007b, Théorème 4.5.67] $\bigcup_{q\in\mathbb{Z}} C^q_{\mathrm{eff}}$ (see (4.1)). Therefore a map $f : E_1 \to E_2$ in $\mathcal{SH}_X$ is an isomorphism if and only if for every $Y \in \mathbf{Sm}_X$ and every $m, n \in \mathbb{Z}$ the induced map of abelian groups $\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n} Y_+, E_1) \to \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n} Y_+, E_2)$ is an isomorphism. Thus, we conclude that $E \cong \mathrm{hocolim}\, f_q E$ in $\mathcal{SH}_X$.

Therefore, we deduce that for every $a, b \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, there exist the isomorphisms

$$\begin{aligned}\operatorname*{colim}_{n\to-\infty} F^n E^{a,b}(Y) &\cong \operatorname*{colim}_{n\to-\infty} \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma^\infty_T Y_+, \Sigma^{a,b} f_n E)\\ &\cong \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma^\infty_T Y_+, \Sigma^{a,b}\, \mathrm{hocolim}\, f_q E) \cong E^{a,b}(Y)\end{aligned}$$

[Neeman 1996, Lemma 2.8; Isaksen 2005, Theorem 6.8], so the filtration $F^\bullet$ is exhaustive. □

4.5. *The slice spectral sequence.* Consider $Y \in \mathbf{Sm}_X$ a smooth X-scheme and $G \in \mathcal{SH}_X$. Since $\mathcal{SH}_X$ is a triangulated category, the collection of distinguished triangles $\{f_{q+1}G \to f_q G \to s_q G\}_{q\in\mathbb{Z}}$ determines a (slice) spectral sequence

$$E_1^{p,q} = \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma^\infty_T Y_+, \Sigma^{p+q}_s s_p G)$$

with $G^{*,*}(Y)$ as its abutment and differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

In order to study the convergence of this spectral sequence, recall from [Voevodsky 2002b, p. 22] that $G \in \mathcal{SH}_X$ is called *bounded* with respect to the slice filtration if for every $m, n \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, there exists $q \in \mathbb{Z}$ such that

$$\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n}\Sigma^\infty_T Y_+, f_{q+i}G) = 0 \tag{4.6}$$

for every $i > 0$. Clearly the slice spectral sequence is strongly convergent when G is bounded.

Proposition 4.7. *Let k be a field with resolution of singularities. Let $F \in \mathcal{SH}$ be bounded with respect to the slice filtration and let $G = \boldsymbol{L}v^* F \in \mathcal{SH}_X$ with $v : X \to \mathrm{Spec}\, k$. Then G is bounded with respect to the slice filtration.*

Proof. Since the base field k admits resolution of singularities, we deduce by [Pelaez 2013, Theorem 3.7] that $f_q G \cong \boldsymbol{L}v^* f_q F$ in $\mathcal{SH}_X$ for every $q \in \mathbb{Z}$. It follows from Theorem 2.14 that for every $m, n \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, we have

$$\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n}\Sigma^\infty_T Y_+, f_{q+i}G) \cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma^\infty_T Y_+, \boldsymbol{L}\pi^*(f_{q+i}F))$$

for every $i > 0$. If $X \in \mathbf{Sm}_k$, then $Y \in \mathbf{Sm}_k$ and we have

$$\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma^\infty_T Y_+, \boldsymbol{L}\pi^*(f_{q+i}F)) \cong \mathrm{Hom}_{\mathcal{SH}}(\Sigma^{m,n}\Sigma^\infty_T Y_+, f_{q+i}F)$$

for every $i > 0$ by Corollary 2.17. Since F is bounded with respect to the slice filtration, we deduce from (4.6) that G is also bounded in $\mathcal{SH}_X$ in this case.

Finally, we proceed by induction on the dimension of Y, and assume that for every $m, n \in \mathbb{Z}$ and every $Y' \in \mathbf{Sch}_k$ with $\dim(Y') < \dim(Y)$, there exists $q \in \mathbb{Z}$ such that

$$\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma_T^\infty Y'_+, \boldsymbol{L}\pi^*(f_{q+i}F)) = 0$$

for every $i > 0$. Since the base field k admits resolution of singularities, there exists a cdh-cover $\{X' \amalg Z \to Y\}$ of Y such that $X' \in \mathbf{Sm}_k$, $\dim(Z) < \dim(Y)$ and $\dim(W) < \dim(Y)$, where we set $W = X' \times_Y Z$.

Let q_1, q_2 and q_3 be the integers such that the vanishing condition (4.6) holds for (X', m, n), (Z, m, n) and $(W, m+1, n)$, respectively. Let q be the maximum of q_1, q_2 and q_3. Then by cdh-excision, for every $i > 0$, the following diagram is exact:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m+1,n}\Sigma_T^\infty W, \boldsymbol{L}\pi^*(f_{q+i}F)) \\
&\qquad\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma_T^\infty Y_+, \boldsymbol{L}\pi^*(f_{q+i}F)) \\
&\qquad\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma_T^\infty X'_+, \boldsymbol{L}\pi^*(f_{q+i}F)) \\
&\qquad\qquad\oplus \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma_T^\infty Z_+, \boldsymbol{L}\pi^*(f_{q+i}F)).
\end{aligned}$$

By choice of q, both ends in the diagram vanish. Hence the group in the middle also vanishes as we wanted. □

In order to get convergence results in positive characteristic, we need to restrict to spectra $E \in \mathcal{SH}$ which admit a structure of traces [Kelly 2012, Definitions 4.2.27 and 4.3.1].

Lemma 4.8. *With the notation of (2.11), let $X \in \mathbf{Sch}_k$.*

(1) *For every $E \in \mathcal{SH}$, $\boldsymbol{L}\pi^*\big(E\big[\frac{1}{p}\big]\big) \cong (\boldsymbol{L}\pi^*E)\big[\frac{1}{p}\big]$ and $\boldsymbol{L}v^*\big(E\big[\frac{1}{p}\big]\big) \cong (\boldsymbol{L}v^*E)\big[\frac{1}{p}\big]$.*

(2) *For every $E \in \mathcal{SH}_{\mathrm{cdh}}$ and every $a, b \in \mathbb{Z}$,*

$$\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{a,b}\Sigma_T^\infty(X_+), E\big[\tfrac{1}{p}\big]\big) \cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{a,b}\Sigma_T^\infty(X_+), E) \otimes \mathbb{Z}\big[\tfrac{1}{p}\big].$$

Proof. (1): It follows from the definition of homotopy colimit [Neeman 2001, Definition 1.6.4] that $\boldsymbol{L}\pi^*$ and $\boldsymbol{L}v^*$ commute with homotopy colimits since they are left adjoint. This implies the result since $E[1/p]$ is given in terms of homotopy colimits.

(2): Since $\Sigma^{a,b}\Sigma_T^\infty(X_+)$ is compact in $\mathcal{SH}_{\mathrm{cdh}}$ [Ayoub 2007b, Théorème 4.5.67], the result follows from [Neeman 1996, Lemma 2.8]. □

Lemma 4.9. *Let $X \in \mathbf{Sch}_k$ and $E \in \mathcal{SH}_X$. Then for every $r \in \mathbb{Z}$,*

$$f_r\big(E\big[\tfrac{1}{p}\big]\big) \cong (f_rE)\big[\tfrac{1}{p}\big] \quad \textit{and} \quad s_r\big(E\big[\tfrac{1}{p}\big]\big) \cong (s_rE)\big[\tfrac{1}{p}\big].$$

Proof. Since the effective categories $\Sigma_T^q \mathcal{SH}_X^{\mathrm{eff}}$ are closed under infinite direct sums, we conclude that the functors f_r, s_r commute with homotopy colimits. $\square$

Proposition 4.10. *Let $F \in \mathcal{SH}$ and $G = \boldsymbol{L}v^* F \in \mathcal{SH}_X$ with $v : X \to \operatorname{Spec} k$. Assume that for every $r \in \mathbb{Z}$, $s_r(F[1/p])$ has a weak structure of smooth traces* (in the sense of [Kelly 2012, Definition 4.2.27]), *and that $F[1/p]$ has a structure of traces* (in the sense of [Kelly 2012, Definition 4.3.1]). *If $F[1/p]$ is bounded with respect to the slice filtration, then $G[1/p]$ is bounded as well.*

Proof. Since the base field k is perfect and $F[1/p]$ is clearly $\mathbb{Z}[1/p]$-local, combining [Kelly 2012, Theorem 4.2.29] and Lemma 4.9, we conclude that $f_q G[1/p] \cong \boldsymbol{L}v^* f_q F[1/p]$ in $\mathcal{SH}_X$ for every $q \in \mathbb{Z}$.

It follows from Theorem 2.14 that for every $m, n \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, we have

$$\operatorname{Hom}_{\mathcal{SH}_X}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+),\, f_{q+i}G\big[\tfrac{1}{p}\big]\big) \cong \operatorname{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+),\, \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)$$

for every $i > 0$. If $X \in \mathbf{Sm}_k$, then $Y \in \mathbf{Sm}_k$ and we have

$$\operatorname{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+),\, \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big) \cong \operatorname{Hom}_{\mathcal{SH}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+),\, f_{q+i}F\big[\tfrac{1}{p}\big]\big)$$

for every $i > 0$ by Corollary 2.17. Since $F[1/p]$ is bounded with respect to the slice filtration, we deduce from (4.6) that $G[1/p]$ is also bounded with respect to the slice filtration in $\mathcal{SH}_X$ in this case.

Finally, we proceed by induction on the dimension of Y, and assume that for every $m, n \in \mathbb{Z}$ and every $Z \in \mathbf{Sch}_k$ with $\dim_k(Z) < \dim_k(Y)$, there exists $q \in \mathbb{Z}$ such that

$$\operatorname{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Z_+),\, \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big) = 0$$

for every $i > 0$.

Since k is perfect, by a theorem of Gabber [Illusie et al. 2014, Théorème 3(1)] and Temkin's strengthening [2017, Theorem 1.2.9] of Gabber's result, there exists $W \in \mathbf{Sm}_k$ and a surjective proper map $h : W \to Y$, which is generically étale of degree p^r, $r \geq 1$. In particular, h is generically flat, and thus by a theorem of Raynaud and Gruson [1971, Théorème 5.2.2], there exists a blow-up $g : Y' \to Y$ with center Z such that the following diagram commutes, where h' is finite flat surjective of degree p^r and $g' : W' \to W$ is the blow-up of W with center $h^{-1}(Z)$:

$$\begin{tikzcd} W' \arrow[r, "h'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ W \arrow[r, "h"'] & Y \end{tikzcd} \tag{4.11}$$

Thus we have a cdh-cover $\{Y' \amalg Z \to Y\}$ of Y such that $\dim_k(Z) < \dim_k(Y)$ and $\dim_k(E) < \dim_k(Y)$, where we set $E = Y' \times_Y Z$.

Let q_1 (resp. q_2, q_3) be the integers such that the vanishing condition (4.6) holds for (W, m, n) (resp. (Z, m, n), $(E, m+1, n)$). Let q be the maximum of q_1, q_2 and q_3. Then by cdh-excision, for every $i > 0$, the following diagram is exact:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m+1,n}\Sigma^\infty_T(E_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\
&\qquad\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma^\infty_T(Y_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\
&\qquad\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma^\infty_T(Y'_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\
&\qquad\qquad\oplus \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma^\infty_T(Z_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big).
\end{aligned}$$

By the choice of q, this reduces to the exact diagram

$$\begin{aligned}
0 &\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma^\infty_T(Y_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\
&\qquad\xrightarrow{g^*} \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma^\infty_T(Y'_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big).
\end{aligned}$$

So it suffices to show that $g^* = 0$. In order to prove this, we observe that the diagram (4.11) commutes. Therefore, by the choice of q,

$$\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma^\infty_T(W_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big) = 0,$$

and we conclude that $h'^* \circ g^* = g'^* \circ h^* = 0$. Thus, it is enough to see that

$$\begin{aligned}
h'^* &: \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma^\infty_T(Y'_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\
&\qquad\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma^\infty_T(W'_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)
\end{aligned}$$

is injective. Let $v' : Y' \to \operatorname{Spec} k$, and let

$$\epsilon : \boldsymbol{L}v'^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big) \to \boldsymbol{R}h'_*\boldsymbol{L}h'^*\boldsymbol{L}v'^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)$$

be the map given by the unit of the adjunction $(\boldsymbol{L}h'^*, \boldsymbol{R}h'_*)$. By the naturality of the isomorphism in Proposition 2.13 we deduce that h'^* gets identified with the map induced by ϵ:

$$\begin{aligned}
\epsilon_* &: \mathrm{Hom}_{\mathcal{SH}_{Y'}}\big(\Sigma^{m,n}\Sigma^\infty_T Y'_+, \boldsymbol{L}v'^* f_{q+i}F\big[\tfrac{1}{p}\big]\big)\\
&\qquad\to \mathrm{Hom}_{\mathcal{SH}_{Y'}}\big(\Sigma^{m,n}\Sigma^\infty_T Y'_+, \boldsymbol{R}h'_*\boldsymbol{L}h'^*\boldsymbol{L}v'^* f_{q+i}F\big[\tfrac{1}{p}\big]\big).
\end{aligned}$$

Since $F[1/p]$ has a structure of traces and $s_r(F[1/p])$ has a weak structure of smooth traces for every $r \in \mathbb{Z}$, it follows from [Kelly 2012, Proposition 4.3.7] that $f_{q+i}(F[1/p])$ has a structure of traces in the sense of [Kelly 2012, Definition 4.3.1]. Thus, we deduce from [Kelly 2012, Definition 4.3.1(Deg), p. 101] that ϵ_* is injective, since h' is finite flat surjective of degree p^r. This finishes the proof. □

If we only assume that the slices $s_r E$ have a structure of traces, then we get the weaker conditions of Proposition 4.15.

Corollary 4.12. *Let $F \in \mathcal{SH}$ and $G = \boldsymbol{L}v^* F \in \mathcal{SH}_X$, where $v : X \to \operatorname{Spec} k$ is the structure map. Assume that the following hold.*

(1) *For every $r \in \mathbb{Z}$, $s_r(F[1/p])$ has a structure of traces* (in the sense of [Kelly 2012, Definition 4.3.1]).

(2) *$F[1/p]$ is bounded with respect to the slice filtration.*

Then for every m, n in $\mathbb{Z}$, there exists $q \in \mathbb{Z}$ such that

$$\operatorname{Hom}_{\mathcal{SH}_X}\big(\Sigma^{m,n}\Sigma_T^\infty X_+, s_{q+i}G\big[\tfrac{1}{p}\big]\big) = 0$$

for every $i > 0$ (see (4.6)).

Proof. Since $s_r(F[1/p])$ has a structure of traces, we observe that in particular $s_r(F[1/p])$ has a weak structure of smooth traces [Kelly 2012, Definition 4.2.27]. Thus, combining Lemma 4.8, Lemma 4.9 and [Kelly 2012, Theorem 4.2.29] we conclude that for every $r \in \mathbb{Z}$,

$$s_r G\big[\tfrac{1}{p}\big] \cong \boldsymbol{L}v^* s_r F\big[\tfrac{1}{p}\big] \quad \text{and} \quad f_r G\big[\tfrac{1}{p}\big] \cong \boldsymbol{L}v^* f_r F\big[\tfrac{1}{p}\big].$$

If $X \in \mathbf{Sm}_k$, we have

$$\operatorname{Hom}_{\mathcal{SH}_X}\big(\Sigma^{m,n}\Sigma_T^\infty X_+, \boldsymbol{L}v^*\big(s_{q+i}F\big[\tfrac{1}{p}\big]\big)\big) \cong \operatorname{Hom}_{\mathcal{SH}}\big(\Sigma^{m,n}\Sigma_T^\infty X_+, s_{q+i}F\big[\tfrac{1}{p}\big]\big)$$

for every $i > 0$ by Corollary 2.17. Since $F[1/p]$ is bounded with respect to the slice filtration, there exist q_1 and $q_2 \in \mathbb{Z}$ such that the vanishing condition (4.6) holds for (X, m, n) and $(X, m-1, n)$, respectively. Let q be the maximum of q_1 and q_2. Then using the distinguished triangle $f_{q+i}F[1/p] \to s_{q+i}F[1/p] \to \Sigma_s^1 f_{q+i+1}F[1/p]$ in $\mathcal{SH}$ we conclude that $\operatorname{Hom}_{\mathcal{SH}}(\Sigma^{m,n}\Sigma_T^\infty(X_+), s_{q+i}F[1/p]) = 0$ for every $i > 0$, as we wanted.

When $X \in \mathbf{Sch}_k$, the argument in the proof of Proposition 4.10 works mutatis mutandis replacing $f_{q+i}F[1/p]$ with $s_{q+i}F[1/p]$, since for every $j \in \mathbb{Z}$, $s_j F[1/p]$ has a structure of traces. □

Corollary 4.13. *Assume the conditions (1) and (2) of Corollary 4.12 hold. Then for every $m, n \in \mathbb{Z}$, there exists $q \in \mathbb{Z}$ such that the map $f_{q+i+1}G[1/p] \to f_{q+i}G[1/p]$ induces an isomorphism*

$$\operatorname{Hom}_{\mathcal{SH}_X}\big(\Sigma^{m,n}\Sigma_T^\infty X_+, f_{q+i+1}G\big[\tfrac{1}{p}\big]\big) \cong \operatorname{Hom}_{\mathcal{SH}_X}\big(\Sigma^{m,n}\Sigma_T^\infty X_+, f_{q+i}G\big[\tfrac{1}{p}\big]\big)$$

for every $i > 0$.

Proof. Let $q_1, q_2 \in \mathbb{Z}$ be the integers corresponding to (m, n), $(m+1, n)$ in Corollary 4.12, respectively. Let q be the maximum of q_1 and q_2. Then the result follows by combining the vanishing in Corollary 4.12 with the distinguished triangle

$$\Sigma_s^{-1} s_{q+i}\big[\tfrac{1}{p}\big] \to f_{q+i+1}G\big[\tfrac{1}{p}\big] \to f_{q+i}G\big[\tfrac{1}{p}\big] \to s_{q+i}G\big[\tfrac{1}{p}\big]$$

in $\mathcal{SH}_X$. □

Remark 4.14. Combining Definition 4.3 and Corollary 4.13, we deduce that for every $a, b \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that

$$F^n G\left[\tfrac{1}{p}\right]^{a,b}(X) = F^m G\left[\tfrac{1}{p}\right]^{a,b}(X)$$

for every $n \geq m$.

Proposition 4.15. *Assume the conditions (1) and (2) of Corollary 4.12 hold. Then for every $n \in \mathbb{Z}$, the slice spectral sequence*

$$E_1^{a,b}(X, n) = \mathrm{Hom}_{\mathcal{SH}_X}\left(\Sigma_T^\infty X_+, \Sigma^{a+b+n,n} s_a G\left[\tfrac{1}{p}\right]\right) \Rightarrow G\left[\tfrac{1}{p}\right]^{a+b+n,n}(X)$$

(see Section 4.5) *satisfies the following.*

(1) *For every $a, b \in \mathbb{Z}$, there exists $N > 0$ such that $E_r^{a,b} = E_\infty^{a,b}$ for $r \geq N$, where $E_\infty^{a,b}$ is the associated graded $\mathrm{gr}^a F^\bullet$ with respect to the descending filtration $F^\bullet$ on $G[1/p]^{a+b+n,n}(X)$* (see Definition 4.3).

(2) *For every $m, n \in \mathbb{Z}$, the descending filtration $F^\bullet$ on $G[1/p]^{m,n}(X)$ is exhaustive and complete* (see [Boardman 1999, Definition 2.1]).

Proof. (1): It suffices to show that for every $a, b \in \mathbb{Z}$ only finitely many of the differentials $d_r : E_r^{a,b} \to E_r^{a+r,b-r+1}$ are nonzero. But this follows from Corollary 4.12.

(2): By Proposition 4.4, the filtration $F^\bullet$ on $G[1/p]^{m,n}(X)$ is exhaustive. Finally, the completeness of $F^\bullet$ follows by combining Remark 4.14 with [Boardman 1999, Propositions 1.8 and 2.2(c)]. □

4.16. ***The slice spectral sequence for* MGL(X).** Our aim here is to apply the results of the previous sections to obtain a Hopkins–Morel type spectral sequence for $\mathrm{MGL}^{*,*}(X)$ when X is a singular scheme. For smooth schemes, the Hopkins–Morel spectral sequence has been studied in [Levine 2009; Hoyois 2015], and over Dedekind domains in [Spitzweck 2014].

Recall from [Voevodsky 1998, §6.3] that for any noetherian scheme S of finite Krull dimension, the scheme $\mathrm{Gr}_S(N, n)$ parametrizes n-dimensional linear subspaces of $\mathbb{A}_S^N$, and one writes $\mathrm{BGL}_{S,n} = \mathrm{colim}_N \mathrm{Gr}_S(N, n)$. There is a universal rank n bundle $U_{S,n} \to \mathrm{BGL}_{S,n}$, and one denotes the Thom space $\mathrm{Th}(U_{S,n})$ of this bundle by $\mathrm{MGL}_{S,n}$. Using the fact that the Thom space of a direct sum is the smash product of the corresponding Thom spaces and $T = \mathrm{Th}(\mathcal{O}_S)$, one gets a T-spectrum $\mathrm{MGL}_S = (\mathrm{MGL}_{S,0}, \mathrm{MGL}_{S,1}, \dots) \in \mathrm{Spt}(\mathcal{M}_S)$. There is a structure of symmetric spectrum on MGL_S, for which we refer to [Panin et al. 2008, §2.1].

We now let k be a field of characteristic zero and let $X \in \mathbf{Sch}_k$. We use MGL as a short hand for MGL_k throughout this text. It follows from the above definition of MGL_X (which shows that MGL_X is constructed from presheaves represented by smooth schemes) and Proposition 2.12 that the canonical map $\boldsymbol{L}v^*(\mathrm{MGL}) \to \mathrm{MGL}_X$ is an isomorphism.

Definition 4.17. We define $\mathrm{MGL}^{*,*}(X)$ to be the generalized cohomology groups

$$\begin{aligned}\mathrm{MGL}^{p,q}(X) &:= \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^{\infty} X_+, \Sigma^{p,q}\,\mathrm{MGL}_X)\\ &\cong \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^{\infty} X_+, \Sigma^{p,q}\boldsymbol{L}v^*\,\mathrm{MGL}).\end{aligned}$$

It follows from Theorem 2.14 that

$$\mathrm{MGL}^{p,q}(X) \cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma_T^{\infty} X_+, \Sigma^{p,q}\boldsymbol{L}\pi^*\,\mathrm{MGL}). \tag{4.18}$$

We now construct the spectral sequence for $\mathrm{MGL}^{*,*}(X)$ using the exact couple technique as follows. For $p, q, n \in \mathbb{Z}$, define

$$\begin{aligned}A^{p,q}(X, n) &:= [\Sigma_T^{\infty} X_+, \Sigma_s^{p+q-n}\Sigma_t^n (f_p\,\mathrm{MGL}_X)],\\ E^{p,q}(X, n) &:= [\Sigma_T^{\infty} X_+, \Sigma_s^{p+q-n}\Sigma_t^n s_p\,\mathrm{MGL}_X].\end{aligned}$$

Here, $[-,-]$ denotes the morphisms in $\mathcal{SH}_X$. It follows from (4.2) that there is an exact sequence

$$A^{p+1,q-1}(X, n) \xrightarrow{a_n^{p,q}} A^{p,q}(X, n) \xrightarrow{b_n^{p,q}} E^{p,q}(X, n) \xrightarrow{c_n^{p,q}} A^{p+1,q}(X, n). \tag{4.19}$$

Set $D_1(X, n) := \bigoplus_{p,q} A^{p,q}(X, n)$ and $E_1(X, n) := \bigoplus_{p,q} E^{p,q}(X, n)$. Write $a_n^1 := \bigoplus a_n^{p,q}$, $b_n^1 := \bigoplus b_n^{p,q}$ and $c_n^1 := \bigoplus c_n^{p,q}$. This gives an exact couple $\{D_n^1, E_n^1, a_n^1, b_n^1, c_n^1\}$ and the map $d_n^1 = b_n^1 \circ c_n^1 : E_n^1 \to E_n^1$ shows that (E_1, d_1) is a complex. Thus, by repeatedly taking the homology functors, we obtain a spectral sequence.

For the target of the spectral sequence, let $A^m(X, n) := \mathrm{colim}_{q\to\infty} A^{m-q,q}(X, n)$. Since X is a compact object of $\mathcal{SH}_X$ (see [Voevodsky 1998, Proposition 5.5; Ayoub 2007b, Théorème 4.5.67]), the colimit enters into $[-,-]$ so that

$$A^m(X, n) = [\Sigma_T^{\infty} X_+, \Sigma_s^{m-n}\Sigma_t^n\,\mathrm{MGL}_X] = \mathrm{MGL}_X^{m,n}(X).$$

The formalism of exact couples then yields a spectral sequence

$$E_1^{p,q}(X, n) = E_1^{p,q} \Rightarrow \mathrm{MGL}_X^{m,n}(X). \tag{4.20}$$

We now have

$$\begin{aligned}E_1^{p,q}(X, n) = {}& [\Sigma_T^{\infty} X_+, \Sigma_s^{p+q-n}\Sigma_t^n s_p\,\mathrm{MGL}_X]\\ \cong^1{}& [\Sigma_T^{\infty} X_+, \Sigma_s^{p+q-n}\Sigma_t^n s_p\boldsymbol{L}v^*\,\mathrm{MGL}]\\ \cong^2{}& [\Sigma_T^{\infty} X_+, \Sigma_s^{p+q-n}\Sigma_t^n \boldsymbol{L}v^*(s_p\,\mathrm{MGL})]\\ \cong^3{}& [\Sigma_T^{\infty} X_+, \Sigma_s^{p+q-n}\Sigma_t^n \boldsymbol{L}v^*(\Sigma_T^p H(\mathbb{L}^{-p}))]\\ \cong{}& [\Sigma_T^{\infty} X_+, \Sigma_s^{p+q-n}\Sigma_t^n \Sigma_T^p\boldsymbol{L}v^*(H(\mathbb{L}^{-p}))].\end{aligned} \tag{4.21}$$

In this sequence of isomorphisms, $\cong^1$ is shown above, $\cong^2$ follows from [Pelaez 2013, Theorem 3.7] and $\cong^3$ follows from the isomorphism $s_p\,\mathrm{MGL} \xrightarrow{\cong} \Sigma_T^p H(\mathbb{L}^{-p})$, as shown, for example, in [Hoyois 2015, (8.6)], where $\mathbb{L} = \bigoplus_{i \le 0} \mathbb{L}^i \cong \bigoplus_{i \ge 0} MU_{2i}$ is the Lazard ring.

Since $\mathbb{L}$ is a torsion-free abelian group, it follows from Corollary 3.6 that the last term of (4.21) is the same as $H^{3p+q}(X, \mathbb{Z}(n+p)) \otimes_{\mathbb{Z}} \mathbb{L}^{-p}$.

The spectral sequence (4.20) is actually identical to an E_2-spectral sequence after reindexing. Indeed, letting

$$\widetilde{E}_2^{p',q'} = H^{p'-q'}(X, \mathbb{Z}(n-q')) \otimes_{\mathbb{Z}} \mathbb{L}^{q'}$$

and using (4.21), an elementary calculation shows that the invertible transformation $(3p+q, n+p) \mapsto (p'-q', n-q')$ yields

$$\begin{aligned} E_1^{p+1,q} &\cong [\Sigma_T^\infty X_+, \Sigma_s^{p+q+1-n} \Sigma_t^n s_{p+1} \mathrm{MGL}_X] \\ &\cong H^{(p'+2)-(q'-1)}(X, \mathbb{Z}(n-(q'-1))) \otimes_{\mathbb{Z}} \mathbb{L}^{q'-1} = \widetilde{E}_2^{p'+2,q'-1}. \end{aligned} \tag{4.22}$$

It is clear from (4.19) that the E_1-differential of the above spectral sequence is $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ and (4.22) shows that this differential is identified with the differential

$$d_2^{p',q'} = d_1^{p,q} : \widetilde{E}_2^{p',q'} \to \widetilde{E}_2^{p'+2,q'-1}.$$

Inductively, it follows that the chain complex $\{E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}\}$ is transformed to the chain complex $\{\widetilde{E}_{r+1}^{p',q'} \xrightarrow{d_r} \widetilde{E}_{r+1}^{p'+r+1,q'-r}\}$. Combining this with (4.18), we conclude the following.

Theorem 4.23. *Let k be a field which has characteristic zero and let $X \in \mathbf{Sch}_k$. Then for any integer $n \in \mathbb{Z}$, there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(n-q)) \otimes_{\mathbb{Z}} \mathbb{L}^q \Rightarrow \mathrm{MGL}^{p+q,n}(X). \tag{4.24}$$

The differentials of this spectral sequence are given by $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, and for every $p, q \in \mathbb{Z}$, there exists $N > 0$ such that $E_r^{p,q} = E_\infty^{p,q}$ for $r \ge N$, where $E_\infty^{p,q}$ is the associated graded $\mathrm{gr}^{-q} F^\bullet$ with respect to the descending filtration on $\mathrm{MGL}^{p+q,n}(X)$ (see Definition 4.3). *Furthermore, this spectral sequence degenerates with rational coefficients.*

Proof. The construction of the spectral sequence is shown above. Since MGL is bounded by [Hoyois 2015, Theorem 8.12], it follows from Proposition 4.7 that the spectral sequence (4.24) is strongly convergent. Thus, we deduce the existence of $N > 0$ such that $E_r^{p,q} = E_\infty^{p,q}$ for $r \ge N$.

As for the degeneration with rational coefficients, we observe that the maps $f_p \,\mathrm{MGL} \to s_p \,\mathrm{MGL} \cong \Sigma_T^p H(\mathbb{L}^{-p})$ rationally split to yield an isomorphism of spectra $\mathrm{MGL}_{\mathbb{Q}} \xrightarrow{\cong} \oplus_{p \ge 0} \Sigma_T^p H(\mathbb{L}_{\mathbb{Q}}^{-p})$ in $\mathcal{SH}$ [Naumann et al. 2009, Theorem 10.5 and Corollary 10.6(i)]. The desired degeneration of the spectral sequence now follows immediately from its construction above. □

Remark 4.25. If k is a perfect field of positive characteristic p, we observe that $s_r(\mathrm{MGL}[1/p]) \cong \Sigma_T^r H(\mathbb{L}^{-r})[1/p]$ for every $r \in \mathbb{Z}$ [Hoyois 2015, (8.6)], and so $s_r(\mathrm{MGL}[1/p])$ has a weak structure of smooth traces [Kelly 2012, Corollary 5.2.4]. Thus, we can apply [Kelly 2012, Theorem 4.2.29] to conclude $\boldsymbol{L}v^* s_r(\mathrm{MGL}[1/p]) \cong s_r(\boldsymbol{L}v^* \mathrm{MGL}[1/p])$. Except for this identification, the proof of Theorem 4.23 does not depend on the characteristic of k. We thus obtain a spectral sequence as in (4.24):

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(n-b)) \otimes_{\mathbb{Z}} \mathbb{L}^b\big[\tfrac{1}{p}\big] \Rightarrow \mathrm{MGL}^{a+b,n}(X)\big[\tfrac{1}{p}\big].$$

But we can only guarantee strong convergence when $X \in \mathbf{Sm}_k$ [Hoyois 2015, Theorem 8.12]. In general, for $X \in \mathbf{Sch}_k$, the spectral sequence satisfies the weaker convergence of Proposition 4.15(1)–(2). In this case, the strong convergence would follow if one knew that MGL has a structure of traces.

4.26. ***The slice spectral sequence for* KGL.** For any noetherian scheme X of finite Krull dimension, the motivic T-spectrum $\mathrm{KGL}_X \in \mathrm{Spt}(\mathcal{M}_X)$ was defined by Voevodsky [1998, §6.2]. It has the property that it represents algebraic K-theory of objects in $\mathbf{Sm}_X$ if X is regular. It was later shown by Cisinski [2013] that for X not necessarily regular, KGL_X represents Weibel's homotopy invariant K-theory $KH_*(Y)$ for $Y \in \mathbf{Sm}_X$. Like MGL_X, there is a structure of symmetric spectrum on KGL_X, for which we refer to [Jardine 2009, pp. 157 and 176].

Let k be a field of exponential characteristic p. The map $\boldsymbol{L}v^*(\mathrm{KGL}_k) \to \mathrm{KGL}_X$ is an isomorphism by [Cisinski 2013, Proposition 3.8]. It is also known that $s_r\,\mathrm{KGL}_k \cong \Sigma_T^r H\mathbb{Z}$ for $r \in \mathbb{Z}$; see [Levine 2008, Theorem 6.4.2] if k is perfect and [Röndigs and Østvær 2016, §1, p. 1158] in general. It follows from [Pelaez 2013, Theorem 3.7] (in positive characteristic we use [Kelly 2012, Theorem 4.2.29] instead) that $\boldsymbol{L}v^*(s_r\,\mathrm{KGL}[1/p]_k) \cong s_r(\boldsymbol{L}v^*\,\mathrm{KGL}[1/p]_k) \cong s_r\,\mathrm{KGL}[1/p]_X$. One also knows that $(\mathrm{KGL}_k)_{\mathbb{Q}} \cong \bigoplus_{p\in\mathbb{Z}} \Sigma_T^p H\mathbb{Q}$ in $\mathcal{SH}$ [Riou 2010, Definition 5.3.17 and Theorem 5.3.10]. We can thus use the Bott periodicity of KGL_X and repeat the construction of Section 4.16 mutatis mutandis (with $n = 0$) to conclude the following.

Theorem 4.27. *Let k be a field that admits resolution of singularities* (*resp. a field of exponential characteristic $p > 1$*), *and let $X \in \mathbf{Sch}_k$. Then there is a strongly convergent spectral sequence*

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(-b)) \Rightarrow KH_{-a-b}(X) \tag{4.28}$$

$$\big(\textit{resp.}\quad E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(-b)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \Rightarrow KH_{-a-b}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]\big). \tag{4.29}$$

The differentials of this spectral sequence are given by $d_r : E_r^{a,b} \to E_r^{a+r,b-r+1}$, and for every $a, b \in \mathbb{Z}$, there exists $N > 0$ such that $E_r^{a,b} = E_\infty^{a,b}$ for $r \geq N$, where $E_\infty^{a,b}$ is the associated graded $\mathrm{gr}^{-b} F^\bullet$ with respect to the descending filtration on

$KH_{-a-b}(X)$ (*resp.* $KH[1/p]_{-a-b}(X)$) (see Definition 4.3). *Furthermore, this spectral sequence degenerates with rational coefficients.*

Proof. If k admits resolution of singularities, we just need to show that the spectral sequence is convergent. For this, we observe that KGL_k is the spectrum associated to the Landweber exact $\mathbb{L}$-algebra $\mathbb{Z}[\beta, \beta^{-1}]$ that classifies the multiplicative formal group law [Spitzweck and Østvær 2009, Theorem 1.2]. Thus [Hoyois 2015, Theorem 8.12] implies that KGL_k is bounded with respect to the slice filtration (this argument also applies in positive characteristic). Hence, the convergence follows from Proposition 4.7.

In the case of positive characteristic, the existence of the spectral sequence follows by combining the argument of Section 4.16 with Lemmas 4.8 and 4.9. To establish the convergence, it suffices to check that $\mathrm{KGL}[1/p]_k$ satisfies the conditions in Proposition 4.10.

We have already seen that KGL_k is bounded with respect to the slice filtration. Thus, by Lemma 4.8(2) we conclude that $\mathrm{KGL}[1/p]_k$ is bounded with respect to the slice filtration as well. On the other hand, it follows from [Kelly 2012, Proposition 5.2.3] that $\mathrm{KGL}[1/p]_k$ has a structure of traces in the sense of [Kelly 2012, Definition 4.3.1]. Finally, since $s_r\,\mathrm{KGL}_k \cong \Sigma_T^r H\mathbb{Z}$ for $r \in \mathbb{Z}$, combining [Kelly 2012, Corollary 5.2.4] and Lemma 4.9, we deduce that $s_r(\mathrm{KGL}[1/p])_k$ has a weak structure of smooth traces in the sense of [Kelly 2012, Definition 4.2.27]. This finishes the proof. □

Remark 4.30. For $\operatorname{char} k = 0$, the spectral sequence of Theorem 4.27 is not new and was constructed by Haesemeyer [2004, Theorem 7.3] using a different approach. However, the expected degeneration (rationally) of this spectral sequence and its positive characteristic analogue are new.

As a combination of Theorem 4.27 and [Thomason and Trobaugh 1990, Theorems 9.5 and 9.6], we obtain the following spectral sequence for the algebraic K-theory $K^B(-)$ of singular schemes [Thomason and Trobaugh 1990].

Corollary 4.31. *Let k be a field of exponential characteristic $p > 1$. Let $\ell \neq p$ be a prime and $m \geq 0$ any integer. Given any $X \in \mathbf{Sch}_k$, there exist strongly convergent spectral sequences*

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(-b)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \Rightarrow K^B_{-a-b}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right], \tag{4.32}$$

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}/\ell^m(-b)) \Rightarrow K^B/\ell^m{}_{-a-b}(X). \tag{4.33}$$

5. Applications I: Comparing cobordism, K-theory and cohomology

In this section, we deduce some geometric applications of the slice spectral sequences for singular schemes. More applications will appear in the subsequent sections.

Consider the edge map $\mathrm{MGL} = f_0\,\mathrm{MGL} \to s_0\,\mathrm{MGL} \cong H\mathbb{Z}$ in the spectral sequence (4.24). This induces a natural map $\nu_X : \mathrm{MGL}^{i,j}(X) \to H^i(X, \mathbb{Z}(j))$ for every $X \in \mathbf{Sch}_k$ and $i, j \in \mathbb{Z}$.

The following result shows that there is no distinction between algebraic cycles and cobordism cycles at the level of 0-cycles.

Theorem 5.1. *Let k be a field which admits resolution of singularities (resp. a perfect field of positive characteristic p). Then for any $X \in \mathbf{Sch}_k$ of dimension d, we have $H^{2a-b}(X, \mathbb{Z}(a)) = 0$ (resp. $H^{2a-b}(X, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$) whenever $a > d + b$. In particular, for every $X \in \mathbf{Sch}_k$ (resp. $X \in \mathbf{Sm}_k$), the map*

$$\nu_X \colon \mathrm{MGL}^{2d+i,d+i}(X) \to H^{2d+i}(X, \mathbb{Z}(d+i)) \tag{5.2}$$

$$\Big(\textit{resp.}\quad \nu_X \colon \mathrm{MGL}^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]\Big) \tag{5.3}$$

is an isomorphism for all $i \geq 0$.

Proof. Using the spectral sequence (4.24) (resp. Remark 4.25) and the fact that $\mathbb{L}^{>0} = 0$, the isomorphism of (5.2) (resp. (5.3)) follows immediately from the vanishing assertion for the motivic cohomology.

To prove the vanishing result, we note that for $X \in \mathbf{Sm}_k$, there is an isomorphism $H^{2a-b}(X, \mathbb{Z}(a)) \cong \mathrm{CH}^a(X, b)$ by [Voevodsky 2002a], and the latter group is clearly zero if $a > d + b$ by definition of Bloch's higher Chow groups.

If X is not smooth and k admits resolution of singularities, our assumption on k implies that there exists a cdh-cover $\{X' \amalg Z \to X\}$ of X such that $X' \in \mathbf{Sm}_k$, $\dim(Z) < \dim(X)$ and $\dim(W) < \dim(X)$, where we set $W = X' \times_X Z$. The cdh-descent for the motivic cohomology yields an exact sequence

$$H^{2a-b-1}(W, \mathbb{Z}(a)) \xrightarrow{\partial} H^{2a-b}(X, \mathbb{Z}(a)) \to H^{2a-b}(X', \mathbb{Z}(a)) \oplus H^{2a-b}(Z, \mathbb{Z}(a)).$$

The smooth case of our vanishing result shown above and an induction on the dimension together imply that the two end terms of this exact sequence vanish. Hence, the middle term vanishes too.

If X is not smooth and k is perfect of positive characteristic, we argue as in Proposition 4.10. Namely, by a theorem of Gabber [Illusie et al. 2014, Théorème 3(1)] and Temkin's strengthening [2017, Theorem 1.2.9] of Gabber's result, there exists $W \in \mathbf{Sm}_k$ and a surjective proper map $h : W \to X$, which is generically étale of degree p^r, $r \geq 1$. Then by a theorem of Raynaud and Gruson [1971, Theorem 5.2.2], there exists a blow-up $g : X' \to X$ with center Z such that the diagram

$$\begin{tikzcd} W' \arrow[r, "h'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ W \arrow[r, "h"'] & X \end{tikzcd} \tag{5.4}$$

commutes, where h' is finite flat surjective of degree p^r and $g' : W' \to W$ is the blow-up of W with center $h^{-1}(Z)$.

Thus we have a cdh-cover $\{X' \amalg Z \to X\}$ of X, such that $\dim_k(Z) < \dim_k(X)$ and $\dim_k(E) < \dim_k(X)$, where we set $E = X' \times_X Z$. Then by cdh-excision, the following diagram is exact:

$$H^{2a-b-1}(E, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^{2a-b}(X, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \\ \to H^{2a-b}(X', \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \oplus H^{2a-b}(Z, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big].$$

By induction on the dimension, this reduces to the exact sequence

$$0 \to H^{2a-b}(X, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \xrightarrow{g^*} H^{2a-b}(X', \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big].$$

So it suffices to show that $g^* = 0$. In order to prove this, we observe that (5.4) commutes. Therefore, since $W \in \mathbf{Sm}_k$, $H^{2a-b}(W, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$. We conclude that $h'^* \circ g^* = g'^* \circ h^* = 0$. Thus, it is enough to see that

$$h'^* : H^{2a-b}(X', \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^{2a-b}(W', \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]$$

is injective. Let $v' : X' \to \operatorname{Spec} k$, and $\epsilon : \boldsymbol{L}v'^* H\mathbb{Z}[1/p] \to \boldsymbol{R}h'_* \boldsymbol{L}h'^* \boldsymbol{L}v'^* H\mathbb{Z}[1/p]$ be the map given by the unit of the adjunction $(\boldsymbol{L}h'^*, \boldsymbol{R}h'_*)$. By the naturality of the isomorphism in Proposition 2.13, we deduce that h'^* gets identified with the map induced by ϵ (see Corollary 3.6):

$$\epsilon_* : \operatorname{Hom}_{\mathcal{SH}_{X'}}\big(\Sigma^{m,n} \Sigma_T^{\infty}(X'_+), \boldsymbol{L}v'^* H\mathbb{Z}\big[\tfrac{1}{p}\big]\big) \\ \to \operatorname{Hom}_{\mathcal{SH}_{X'}}\big(\Sigma^{m,n} \Sigma_T^{\infty}(X'_+), \boldsymbol{R}h'_* \boldsymbol{L}h'^* \boldsymbol{L}v'^* H\mathbb{Z}\big[\tfrac{1}{p}\big]\big).$$

By [Kelly 2012, Corollary 5.2.4], $H\mathbb{Z}[1/p]$ has a structure of traces in the sense of [Kelly 2012, Definition 4.3.1]. Thus, we deduce from [Kelly 2012, Definition 4.3.1(Deg), p. 101] that ϵ_* is injective since h' is finite flat surjective of degree p^r. This finishes the proof. □

Remark 5.5. For $X \in \mathbf{Sm}_k$ and $i = 0$, the isomorphism of (5.2) was proved by Déglise [2013, Corollary 4.3.4].

When A is a field, the following result was proven by Morel [2012, Corollary 1.25] using methods of unstable motivic homotopy theory. Taking for granted the result for fields, Déglise [2013] proved Theorem 5.6 using homotopy modules. Spitzweck [2014, Corollary 7.3] proved Theorem 5.6 for localizations of a Dedekind domain.

Theorem 5.6. *Let k be a perfect field of exponential characteristic p. Then for any regular semilocal ring A which is essentially of finite type over k, and for any* *integer $n \geq 0$, the map*

$$\mathrm{MGL}^{n,n}(A) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \to H^n(A, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \tag{5.7}$$

is an isomorphism. In particular, there is a natural isomorphism

$$\mathrm{MGL}^{n,n}(A) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \cong K_n^M(A) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]$$

if k is also infinite.

Proof. Using the spectral sequence (4.24) and the fact that $\mathbb{L}^{>0} = 0$, it suffices to prove that $E_2^{n+i+j,-i}(A) = 0$ for every $j \geq 0$ and $i \geq 1$. In positive characteristic, we can use Remark 4.25 since A is regular. Notice that (4.24) and the spectral sequence in Remark 4.25 are strongly convergent for A by [Hoyois 2015, Lemmas 8.9 and 8.10].

On the one hand, we have isomorphisms

$$\begin{aligned} E_2^{n+i+j,-i}(A) &= H^{n+2i+j}(A, \mathbb{Z}(n+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \\ &\cong \mathrm{CH}^{n+i}(A, 2n+2i-n-2i-j) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \\ &= \mathrm{CH}^{n+i}(A, n-j) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]. \end{aligned}$$

On the other hand, letting F denote the fraction field of A, the Gersten resolution for the higher Chow groups (see [Bloch 1986, Theorem 10.1]) shows that the restriction map $\mathrm{CH}^{n+i}(A, n-j) \to \mathrm{CH}^{n+i}(F, n-j)$ is injective. But the term $\mathrm{CH}^{n+i}(F, n-j)$ is zero whenever $j \geq 0$, $i \geq 1$ for dimensional reasons. We conclude that $E_2^{n+i+j,-i}(A) = 0$. The last assertion of the theorem now follows from the isomorphism $\mathrm{CH}^n(A, n) \cong K_n^M(A)$ by [Kerz 2009, Theorem 1.1]. ☐

5.8. *Connective K-theory.* Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$. Recall that the *connective K-theory* spectrum KGL_X^0 is defined to be the motivic T-spectrum $f_0 \,\mathrm{KGL}_X$ in $\mathcal{SH}_X$ (see (4.2)). Strictly speaking, KGL_X^0 should be called effective K-theory. Nevertheless, we follow the terminology of [Dai and Levine 2014].

In particular, there is a canonical map $u_X : \mathrm{KGL}_X^0 \to \mathrm{KGL}_X$ which is universal for morphisms from objects of $\mathcal{SH}_X^{\mathrm{eff}}$ to KGL_X. For any $Y \in \mathbf{Sm}_X$, we let $CKH^{p,q}(Y) = \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{p,q}\,\mathrm{KGL}_X^0)$. Using an analogue of Theorem 4.23 for KGL_X^0, one can prove the existence of the cycle class map for the higher Chow groups as follows.

Theorem 5.9. *Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$ have dimension d. Then the map $\mathrm{KGL}_X^0[1/p] \to s_0\,\mathrm{KGL}_X[1/p] \cong H\mathbb{Z}[1/p]$ induces for every integer $i \geq 0$, an isomorphism*

$$CKH^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \xrightarrow{\cong} H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]. \tag{5.10}$$

In particular, *the canonical map* $\mathrm{KGL}^0_X \to \mathrm{KGL}_X$ *induces a natural cycle class map*

$$\mathrm{cyc}_i : H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to KH_i(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]. \tag{5.11}$$

Proof. First we assume that k admits resolution of singularities. It follows from the definition that KGL^0_X is a connective T-spectrum, and $\boldsymbol{L}v^*(\mathrm{KGL}^0_k) \xrightarrow{\cong} \mathrm{KGL}^0_X$ by [Pelaez 2013, Theorem 3.7]. One also knows that $s_r\,\mathrm{KGL}^0_k \cong \Sigma^r_T H\mathbb{Z}$ for $r \geq 0$ [Levine 2008, Theorem 6.4.2] and is zero otherwise. The proof of Theorem 4.23 can now be repeated verbatim to conclude that for each $n \in \mathbb{Z}$, there is a strongly convergent spectral sequence

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(n-b)) \otimes_{\mathbb{Z}} \mathbb{Z}_{b\leq 0} \Rightarrow CKH^{a+b,n}(X), \tag{5.12}$$

where $\mathbb{Z}_{b\leq 0} = \mathbb{Z}$ if $b \leq 0$ and is zero otherwise. Furthermore, this spectral sequence degenerates with rational coefficients.

One now repeats the proof of Theorem 5.1 to conclude that the edge map $CKH^{2d+i,d+i}(X) \to H^{2d+i}(X, \mathbb{Z}(d+i))$ is an isomorphism for every $i \geq 0$. Finally, to get the desired cycle class map, we compose the inverse of this isomorphism with the canonical map $CKH^{2d+i,d+i}(X) \to KH_i(X)$.

If the characteristic of k is positive, then $s_r(\mathrm{KGL}^0_k) \cong \Sigma^r_T H\mathbb{Z}$ for every $r \geq 0$ and is zero otherwise [Levine 2008, Theorem 6.4.2]. So $s_r(\mathrm{KGL}^0_k[1/p])$ has a weak structure of traces [Kelly 2012, Corollary 5.2.4]. By Lemma 4.9, we deduce that $s_r(\mathrm{KGL}^0_k[1/p]) \cong \Sigma^r_T H\mathbb{Z}[1/p]$ for every $r \geq 0$ and is zero otherwise. Thus, we can apply [Kelly 2012, Theorem 4.2.29] to conclude $\boldsymbol{L}v^*(\mathrm{KGL}^0_k[1/p]) \cong \mathrm{KGL}^0_X[1/p]$. Then the argument of Theorem 4.27 applies, and we conclude that for each $n \in \mathbb{Z}$, there is a strongly convergent spectral sequence

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(n-b)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]_{b\leq 0} \Rightarrow CKH^{a+b,n}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]. \tag{5.13}$$

By Theorem 5.1, $H^{2a-b}(X, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$ whenever $a > d+b$. Thus, combining the spectral sequence (5.13) and the fact that $\mathbb{L}^{>0} = 0$, we deduce the isomorphism of (5.10) with $\mathbb{Z}[1/p]$-coefficients:

$$CKH^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \xrightarrow{\cong} H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]. \qquad \square$$

An argument identical to the proof of Theorem 5.6 shows that for any regular semilocal ring A which is essentially of finite type over an infinite field k and any integer $n \geq 0$, there is a natural isomorphism

$$CKH^{n,n}(A) \xrightarrow{\cong} K^M_n(A) \tag{5.14}$$

(notice that in positive characteristic, the spectral sequence is also strongly convergent integrally since A is regular).

Moreover, the canonical map $CKH^{n,n}(A) \to K_n(A)$ respects products [Pelaez 2011, Theorem 3.6.9], and hence coincides with the known map $K_n^M(A) \to K_n(A)$. This shows that the Milnor K-theory is represented by the connective K-theory, and one gets a lifting of the relation between the Milnor and Quillen K-theory of smooth semilocal schemes to the level of $\mathcal{SH}$. In particular, it is possible to recover Milnor K-theory and its map into Quillen K-theory from the T-spectrum KGL (which represents Quillen K-theory in $\mathcal{SH}$ for smooth k-schemes) by passing to its (-1)-effective cover $f_0\,\mathrm{KGL}_k \to \mathrm{KGL}_k$.

As another consequence of the slice spectral sequence, one gets the following comparison result between the connective and nonconnective versions of the homotopy K-theory. The homological analogue of this result was shown in [Dai and Levine 2014, Corollary 5.5].

Theorem 5.15. *Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$ have dimension d. Then the canonical map*

$$CKH^{2n,n}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \to KH_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]$$

is an isomorphism for every integer $n \le 0$.

Proof. If k admits resolution of singularities, we observe that the slice spectral sequence is functorial for morphisms of motivic T-spectra. Since $H^{2q}(X, \mathbb{Z}(q)) = 0$ for $q < 0$, a comparison of the spectral sequences (4.28) and (5.12) shows that it is enough to prove that for every $r \ge 2$ and $q \le 0$, either $q + r - 1 \le 0$ or

$$H^{-q-r-(q+r-1)}(X, \mathbb{Z}(1-r-q)) = H^{1-2r-2q}(X, \mathbb{Z}(1-r-q)) = 0.$$

But this is true because $H^{1-2r-2q}(X, \mathbb{Z}(s)) = 0$ if $s < 0$.

In positive characteristic, we use the same argument as above for the spectral sequences (4.29) and (5.13). $\square$

Yet another consequence of the above spectral sequences is the following direct verification of Weibel's vanishing conjecture for negative KH-theory and negative CKH-theory of singular schemes. For KH-theory, there are other proofs of this conjecture by Haesemeyer [2004, Theorem 7.1] in characteristic zero and Kelly [2014, Theorem 3.5] and Kerz and Strunk [2017] in positive characteristic using different methods. We refer the reader to [Cisinski 2013; Cortiñas et al. 2008a; Geisser and Hesselholt 2010; Kerz et al. 2018; Krishna 2009; Weibel 2001] for more results associated to Weibel's conjecture. The vanishing result below for CKH-theory is new in any characteristic.

Theorem 5.16. *Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$ have dimension d. Then $CKH^{m,n}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = KH_{2n-m}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$ whenever $2n - m < -d$ and $KH_{-d}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$.*

Proof. When k admits resolution of singularities, using the spectral sequences (4.28) and (5.10), it suffices to show $H^{p-q}(X, \mathbb{Z}(n-q)) = 0$ whenever $2n-p-q+d < 0$.

If $n-q < 0$, then we already know that this motivic cohomology group is zero. So we can assume $n-q \geq 0$. We set $a = n-q$ and $b = 2n-p-q$ so that $2a-b = 2n-2q-2n+p+q = p-q$. Since $2n-p-q+d < 0$ and $n-q \geq 0$ by our assumption, we get

$$b+d-a = 2n-p-q+d-n+q = n-p+d = (2n-p-q+d)-(n-q) < 0.$$

The theorem now follows because we have shown in the proof of Theorem 5.1 that $H^{p-q}(X, \mathbb{Z}(n-q)) = H^{2a-b}(X, \mathbb{Z}(a)) = 0$ as $a > b+d$. This argument also shows that $KH_{-d}(X) \cong H^d(X, \mathbb{Z}(0)) \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z})$.

In positive characteristic, the same argument with the spectral sequences (4.29) and (5.13) gives that $CKH^{m,n}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = KH_{2n-m}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$ whenever $2n-m < -d$ and $KH_{-d}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$. □

Weibel's conjecture on the vanishing of certain negative K-theory was proven (after inverting the characteristic) by Kelly [2014]. Using our spectral sequence (which uses the methods of [Kelly 2012]), we can obtain the following result (which follows as well from [Kelly 2014] via the cdh-descent spectral sequence). The characteristic zero version of this computation was proven in [Cortiñas et al. 2008b, Theorem 0.2], and for arbitrary noetherian schemes, we refer the reader to [Kerz et al. 2018, Corollary D].

Corollary 5.17. *Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$ have dimension d. Then*

$$K^B_{-d}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right].$$

6. The Chern classes on *KH*-theory

In order to obtain more applications of the slice spectral sequence for KH-theory and the cycle class map (see Theorem 5.9), we need to have a theory of Chern classes on the KH-theory of singular schemes.

Gillet [1981] showed that any cohomology theory satisfying the projective bundle formula and some other standard admissibility axioms admits a theory of Chern classes from algebraic K-theory of schemes over a field. These Chern classes are very powerful tools for understanding algebraic K-theory groups in terms of various cohomology theories such as motivic cohomology and Hodge theory. The Chern classes in Deligne cohomology are used to define various regulator maps on K-theory and they also give rise to the construction of intermediate Jacobians of smooth projective varieties over $\mathbb{C}$.

For a perfect field k of exponential characteristic $p \geq 1$, Kelly [2012, Corollary 5.5.10] showed that the motivic cohomology functor $X \mapsto \{H^i(X, \mathbb{Z}(j))[1/p]\}_{i,j \in \mathbb{Z}}$ satisfies the projective bundle formula in $\mathbf{Sch}_k$. This implies in particular by Gillet's theory that there are functorial Chern class maps

$$c_{i,j} : K_j(X) \to H^{2i-j}(X, \mathbb{Z}(i))\left[\tfrac{1}{p}\right]. \tag{6.1}$$

In this section, we show that in characteristic zero, Gillet's technique can be used to construct the above Chern classes on the homotopy invariant K-theory of singular schemes. Applications of these Chern classes to the understanding of the motivic cohomology and KH-theory of singular schemes will be given in the following two sections.

Let k be a field of characteristic zero and let $\mathbf{Sch}_{\mathrm{Zar}/k}$ denote the category of separated schemes of finite type over k equipped with the Zariski topology. Let $\mathbf{Sm}_{\mathrm{Zar}/k}$ denote the full subcategory of smooth schemes over k equipped with the Zariski topology. For any $X \in \mathbf{Sch}_k$, let X_{Zar} denote the small Zariski site of X. A presheaf of spectra on $\mathbf{Sch}_k$ or $\mathbf{Sm}_k$ means a presheaf of S^1-spectra.

Let $\mathrm{Pre}(\mathbf{Sch}_{\mathrm{Zar}/k})$ be the category of presheaves of simplicial sets on $\mathbf{Sch}_{\mathrm{Zar}/k}$ equipped with the injective Zariski local model structure, i.e., the weak equivalences are the maps that induce a weak equivalence of simplicial sets at every Zariski stalk and the cofibrations are given by monomorphisms. This model structure restricts to a similar model structure on the category $\mathrm{Pre}(X_{\mathrm{Zar}})$ of presheaves of simplicial sets on X_{Zar} for every $X \in \mathbf{Sch}_k$. We write $\mathcal{H}^{\mathrm{big}}_{\mathrm{Zar}}(k)$ and $\mathcal{H}^{\mathrm{sml}}_{\mathrm{Zar}}(X)$ for the homotopy categories of $\mathrm{Pre}(\mathbf{Sch}_{\mathrm{Zar}/k})$ and $\mathrm{Pre}(X_{\mathrm{Zar}})$, respectively.

6.2. ***Chern classes from KH-theory to motivic cohomology.*** For any $X \in \mathbf{Sch}_k$, let $\Omega BQP(X)$ denote the simplicial set obtained by taking the loop space of the nerve of the category $QP(X)$ obtained by applying Quillen's Q-construction to the exact category of locally free sheaves on X_{Zar}. Let $\mathcal{K}$ denote the presheaf of simplicial sets on $\mathbf{Sch}_{\mathrm{Zar}/k}$ given by $X \mapsto \Omega BQP(X)$. One knows that $\mathcal{K}$ is a presheaf of infinite loop spaces so that there is a presheaf of spectra $\widetilde{\mathcal{K}}$ on $\mathbf{Sch}_k$ such that $\mathcal{K} = (\widetilde{\mathcal{K}})_0$. Let $\widetilde{\mathcal{K}}^B$ denote the Thomason–Trobaugh presheaf of spectra on $\mathbf{Sch}_k$ such that $\widetilde{\mathcal{K}}^B(X) = K^B(X)$ for every $X \in \mathbf{Sch}_k$. There is a natural map of presheaves of spectra $\widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}^B$ which induces isomorphism between the nonnegative homotopy group presheaves.

Recall from [Jardine 1997, Theorem 2.34] that the category of presheaves of spectra on $\mathbf{Sch}_{\mathrm{Zar}/k}$ has a closed model structure, where the weak equivalences are given by the stalkwise stable equivalence of spectra, and a map $f : E \to F$ is a cofibration if f_0 is a monomorphism and $E_{n+1} \amalg_{S^1 \wedge E_n} S^1 \wedge F_n \to F_{n+1}$ is a monomorphism for each $n \geq 0$. Let $\mathcal{H}^s_{\mathrm{Zar}}(k)$ denote the associated homotopy category. There is a functor $\Sigma^\infty_s : \mathcal{H}^{\mathrm{big}}_{\mathrm{Zar}}(k) \to \mathcal{H}^s_{\mathrm{Zar}}(k)$ which has a right adjoint. We can consider the above model structure and the corresponding homotopy categories with respect to the Nisnevich and cdh-sites as well.

Let $\widetilde{\mathcal{K}}_{\mathrm{cdh}} \to \widetilde{\mathcal{K}}^B_{\mathrm{cdh}}$ denote the map between the functorial fibrant replacements in the above model structure on presheaves of spectra on $\mathbf{Sch}_k$ with respect to the cdh-topology. Let KH denote the presheaf of spectra on $\mathbf{Sch}_k$ such that $KH(X)$ is Weibel's homotopy invariant K-theory of X [Weibel 1989].

The following is a direct consequence of the main result of [Haesemeyer 2004].

Lemma 6.3. *Let k be a field of characteristic zero. For every $X \in \mathbf{Sch}_k$ and integer $p \in \mathbb{Z}$, there is a natural isomorphism $KH_p(X) \xrightarrow{\cong} \mathbb{H}^{-p}_{\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}})$.*

Proof. We have a natural isomorphism

$$\begin{aligned} \pi_p(\widetilde{\mathcal{K}}_{\mathrm{cdh}}(X)) &= \mathrm{Hom}_{\mathcal{H}^s_{\mathrm{cdh}}(k)}(\Sigma^\infty_s(S^p_s \wedge X), \widetilde{\mathcal{K}}) \\ &\cong \mathrm{Hom}_{\mathcal{H}_{\mathrm{cdh}}(k)}(S^p_s \wedge X, \mathcal{K}) \\ &\cong \mathbb{H}^{-p}_{\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}}). \end{aligned} \tag{6.4}$$

It is well known that the natural maps $K_p(X) \to \pi_p(\widetilde{\mathcal{K}}_{\mathrm{cdh}}(X)) \to \pi_p(\widetilde{\mathcal{K}}^B_{\mathrm{cdh}}(X))$ are isomorphisms for all $p \in \mathbb{Z}$ when X is smooth over k. In general, let $X \in \mathbf{Sch}_k$. We can find a Cartesian square

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd} \tag{6.5}$$

where $X' \in \mathbf{Sm}_k$ and f is a proper birational morphism which is an isomorphism outside the closed immersion $Z \hookrightarrow X$. Induction on dimension of X and cdh-descent for $\widetilde{\mathcal{K}}_{\mathrm{cdh}}$ as well as $\widetilde{\mathcal{K}}^B_{\mathrm{cdh}}$ now show that the map $\pi_p(\widetilde{\mathcal{K}}_{\mathrm{cdh}}(X)) \to \pi_p(\widetilde{\mathcal{K}}^B_{\mathrm{cdh}}(X))$ is an isomorphism for all $p \in \mathbb{Z}$. Composing the inverse of this isomorphism with the map in (6.4), we get a natural isomorphism $\pi_p(\widetilde{\mathcal{K}}^B_{\mathrm{cdh}}(X)) \xrightarrow{\cong} \mathbb{H}^{-p}_{\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}})$.

On the other hand, it follows from [Haesemeyer 2004, Theorem 6.4] that the natural map $KH(X) \to \widetilde{\mathcal{K}}^B_{\mathrm{cdh}}(X)$ is a homotopy equivalence. We conclude that there is a natural isomorphism $\nu_X : KH_p(X) \xrightarrow{\cong} \mathbb{H}^{-p}_{\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}})$ for every $X \in \mathbf{Sch}_k$ and $p \in \mathbb{Z}$. □

Let $\mathcal{BGL}$ be the simplicial presheaf on $\mathbf{Sch}_k$ with $\mathcal{BGL}(X) = \mathrm{colim}_n\, \mathrm{BGL}_n(\mathcal{O}(X))$. It is known (see [Gillet 1981, Proposition 2.15]) that there is a natural sectionwise weak equivalence $\mathcal{K}|_X \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}_\infty \mathcal{BGL}|_X$ in $\mathrm{Pre}(\mathbf{Sch}_{\mathrm{Zar}/k})$ (see Section 6.2), where $\mathbb{Z}_\infty(-)$ is the $\mathbb{Z}$-completion functor of Bousfield–Kan.

To simplify the notation, for any integer $q \in \mathbb{Z}$, we write $\Gamma(q)$ for the presheaf on $\mathbf{Sch}_{\mathrm{Zar}/k}$ given by

$$\Gamma(q)(U) = \begin{cases} \underline{C}_* z_{\mathrm{equi}}(\mathbb{A}^q_k, 0)(U)[-2q] & \text{if } q \geq 0, \\ 0 & \text{if } q < 0. \end{cases}$$

(see Section 3). It is known that the restriction of $\Gamma(q)$ on $\mathbf{Sm}_{\mathrm{Zar}/k}$ is a sheaf (see, for instance, [Mazza et al. 2006, Definition 16.1]). We let $\Gamma(q)[2q] \to \mathcal{K}(\Gamma(q), 2q)$ denote a functorial fibrant replacement of $\Gamma(q)[2q]$ with respect to the injective Zariski local model structure.

It follows from [Asakura and Sato 2015, Section 3.1] that $\mathcal{K}(\Gamma(q), 2q)$ is a cohomology theory on $\mathbf{Sm}_{\mathrm{Zar}/k}$ which satisfies all of the conditions of [Gillet 1981, Definitions 1.1 and 1.2]. We conclude from Gillet's construction [1981, §2, p. 225] that for any $X \in \mathbf{Sm}_{\mathrm{Zar}/k}$, there is a morphism of simplicial presheaves $C_q : \mathcal{BGL}|_X \to \mathcal{K}(\Gamma(q), 2q)|_X$ in $\mathcal{H}^{\mathrm{sml}}_{\mathrm{Zar}}(X)$ which is natural in X. Composing with $\mathcal{K}|_X \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}_\infty \mathcal{BGL}|_X$ and using the isomorphism $\mathbb{Z}_\infty \mathcal{K}(\Gamma(q), 2q) \cong \mathcal{K}(\Gamma(q), 2q)$, we obtain a map

$$C_q : \mathcal{K}|_X \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}_\infty \mathcal{BGL}|_X \to \mathbb{Z} \times \mathcal{K}(\Gamma(q), 2q)|_X \to \mathcal{K}(\Gamma(q), 2q)|_X$$

in $\mathcal{H}^{\mathrm{sml}}_{\mathrm{Zar}}(X)$, where the last arrow is the projection.

Since $\mathcal{K}(\Gamma(q), 2q)$ is fibrant in $\mathrm{Pre}(\mathbf{Sch}_{\mathrm{Zar}/k})$, it follows from [Jardine 2015, Corollary 5.26] that the restriction $\mathcal{K}(\Gamma(q), 2q)|_X$ is fibrant in $\mathrm{Pre}(X_{\mathrm{Zar}})$. Since $\mathcal{K}|_X$ is cofibrant (in our local injective model structure), Gillet's construction [1981, p. 225] yields a map of simplicial presheaves $C_q : \mathcal{K}|_X \to \mathcal{K}(\Gamma(q), 2q)|_X$ in $\mathrm{Pre}(X_{\mathrm{Zar}})$. In particular, a map $\mathcal{K}(X) \to \mathcal{K}(\Gamma(q), 2q)(X)$. Furthermore, the naturality of the construction gives, for any morphism $f : Y \to X$ in $\mathbf{Sm}_k$, a diagram that commutes up to homotopy

$$\begin{tikzcd} \mathcal{K}(X) \arrow[r, "C_q"] \arrow[d, "f^*"'] & \mathcal{K}(\Gamma(q), 2q)(X) \arrow[d, "f^*"] \\ \mathcal{K}(Y) \arrow[r, "C_q"] & \mathcal{K}(\Gamma(q), 2q)(Y) \end{tikzcd} \tag{6.6}$$

(see, for instance, [Asakura and Sato 2015, (5.6.1)]). Equivalently, there is a morphism of simplicial presheaves $C_q : \mathcal{K} \to \mathcal{K}(\Gamma(q), 2q)$ in $\mathcal{H}^{\mathrm{big}}_{\mathrm{Zar}}(k)$ and hence a morphism in $(\mathbf{Sm}_k)_{\mathrm{Nis}}$ (see Section 2.1). Pulling back C_q via the morphism of sites $\pi : (\mathbf{Sch}_k)_{\mathrm{cdh}} \to (\mathbf{Sm}_k)_{\mathrm{Nis}}$ [Jardine 2015, p. 111], and considering the cohomologies of the associated cdh-sheaves, we obtain for any $X \in \mathbf{Sch}_k$, closed subscheme $Z \subseteq X$ and $p, q \geq 0$, the Chern class maps

$$\begin{aligned} c^Z_{X,p,q} : \mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}}) &:= \mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, \boldsymbol{L}\pi^*(\mathcal{K})) \\ &\to \mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, \boldsymbol{L}\pi^*(\mathcal{K}(\Gamma(q), 2q))) \\ &= \mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, C_* z_{\mathrm{equi}}(\mathbb{A}^q_k, 0)_{\mathrm{cdh}}) := H^{2q-p}_Z(X, \mathbb{Z}(q)). \end{aligned} \tag{6.7}$$

It follows from Lemma 6.3 that $\mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}}) = KH^Z_p(X)$, where the $KH^Z(X)$ is the homotopy fiber of the map $KH(X) \to KH(X \setminus Z)$. Let (X, Z) denote the pair consisting of a scheme $X \in \mathbf{Sch}_k$ and a closed subscheme $Z \subseteq X$. A map of pairs $f:(Y,W)\to(X,Z)$ is a morphism $f:Y\to X$ such that $f^{-1}(Z)\subseteq W$. We have then shown the following.

Theorem 6.8. *Let k be a field of characteristic zero. Then for any pair (X,Z) in $\mathbf{Sch}_k$ and for any $p\geq 0$, $q\in\mathbb{Z}$, there are Chern class homomorphisms*

$$c^Z_{X,p,q}:KH^Z_p(X)\to H^{2q-p}_Z(X,\mathbb{Z}(q))$$

such that the composition of $c^X_{X,0,0}$ with $K_0(X)\to KH_0(X)$ is the rank map. For any map of pairs $f:(Y,W)\to(X,Z)$, there is a commutative diagram

$$\begin{tikzcd} KH^Z_p(X) \arrow[r, "c^Z_{X,p,q}"] \arrow[d, "f^*"'] & H^{2q-p}_Z(X,\mathbb{Z}(q)) \arrow[d, "f^*"] \\ KH^W_p(Y) \arrow[r, "c^W_{Y,p,q}"] & H^{2q-p}_W(Y,\mathbb{Z}(q)) \end{tikzcd} \tag{6.9}$$

6.10. ***Chern classes from KH-theory to Deligne cohomology.*** Let $\mathcal{C}_{\mathrm{Zar}}$ denote the category of schemes which are separated and of finite type over $\mathbb{C}$ with the Zariski topology. We denote by $\mathcal{C}_{\mathrm{Nis}}$ the same category but with the Nisnevich topology. Let $\mathcal{C}_{\mathrm{an}}$ denote the category of complex analytic spaces with the analytic topology. There is a morphism of sites $\epsilon:\mathcal{C}_{\mathrm{an}}\to\mathcal{C}_{\mathrm{Zar}}$. For any $q\in\mathbb{Z}$, let $\Gamma(q)$ denote the complex of sheaves on $\mathcal{C}_{\mathrm{Zar}}$ defined as

$$\Gamma(q)=\begin{cases}\Gamma_{\mathcal{D}}(q) & \text{if } q\geq 0,\\ \boldsymbol{R}\epsilon_*((2\pi\sqrt{-1})\mathbb{Z}) & \text{if } q<0,\end{cases} \tag{6.11}$$

where $\Gamma_{\mathcal{D}}(q)$ is the Deligne–Beilinson complex on $\mathcal{C}_{\mathrm{Zar}}$ in the sense of [Esnault and Viehweg 1988]. Then $\Gamma(q)$ is a cohomology theory on $\mathbf{Sm}_{\mathbb{C}}$ satisfying Gillet's conditions for a theory of Chern classes; see, for instance, [Asakura and Sato 2015, Section 3.4]. Applying the argument of Theorem 6.8 in verbatim, we obtain the Chern class homomorphisms

$$c^Z_{X,p,q}:KH^Z_p(X)\to\mathbb{H}^{2q-p}_{Z,\mathrm{cdh}}(X,(\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \tag{6.12}$$

for a pair of schemes (X,Z) in $\mathbf{Sch}_{\mathbb{C}}$ which is natural in (X,Z).

Let us now fix a scheme $X\in\mathbf{Sch}_{\mathbb{C}}$. Recall from [Deligne 1974, §6.2.5–6.2.8] that a smooth proper hypercovering of X is a smooth simplicial scheme $X_\bullet$ with a map of simplicial schemes $p_X:X_\bullet\to X$ such each map $X_i\to X$ is proper and p_X satisfies the universal cohomological descent in the sense of [Deligne 1974]. The resolution of singularities implies that such a hypercovering exists. The Deligne cohomology of X is defined in [Deligne 1974, §5.1.11] to be

$$H^p_{\mathcal{D}}(X,\mathbb{Z}(q)):=\mathbb{H}^p_{\mathrm{Zar}}(X,\boldsymbol{R}p_{X*}\Gamma_{\mathcal{D}}(q))=\mathbb{H}^p_{\mathrm{Zar}}(X_\bullet,\Gamma_{\mathcal{D}}(q)). \tag{6.13}$$

Gillet's theory of Chern classes gives rise to the Chern class homomorphisms

$$c^{Q}_{X,p,q} : K_p(X) \to H^{2q-p}_{\mathcal{D}}(X, \mathbb{Z}(q)) \tag{6.14}$$

for any $X \in \mathbf{Sch}_{\mathbb{C}}$ which is contravariant functorial, where $K_i(X) = \pi_i(\Omega BQP(X))$ is the Quillen K-theory (see, for instance, [Barbieri-Viale et al. 1996, §2.4]). Our objective is to show that these Chern classes actually factor through the natural map $K_*(X) \to KH_*(X)$.

The construction of the Chern classes from KH-theory to the Deligne cohomology (see Theorem 6.20 below) will be achieved by the cdh-sheafification of Gillet's Chern classes at the level of presheaves of simplicial sets, followed by considering the induced maps on the hypercohomologies. Therefore, in order to factor the classical Chern classes $c^{Q}_{X,p,q}$ on Quillen K-theory through KH-theory, we only need to identify the target of the Chern class maps in (6.12) with the Deligne cohomology.

To do this, for any $X \in \mathbf{Sch}_{\mathbb{C}}$ we let $H^*_{\mathrm{an}}(X, \mathcal{F})$ denote the cohomology of the analytic space X_{an} with coefficients in the sheaf $\mathcal{F}$ on $\mathcal{C}_{\mathrm{an}}$. Let $\mathbb{Z} \to \mathrm{Sing}^*$ denote a fibrant replacement of the sheaf $\mathbb{Z}$ on $\mathcal{C}_{\mathrm{an}}$ so that $\boldsymbol{R}\epsilon_*(\mathbb{Z}) \xrightarrow{\cong} \epsilon_*(\mathrm{Sing}^*)$. Set $\mathbb{Z}(q) = (2\pi\sqrt{-1})^q \epsilon_*(\mathrm{Sing}^*) \cong \boldsymbol{R}\epsilon_*(\mathbb{Z})$.

Lemma 6.15. *For any $X \in \mathbf{Sm}_{\mathbb{C}}$, the map $H^p_{\mathrm{an}}(X, \mathbb{Z}) \to \mathbb{H}^p_{\mathrm{cdh}}(X, \mathbb{Z}(q)_{\mathrm{cdh}})$ is an isomorphism.*

Proof. Since $H^p_{\mathrm{an}}(X, \mathbb{Z}) \cong \mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q))$, it is sufficient to show that the map $\mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q)) \to \mathbb{H}^p_{\mathrm{cdh}}(X, \mathbb{Z}(q)_{\mathrm{cdh}})$ is an isomorphism.

Let $\mathcal{C}_{\mathrm{loc}}$ denote the category of schemes which are separated and of finite type over $\mathbb{C}$. We consider $\mathcal{C}_{\mathrm{loc}}$ as a Grothendieck site with coverings given by maps $Y' \to Y$ where the associated map of the analytic spaces is a local isomorphism of the corresponding topological spaces [SGA 4_3 1973, Exposé XI, p. 9]. Since a Nisnevich cover of schemes is a local isomorphism of the associated analytic spaces, there is a commutative diagram of morphisms of sites:

$$\begin{array}{ccc} \mathcal{C}_{\mathrm{loc}} & \xrightarrow{\delta} & \mathcal{C}_{\mathrm{an}} \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ \mathcal{C}_{\mathrm{Nis}} & \xrightarrow{\tau} & \mathcal{C}_{\mathrm{Zar}} \end{array} \tag{6.16}$$

Since every local isomorphism of analytic spaces is refined by open coverings, it is well known that the map $\mathbb{H}^p_{\mathrm{an}}(X, \mathcal{F}^*) \to H^p_{\mathrm{loc}}(X, \mathcal{F}^*)$ is an isomorphism for any complex of sheaves on $\mathcal{C}_{\mathrm{an}}$; see, for instance, [Milne 1980, Proposition 3.3, Theorem 3.12].

We set $(\mathbb{Z}(q))_{\mathrm{Nis}} = \tau^*(\mathbb{Z}(q)) = \nu_* \circ \delta^*(\mathrm{Sing}^*)$. We observe that for every $i \in \mathbb{Z}$, the cohomology sheaf $\mathcal{H}^i$ associated to the complex $\mathbb{Z}(q)$ is isomorphic to the Zariski (or Nisnevich) sheaf on $\mathbf{Sch}_{\mathbb{C}}$ associated to the presheaf $U \mapsto H^i_{\mathrm{an}}(U, \mathbb{Z})$. But this latter presheaf on $\mathbf{Sm}_{\mathbb{C}}$ is homotopy invariant with transfers. It follows from [Suslin and Voevodsky 2000, Corollary 1.1.1] that $\mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q)) \to \mathbb{H}^p_{\mathrm{Nis}}(X, (\mathbb{Z}(q))_{\mathrm{Nis}})$ is an isomorphism. We are thus reduced to showing that for $X \in \mathbf{Sm}_{\mathbb{C}}$, the map $\mathbb{H}^p_{\mathrm{Nis}}(X, (\mathbb{Z}(q))_{\mathrm{Nis}}) \to \mathbb{H}^p_{\mathrm{cdh}}(X, (\mathbb{Z}(q))_{\mathrm{cdh}})$ is an isomorphism.

But this follows again from [Suslin and Voevodsky 2000, Corollary 1.1.1, 5.12.3, Theorem 5.13] because each $\mathcal{H}^i \cong \boldsymbol{R}^i \nu_*(\mathbb{Z})$ is a Nisnevich sheaf on $\mathbf{Sm}_{\mathbb{C}}$ associated to the homotopy invariant presheaf with transfers $U \mapsto H^i_{\mathrm{an}}(U, \mathbb{Z})$. The proof is therefore complete. $\square$

For any $X \in \mathbf{Sch}_{\mathbb{C}}$, there are natural maps

$$H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \cong \mathbb{H}^p_{\mathrm{Zar}}(X_\bullet, \Gamma_{\mathcal{D}}(q)) \to \mathbb{H}^p_{\mathrm{Nis}}(X_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{Nis}}) \to \mathbb{H}^p_{\mathrm{cdh}}(X_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}). \tag{6.17}$$

Lemma 6.18. *For a projective scheme X over $\mathbb{C}$, the map*

$$H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \to \mathbb{H}^p_{\mathrm{cdh}}(X_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}})$$

is an isomorphism.

Proof. Our assumption implies that each component X_p of the simplicial scheme $X_\bullet$ is smooth and projective. Given a complex of sheaves $\mathcal{F}^*_\bullet$ (in the Zariski or cdh-topology), there is a spectral sequence

$$E_1^{p,q} = \mathbb{H}^q_{\mathrm{Zar/cdh}}(X_p, (\mathcal{F}^*_\bullet)_{\mathrm{Zar/cdh}}) \Rightarrow \mathbb{H}^{p+q}_{\mathrm{Zar/cdh}}(X_\bullet, (\mathcal{F}^*_\bullet)_{\mathrm{Zar/cdh}});$$

see, for instance, [Asakura and Sato 2015, Appendix]. Using this spectral sequence and (6.17), it suffices to show that the map $H^p_{\mathrm{Zar}}(X, \Gamma_{\mathcal{D}}(q)) \to \mathbb{H}^p_{\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}})$ is an isomorphism for any smooth projective scheme X over $\mathbb{C}$. For $q \le 0$, this follows from Lemma 6.15. So we assume $q > 0$.

Since X is smooth and projective, the analytic Deligne complex $\mathbb{Z}(q)_{\mathcal{D}}$ is the complex of analytic sheaves $\mathbb{Z}(q) \to \mathcal{O}_{X_{\mathrm{an}}} \to \Omega^1_{X_{\mathrm{an}}} \to \cdots \to \Omega^{q-1}_{X_{\mathrm{an}}}$. In particular, there is a distinguished triangle

$$\boldsymbol{R}\epsilon_*(\Omega^{<q}_{X_{\mathrm{an}}}[-1]) \to \Gamma_{\mathcal{D}}(q) \to \mathbb{Z}(q) \to \boldsymbol{R}\epsilon_*(\Omega^{<q}_{X_{\mathrm{an}}})$$

in the derived category of sheaves on X_{Zar}.

As X is projective, it follows from GAGA that the natural map $\Omega^{<q}_{X/\mathbb{C}} \to \boldsymbol{R}\epsilon_*(\Omega^{<q}_{X_{\mathrm{an}}})$ is an isomorphism in the derived category of sheaves on X_{Zar}. In particular, we get a distinguished triangle in the derived category of sheaves on X_{Zar}:

$$\Omega^{<q}_{X/\mathbb{C}}[-1] \to \Gamma_{\mathcal{D}}(q) \to \mathbb{Z}(q) \to \Omega^{<q}_{X/\mathbb{C}}. \tag{6.19}$$

We thus have a commutative diagram of exact sequences

$$\begin{tikzcd}
\mathbb{H}^{p-1}_{\mathrm{Zar}}(X, \mathbb{Z}(q)) \arrow[r] \arrow[d] & \mathbb{H}^{p-1}_{\mathrm{Zar}}(X, \Omega^{<q}_{X/\mathbb{C}}) \arrow[r] \arrow[d] & H^{p}_{\mathrm{Zar}}(X, \Gamma_{\mathcal{D}}(q)) \arrow[d] \\
\mathbb{H}^{p-1}_{\mathrm{cdh}}(X, (\mathbb{Z}(q))_{\mathrm{cdh}}) \arrow[r] & \mathbb{H}^{p-1}_{\mathrm{cdh}}(X, (\Omega^{<q}_{X/\mathbb{C}})_{\mathrm{cdh}}) \arrow[r] & H^{p}_{\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}})
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & \mathbb{H}^{p}_{\mathrm{Zar}}(X, \mathbb{Z}(q)) \arrow[r] \arrow[d] & \mathbb{H}^{p}_{\mathrm{Zar}}(X, \Omega^{<q}_{X/\mathbb{C}}) \arrow[d] \\
\arrow[r] & \mathbb{H}^{p}_{\mathrm{cdh}}(X, (\mathbb{Z}(q))_{\mathrm{cdh}}) \arrow[r] & \mathbb{H}^{p}_{\mathrm{cdh}}(X, (\Omega^{<q}_{X/\mathbb{C}})_{\mathrm{cdh}})
\end{tikzcd}$$

It follows from Lemma 6.15 that the first and the fourth vertical arrows from the left are isomorphisms. The second and the fifth vertical arrows are isomorphisms by [Cortiñas et al. 2008b, Corollary 2.5]. We conclude that the middle vertical arrow is also an isomorphism and this completes the proof. □

As a combination of Lemma 6.3, (6.14) and Lemma 6.18, we obtain a theory of Chern classes from KH-theory to Deligne cohomology as follows.

Theorem 6.20. *For every projective scheme X over $\mathbb{C}$, there are Chern class homomorphisms*

$$c_{X,p,q} : KH_p(X) \to H^{2q-p}_{\mathcal{D}}(X, \mathbb{Z}(q))$$

such that for any morphism of projective $\mathbb{C}$-schemes $f : Y \to X$, one has

$$f^* \circ c_{X,p,q} = c_{Y,p,q} \circ f^*.$$

Proof. We only need to show that there is a natural isomorphism

$$\alpha_X : \mathbb{H}^{p}_{\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \xrightarrow{\cong} H^{p}_{\mathcal{D}}(X, \mathbb{Z}(q)).$$

Given a morphism of projective $\mathbb{C}$-schemes $f : Y \to X$, there exists a commutative diagram

$$\begin{tikzcd}
Y_\bullet \arrow[r, "f_\bullet"] \arrow[d, "p_Y"'] & X_\bullet \arrow[d, "p_X"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

where the vertical arrows are the simplicial hypercovering maps. In particular, there is a commutative diagram

$$\begin{tikzcd}[column sep=small]
\mathbb{H}^p_{\mathrm{Zar}}(X, \Gamma_{\mathcal{D}}(q)) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathbb{H}^p_{\mathrm{Zar}}(Y, \Gamma_{\mathcal{D}}(q)) \arrow[dr] \arrow[dd] & \\
& \mathbb{H}^p_{\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \arrow[rr] \arrow[dd] & & \mathbb{H}^p_{\mathrm{cdh}}(Y, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \arrow[dd] \\
H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \arrow[rr] \arrow[dr] & & H^p_{\mathcal{D}}(Y, \mathbb{Z}(q)) \arrow[dr] & \\
& \mathbb{H}^p_{\mathrm{cdh}}(X_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \arrow[rr] & & \mathbb{H}^p_{\mathrm{cdh}}(Y_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}})
\end{tikzcd}$$

Using Lemma 6.18, we get a map $\alpha_X : \mathbb{H}^p_{\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \to H^p_{\mathcal{D}}(X, \mathbb{Z}(q))$ such that $f^* \circ \alpha_X = \alpha_Y \circ f^*$ for any $f : Y \to X$ as above. Moreover, we have shown in the proof of Lemma 6.18 that this map is an isomorphism if $X \in \mathbf{Sm}_{\mathbb{C}}$. Since the source as well as the target of α_X satisfy cdh-descent by Lemma 6.18 (see [Suslin and Voevodsky 2000, Lemma 12.1]), we conclude as in the proof of Lemma 6.3 that α_X is an isomorphism for every projective $\mathbb{C}$-scheme X. $\square$

7. Applications II: Intermediate Jacobian and Abel–Jacobi map for singular schemes

Recall that a very important object in the study of the geometric part of motivic cohomology of smooth projective varieties is an intermediate Jacobian. The intermediate Jacobians were defined by Griffiths and they receive the Abel–Jacobi maps from certain subgroups of the geometric part $H^{2*}(X, \mathbb{Z}(*))$ of the motivic cohomology groups.

A special case of these intermediate Jacobians is the Albanese variety of a smooth projective variety. The most celebrated result about the Albanese variety in the context of algebraic cycles is that the Abel–Jacobi map from the group of 0-cycles of degree zero to the Albanese variety is an isomorphism on the torsion subgroups. This theorem of Roitman tells us that the torsion part of the Chow group of 0-cycles on a smooth projective variety over $\mathbb{C}$ can be identified with the torsion subgroup of an abelian variety.

Roitman's torsion theorem has had enormous applications in the theory of algebraic cycles and algebraic K-theory. For example, it was predicted as part of the conjectures of Bloch and Beilinson that the Chow group of 0-cycles on smooth affine varieties of dimension at least two should be torsion-free. This is now a consequence of Roitman's torsion theorem. We hope to use the Roitman's torsion theorem of this paper to answer the analogous question about the motivic cohomology $H^{2d}(X, \mathbb{Z}(d))$ of a d-dimensional singular affine variety in a future project.

It was predicted as part of the relation between algebraic K-theory and motivic cohomology that the Chow group of 0-cycles should be (integrally) a subgroup of the Grothendieck group. This is also now a consequence of Roitman's theorem. We shall prove the analogue of this for singular schemes in the next section. Recall that the Riemann–Roch theorem says that this inclusion of the Chow group inside the Grothendieck group is always true rationally. For applications concerning the relation between Chow groups and étale cohomology, see [Bloch 1979].

In this section, we apply the theory of Chern classes from KH-theory to Deligne cohomology from Section 6 to construct the intermediate Jacobian and Abel–Jacobi map from the geometric part of the motivic cohomology of any singular projective variety over $\mathbb{C}$. In the next section, we shall use the Abel–Jacobi map to prove a Roitman torsion theorem for singular schemes. As another application of our Chern classes and the Roitman torsion theorem, we shall show that the cycle map from the geometric part of motivic cohomology to the KH groups, constructed in Theorem 5.9, is injective for a large class of schemes.

7.1. ***The Abel–Jacobi map.*** In the rest of this section, we consider all schemes over $\mathbb{C}$ and mostly deal with projective schemes. Let X be a projective scheme over $\mathbb{C}$ of dimension d. Let X_{sing} and X_{reg} denote the singular (with the reduced induced subscheme structure) and the smooth loci of X, respectively. Let r denote the number of d-dimensional irreducible components of X. We fix a resolution of singularities $f : \widetilde{X} \to X$ and let $E = f^{-1}(X_{\mathrm{sing}})$ throughout this section. The following is an immediate consequence of the cdh-descent for Deligne cohomology.

Lemma 7.2. *For any integer $q \geq d+1$, one has $H^{q+d+i}_{\mathcal{D}}(X, \mathbb{Z}(q)) = 0$ for $i \geq 1$.*

Proof. If X is smooth, this follows immediately from (6.19). In general, the cdh-descent for Deligne cohomology (see Lemma 6.18 or [Barbieri-Viale et al. 1996, Variant 3.2]) implies that there is an exact sequence

$$H^{q+d+i-1}_{\mathcal{D}}(E, \mathbb{Z}(q)) \to H^{q+d+i}_{\mathcal{D}}(X, \mathbb{Z}(q)) \\ \to H^{q+d+i}_{\mathcal{D}}(\widetilde{X}, \mathbb{Z}(q)) \oplus H^{q+d+i}_{\mathcal{D}}(X_{\mathrm{sing}}, \mathbb{Z}(q)).$$

We conclude the proof by using this exact sequence and induction on $\dim(X)$. $\square$

It follows from the definition of the Deligne cohomology that there is a natural map of complexes $\Gamma_{\mathcal{D}}(q)|_X \to \mathbb{Z}(q)|_X$ (see (6.19)) and in particular, there is a natural map $H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \xrightarrow{\kappa_X} H^p_{\mathrm{an}}(X, \mathbb{Z}(q))$. For any integer $0 \leq q \leq d$, the *intermediate Jacobian* $J^q(X)$ is defined so that we have an exact sequence

$$0 \to J^q(X) \to H^{2q}_{\mathcal{D}}(X, \mathbb{Z}(q)) \xrightarrow{\kappa_X} H^{2q}_{\mathrm{an}}(X, \mathbb{Z}(q)).$$

It follows from Theorem 6.20 that there is a commutative diagram

$$\begin{tikzcd} KH_0(X) \arrow[r, "c_{X,d,0}"] \arrow[d, "f^*"'] & H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d)) \arrow[r, "\kappa_X"] \arrow[d, "f^*"] & H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d)) \arrow[d, "f^*"] \\ KH_0(\widetilde{X}) \arrow[r, "c_{X,d,0}"] & H^{2d}_{\mathcal{D}}(\widetilde{X}, \mathbb{Z}(d)) \arrow[r, "\kappa_{\widetilde{X}}"] & H^{2d}_{\mathrm{an}}(\widetilde{X}, \mathbb{Z}(d)) \end{tikzcd} \tag{7.3}$$

It follows from (6.19) that $\kappa_{\widetilde{X}}$ is surjective. The cdh-descent for the Deligne cohomology and Lemma 7.2 together imply that the middle vertical arrow in (7.3) is surjective. The cdh-excision property of singular cohomology (see [Deligne 1974, 8.3.10]) yields an exact sequence

$$\begin{aligned} H^{2d-1}_{\mathrm{an}}(E, \mathbb{Z}(d)) &\to H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d)) \\ &\to H^{2d}_{\mathrm{an}}(\widetilde{X}, \mathbb{Z}(d)) \oplus H^{2d}_{\mathrm{an}}(X_{\mathrm{sing}}, \mathbb{Z}(d)) \to H^{2d+1}_{\mathrm{an}}(E, \mathbb{Z}(d)). \end{aligned}$$

Since X_{sing} and E are projective schemes of dimension at most $d-1$, it follows that the right vertical arrow in (7.3) is an isomorphism. We conclude that there is a short exact sequence

$$0 \to J^d(X) \to H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d)) \xrightarrow{\kappa_X} H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d)) \to 0. \tag{7.4}$$

A similar Mayer–Vietoris property of the motivic cohomology yields an exact sequence

$$\begin{aligned} H^{2d-1}(E, \mathbb{Z}(d)) &\to H^{2d}(X, \mathbb{Z}(d)) \\ &\to H^{2d}(\widetilde{X}, \mathbb{Z}(d)) \oplus H^{2d}(X_{\mathrm{sing}}, \mathbb{Z}(d)) \to H^{2d+1}(E, \mathbb{Z}(d)). \end{aligned}$$

It follows from Theorem 5.1 that $H^{2d}(X_{\mathrm{sing}}, \mathbb{Z}(d)) = H^{2d+1}(E, \mathbb{Z}(d)) = 0$. In particular, there exists a short exact sequence

$$\begin{aligned} 0 \to \frac{H^{2d-1}(E, \mathbb{Z}(d))}{H^{2d-1}(\widetilde{X}, \mathbb{Z}(d)) + H^{2d-1}(X_{\mathrm{sing}}, \mathbb{Z}(d))} \\ \to H^{2d}(X, \mathbb{Z}(d)) \to H^{2d}(\widetilde{X}, \mathbb{Z}(d)) \to 0. \end{aligned} \tag{7.5}$$

Since the map $H^{2d}(\widetilde{X}, \mathbb{Z}(d)) \cong \mathrm{CH}^d(\widetilde{X}) \to H^{2d}_{\mathrm{an}}(\widetilde{X}, \mathbb{Z}(d))$ is the degree map, which is surjective, we deduce that the "degree" map $H^{2d}(X, \mathbb{Z}(d)) \to H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d))$ is also surjective. We let $A^d(X)$ denote the kernel of this degree map.

It follows from Theorem 6.20 that there is a Chern class map (take $p = 0$) $c_{X,q} : KH_0(X) \to H^{2q}_{\mathcal{D}}(X, \mathbb{Z}(q))$. Theorem 5.9 says that the spectral sequence (4.28) induces a cycle class map $\mathrm{cyc}_{X,0} : H^{2d}(X, \mathbb{Z}(d)) \to KH_0(X)$. Composing the two maps, we get a cycle class map from motivic to Deligne cohomology

$$\tilde{c}^d_X : H^{2d}(X, \mathbb{Z}(d)) \to H^{2q}_{\mathcal{D}}(X, \mathbb{Z}(q)) \tag{7.6}$$

and a commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A^d(X) \arrow[r] \arrow[d, "\mathrm{AJ}^d_X"'] & H^{2d}(X, \mathbb{Z}(d)) \arrow[r] \arrow[d, "\tilde{c}^d_X"] & H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^d(X) \arrow[r] & H^{2q}_{\mathcal{D}}(X, \mathbb{Z}(q)) \arrow[r] & H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d)) \arrow[r] & 0
\end{tikzcd} \tag{7.7}$$

It is known that $J^d(X)$ is a semiabelian variety whose abelian variety quotient is the classical Albanese variety of $\widetilde{X}$; see [Biswas and Srinivas 1999, Theorem 1.1] or [Barbieri-Viale and Srinivas 2001]. The induced map $\mathrm{AJ}^d_X : A^d(X) \to J^d(X)$ is called the *Abel–Jacobi* map for the singular scheme X. We shall prove our main result about this Abel–Jacobi map in the next section. Here, we recall the following description of $J^d(X)$ in terms of 1-motives. Recall from [Barbieri-Viale and Kahn 2016, §12.12] that every projective scheme X of dimension d over $\mathbb{C}$ has a 1-motive $\mathrm{Alb}^+(X)$ associated to it. This is called the cohomological Albanese 1-motive of X. This is a generalization of the Albanese variety of smooth projective schemes.

Theorem 7.8 [Barbieri-Viale and Srinivas 2001, Corollary 3.3.2]. *For a projective scheme X of dimension d over $\mathbb{C}$, there is a canonical isomorphism*

$$J^d(X) \cong \mathrm{Alb}^+(X).$$

7.9. ***Levine–Weibel Chow group and motivic cohomology.*** In order to prove our main theorem of this section, we need to compare the motivic cohomology of singular schemes with another "motivic cohomology", called the (cohomological) Chow-group of 0-cycles, introduced by Levine and Weibel [1985]. We assume throughout our discussion that X is a reduced projective scheme of dimension d over $\mathbb{C}$. However, we remark that the following definition of the Chow group of 0-cycles makes sense over any ground field. Let $\mathcal{Z}_0(X)$ denote the free abelian group on the closed points of X_{reg}.

Definition 7.10. Let C be a pure dimension one reduced scheme in $\mathbf{Sch}_{\mathbb{C}}$. We say that a pair (C, Z) is *a good curve relative to X* if there exists a finite morphism $\nu : C \to X$ and a closed proper subscheme $Z \subsetneq C$ such that the following hold.

(1) No component of C is contained in Z.

(2) $\nu^{-1}(X_{\mathrm{sing}}) \cup C_{\mathrm{sing}} \subseteq Z$.

(3) ν is a local complete intersection morphism at every point $x \in C$ such that $\nu(x) \in X_{\mathrm{sing}}$.

Let (C, Z) be a good curve relative to X and let $\{\eta_1, \dots, \eta_r\}$ be the set of generic points of C. Let $\mathcal{O}_{C,Z}$ denote the semilocal ring of C at $S = Z \cup \{\eta_1, \dots, \eta_r\}$. Let $\mathbb{C}(C)$ denote the ring of total quotients of C and write $\mathcal{O}^{\times}_{C,Z}$ for the group of units in $\mathcal{O}_{C,Z}$. Notice that $\mathcal{O}_{C,Z}$ coincides with $k(C)$ if $|Z| = \varnothing$. As C is Cohen–Macaulay, $\mathcal{O}_{C,Z}^{\times}$ is the subgroup of $k(C)^{\times}$ consisting of those f which are regular and invertible in the local rings $\mathcal{O}_{C,x}$ for every $x \in Z$.

Given any $f \in \mathcal{O}_{C,Z}^{\times} \hookrightarrow \mathbb{C}(C)^{\times}$, we denote by $\mathrm{div}(f)$ the divisor of zeros and poles of f on C, which is defined as follows. If $C_1, \dots, C_r$ are the irreducible components of C, we set $\mathrm{div}(f)$ to be the 0-cycle $\sum_{i=1}^{r} \mathrm{div}(f_i)$, where $(f_1, \dots, f_r) = \theta_{(C,Z)}(f)$ and $\mathrm{div}(f_i)$ is the usual divisor of a rational function on an integral curve in the sense of [Fulton 1998]. Let $\mathcal{Z}_0(C, Z)$ denote the free abelian group on the closed points of $C \setminus Z$. As f is an invertible regular function on C along Z, $\mathrm{div}(f) \in \mathcal{Z}_0(C, Z)$.

By definition, given any good curve (C, Z) relative to X, we have a pushforward map $\mathcal{Z}_0(C, Z) \xrightarrow{\nu_*} \mathcal{Z}_0(X)$. We write $\mathcal{R}_0(C, Z, X)$ for the subgroup of $\mathcal{Z}_0(X)$ generated by the set $\{\nu_*(\mathrm{div}(f)) \mid f \in \mathcal{O}_{C,Z}^{\times}\}$. Let $\mathcal{R}_0^{\mathrm{BK}}(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the image of the map $\mathcal{Z}_0(C, Z, X) \to \mathcal{Z}_0(X)$, where $\mathcal{Z}_0(C, Z, X)$ runs through all good curves. We let $\mathrm{CH}_0^{\mathrm{BK}}(X) = \mathcal{Z}_0(X)/\mathcal{R}_0^{\mathrm{BK}}(X)$.

If we let $\mathcal{R}_0^{\mathrm{LW}}(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the divisors of rational functions on good curves as above, where we further assume that the map $\nu : C \to X$ is a closed immersion, then the resulting quotient group $\mathcal{Z}_0(X)/\mathcal{R}_0^{\mathrm{LW}}(X)$ is denoted by $\mathrm{CH}_0^{\mathrm{LW}}(X)$. There is a canonical surjection $\mathrm{CH}_0^{\mathrm{LW}}(X) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)$. However, we can say more about this map in the present context. This comparison will be an essential ingredient in the proof of Theorem 8.4.

Theorem 7.11. *For a projective scheme X over $\mathbb{C}$, the map $\mathrm{CH}_0^{\mathrm{LW}}(X) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)$ is an isomorphism.*

Proof. By [Binda and Krishna 2018, Lemma 3.13], there are cycle class maps $\mathrm{CH}_0^{\mathrm{LW}}(X) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X) \to K_0(X)$, and one knows from [Levine 1987, Corollary 2.7] that the kernel of the composite map is $(d-1)!$-torsion. It follows that $\mathrm{Ker}(\mathrm{CH}_0^{\mathrm{LW}}(X) \to \mathrm{CH}_0^{\mathrm{BK}}(X))$ is torsion. In particular, it lies in $\mathrm{CH}_0^{\mathrm{LW}}(X)_{\deg 0}$.

On the other hand, it follows from [Binda and Krishna 2018, Proposition 9.7] that the Abel–Jacobi map $\mathrm{CH}_0^{\mathrm{LW}}(X)_{\deg 0} \to J^d(X)$ (see [Biswas and Srinivas 1999, Theorem 1.1]) factors through $\mathrm{CH}_0^{\mathrm{LW}}(X)_{\deg 0} \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)_{\deg 0} \to J^d(X)$. Moreover, it follows from [Biswas and Srinivas 1999, Theorem 1.1] that the composite map is an isomorphism on the torsion subgroups. In particular,

$$\mathrm{Ker}(\mathrm{CH}_0^{\mathrm{LW}}(X)_{\deg 0} \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)_{\deg 0})$$

is torsion-free. It must therefore be zero. □

In the rest of this text, we identify the above two Chow groups for projective schemes over $\mathbb{C}$ and write them as $\mathrm{CH}^d(X)$. There is a degree map

$$\deg_X : \mathrm{CH}^d(X) \to H_{\mathrm{an}}^{2d}(X, \mathbb{Z}(d)) \cong \mathbb{Z}^r.$$

Let $\mathrm{CH}^d(X)_{\deg 0}$ denote the kernel of this degree map. In order to obtain applications of the above Abel–Jacobi map, we connect $\mathrm{CH}^d(X)$ with the motivic cohomology as follows.

Lemma 7.12. *There is a canonical map $\gamma_X : \mathrm{CH}^d(X) \to H^{2d}(X, \mathbb{Z}(d))$ which restricts to a map $\gamma_X : \mathrm{CH}^d(X)_{\deg 0} \to A^d(X)$.*

Proof. We let U denote the smooth locus of X and let $x \in U$ be a closed point. The excision for the local cohomology with support in a closed subscheme tells us that the map

$$\mathbb{H}^0_{\{x\},\mathrm{cdh}}(X, C_* z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}}) \to \mathbb{H}^0_{\{x\},\mathrm{cdh}}(U, C_* z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}})$$

is an isomorphism. On the other hand, the purity theorem for the motivic cohomology of smooth schemes and the isomorphism between the motivic cohomology and higher Chow groups [Voevodsky 2002a] imply that the map

$$\mathbb{H}^0_{\{x\},\mathrm{cdh}}(U, C_* z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}}) \to \mathbb{H}^0_{\mathrm{cdh}}(U, C_* z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}})$$

is same as the map of the Chow groups $\mathbb{Z} \cong \mathrm{CH}_0(\{x\}) \to \mathrm{CH}_0(U)$. In particular, we obtain a map

$$\begin{aligned}\gamma_x : \mathbb{Z} &\to \mathbb{H}^0_{\{x\},\mathrm{cdh}}(X, C_* z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}}) \\ &\to \mathbb{H}^0_{\mathrm{cdh}}(X, C_* z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}}) = H^{2d}(X, \mathbb{Z}(d)).\end{aligned}$$

We let $\gamma_X([x])$ be the image of $1 \in \mathbb{Z}$ under this map. This yields a homomorphism $\gamma_X : \mathcal{Z}_0(X) \to H^{2d}(X, \mathbb{Z}(d))$. We now show that this map kills $\mathcal{R}_0(X)$.

We first assume that X is a reduced curve. In this case, an easy application of the spectral sequence of Theorem 4.27 and the vanishing result of Theorem 5.1 shows that there is a short exact sequence

$$0 \to H^2(X, \mathbb{Z}(1)) \to KH_0(X) \to H^0(X, \mathbb{Z}(0)) \to 0. \tag{7.13}$$

Using $H^0(X, \mathbb{Z}(0)) \xrightarrow{\cong} H^0_{\mathrm{an}}(X, \mathbb{Z})$ and the natural map $K_*(X) \to KH_*(X)$, we have a commutative diagram of the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(X) \arrow[r] & K_0(X) \arrow[r] \arrow[d] & H^0_{\mathrm{an}}(X, \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^2(X, \mathbb{Z}(1)) \arrow[r] & KH_0(X) \arrow[r] & H^0_{\mathrm{an}}(X, \mathbb{Z}) \arrow[r] & 0 \end{tikzcd} \tag{7.14}$$

It follows from [Binda and Krishna 2018, Lemma 3.11] that the map $\mathcal{Z}_0(X) \to K_0(X)$ given by $\mathrm{cyc}_X([x]) = [\mathcal{O}_{\{x\}}] \in K_0(X)$ defines an isomorphism $\mathrm{CH}^1(X) \xrightarrow{\cong} \mathrm{Pic}(X)$. Note that $x \in U$ and hence the class $[\mathcal{O}_{\{x\}}]$ in $K_0(X)$ makes sense. We conclude from this isomorphism and (7.14) that the composite map $\mathcal{Z}_0(X) \to K_0(X) \to KH_0(X)$ has image in $H^2(X, \mathbb{Z}(1))$ and it factors through $\mathrm{CH}^1(X)$.

We now assume $d \geq 2$ and $\nu : (C, Z) \to X$ be a good curve and let $f \in \mathcal{O}^{\times}_{C,Z}$. We need to show that $\gamma_X(\nu_*(\mathrm{div}(f))) = 0$. By [Binda and Krishna 2018, Lemma 3.4], we can assume that ν is an lci morphism. In particular, there is a functorial pushforward map $\nu_* : H^2(C, \mathbb{Z}(1)) \to H^{2d}(X, \mathbb{Z}(d))$ by Corollary 3.6 and [Navarro 2018, Definition 2.32, Theorem 2.33]. We thus have a commutative diagram

$$\begin{tikzcd} \mathcal{Z}_0(C, Z) \arrow[r, "\cong"'] \arrow[d, "\nu_*"'] \arrow[rr, bend left, "\gamma_C"] & \bigoplus_{x \notin Z} H^0(\{x\}, \mathbb{Z}(0)) \arrow[r] \arrow[d, "\nu_*"] & H^2(C, \mathbb{Z}(1)) \arrow[d, "\nu_*"] \\ \mathcal{Z}_0(X) \arrow[r, "\cong"] \arrow[rr, bend right, "\gamma_X"'] & \bigoplus_{x \notin X_{\mathrm{sing}}} H^0(\{x\}, \mathbb{Z}(0)) \arrow[r] & H^{2d}(X, \mathbb{Z}(d)) \end{tikzcd} \tag{7.15}$$

The two horizontal arrows on the right are the pushforward maps on the motivic cohomology since the inclusion $\{x\} \hookrightarrow X$ is an lci morphism for every $x \notin X_{\mathrm{sing}}$. We have shown that $\gamma_C(\mathrm{div}(f)) = 0$ and hence $\gamma_X(\nu_*(\mathrm{div}(f))) = \nu_*(\gamma_C(\mathrm{div}(f))) = 0$. Furthermore, the composite

$$\mathcal{Z}_0(X) \to H^{2d}(X, \mathbb{Z}(d)) \to H^{2d}(\widetilde{X}, \mathbb{Z}(d)) \to H^{2d}_{\mathrm{an}}(\widetilde{X}, \mathbb{Z}(d)) \cong \mathbb{Z}^r$$

is the degree map. This shows that $\gamma_X(\mathcal{Z}_0(X)_{\deg 0}) \subseteq A^d(X)$. $\square$

8. Applications III: Roitman torsion and cycle class map

We now consider a projective scheme X of dimension d over $\mathbb{C}$. Using the map $\gamma_X : \mathrm{CH}^d(X) \to H^{2d}(X, \mathbb{Z}(d))$ and the Abel–Jacobi map AJ^d_X of (7.7), we now prove our main result on the Abel–Jacobi map and Roitman torsion for singular schemes. We shall use the following lemma in the proof.

Lemma 8.1. *Let X be a reduced projective scheme of dimension d over $\mathbb{C}$. There is a cycle class map $\mathrm{cyc}^Q_{X,0} : \mathrm{CH}^d(X) \to K_0(X)$ and a commutative diagram*

$$\begin{tikzcd} \mathrm{CH}^d(X) \arrow[r, "\mathrm{cyc}^Q_{X,0}"] \arrow[d, "\gamma_X"'] & K_0(X) \arrow[d] \\ H^{2d}(X, \mathbb{Z}(d)) \arrow[r, "\mathrm{cyc}_{X,0}"] & KH_0(X) \end{tikzcd} \tag{8.2}$$

Proof. Every closed point $x \in U$ defines the natural map

$$\mathbb{Z} = K_0(\{x\}) = K_0^{\{x\}}(X) \to K_0(X)$$

and hence a class $[\mathcal{O}_{\{x\}}] \in K_0(X)$. This defines a map $\mathrm{cyc}^Q_{X,0} : \mathcal{Z}_0(X) \to K_0(X)$ and it factors through $\mathrm{CH}^d(X)$ by [Levine and Weibel 1985, Proposition 2.1]. Since $\mathrm{CH}^d(X)$ is generated by the closed points in U, it suffices to show that for every closed point $x \in U$, the diagram

$$\begin{tikzcd} K_0^{\{x\}}(X) \arrow[r] \arrow[d, equal] & K_0(X) \arrow[d] \\ KH_0^{\{x\}}(X) \arrow[r] & KH_0(X) \end{tikzcd} \tag{8.3}$$

commutes. But this is clear from the functorial properties of the map of presheaves $K(-) \to KH(-)$ on $\mathbf{Sch}_{\mathbb{C}}$. ☐

We can now prove:

Theorem 8.4. *Let X be a projective scheme over $\mathbb{C}$ of dimension d. Assume that either $d \leq 2$ or X is regular in codimension one. Then there is a semiabelian variety $J^d(X)$ and an Abel–Jacobi map $\mathrm{AJ}^d_X : A^d(X) \to J^d(X)$ which is surjective and whose restriction to the torsion subgroups $\mathrm{AJ}^d_X : A^d(X)_{\mathrm{tors}} \to J^d(X)_{\mathrm{tors}}$ is an isomorphism.*

Proof. We can assume that X is reduced. We first consider the case when X has dimension at most two but has arbitrary singularity. In this case, we only need to prove that AJ^d_X is surjective and its restriction to the torsion subgroups is an isomorphism.

The map AJ^d_X is induced by the Chern class map $c_{X,d,0} : KH_0(X) \to H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d))$ and the composite map $K_0(X) \to KH_0(X) \to H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d))$ is Gillet's Chern class map $C^Q_{X,d,0}$ of (6.14). Composing these maps with the cycle class maps and using Lemma 8.1, we get a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^d(X)_{\deg 0} \arrow[r, "\gamma_X"] \arrow[dr, "\mathrm{AJ}^{d,Q}_X"'] & A^d(X) \arrow[d, "\mathrm{AJ}^d_X"] \\ & J^d(X) \end{tikzcd} \tag{8.5}$$

The map $\mathrm{AJ}^{d,Q}_X$ is surjective and is an isomorphism on the torsion subgroups by [Barbieri-Viale et al. 1996, Main Theorem]. It follows that AJ^d_X is also surjective. To prove that it is an isomorphism on the torsion subgroups, we apply Theorem 7.8 and [Barbieri-Viale and Kahn 2016, Corollary 13.7.5]. It follows from these results that there is indeed an isomorphism $\phi^d_X : J^d(X)_{\mathrm{tor}} \xrightarrow{\cong} A^d(X)_{\mathrm{tor}}$. Since $J^d(X)$ is a semiabelian variety, we know that for any given integer $n \geq 1$, the n-torsion subgroup ${}_nJ^d(X)$ is finite. It follows that ${}_nA^d(X)$ and ${}_nJ^d(X)$ are finite abelian groups of the same order. We conclude that the Abel–Jacobi map $\mathrm{AJ}^d_X : A^d(X) \to J^d(X)$ induces the map $\mathrm{AJ}^d_X : {}_nA^d(X) \to {}_nJ^d(X)$ between finite abelian groups which have same order. Therefore, this map is an isomorphism if and only if it is a surjection. But this is true by (8.5) because we have seen above that the composite map $\mathrm{AJ}^{d,Q}_X$

is an isomorphism between the n-torsion subgroups. Since $n \geq 1$ is arbitrary in this argument, we conclude the proof of the theorem.

We now consider the case when X has arbitrary dimension but is regular in codimension one. Let $f : \widetilde{X} \to X$ be a resolution of singularities of X. It is then known that $J^d(X) \cong J^d(\widetilde{X}) = \mathrm{Alb}(\widetilde{X})$; see [Mallick 2009, Remark 2, p. 505]. We have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^d(X)_{\deg 0} \arrow[r, "\gamma_X"] \arrow[dr, "\mathrm{AJ}^{\mathrm{LW}}_X"'] & A^d(X) \arrow[r, two heads, "f^*"] \arrow[d, dotted, "\mathrm{AJ}^d_X"] & A^d(\widetilde{X}) \arrow[d, two heads, "\mathrm{AJ}^d_{\widetilde{X}}"] \\ & J^d(X) \arrow[r, "\cong"] & J^d(\widetilde{X}) \end{tikzcd} \tag{8.6}$$

Since the lower horizontal arrow in this diagram is an isomorphism, it uniquely defines the Abel–Jacobi map AJ^d_X. The map $f^* \circ \gamma_X$ is known to be surjective by the moving lemma for 0-cycles on smooth schemes. In particular, f^* is surjective. The map $\mathrm{AJ}^d_{\widetilde{X}}$ is also known to be surjective. It follows that AJ^d_X is surjective.

To prove that this is an isomorphism on the torsion subgroups, we can argue exactly as in the first case of the theorem. This reduces us to showing that AJ^d_X is surjective on the n-torsion subgroups for every given integer $n \geq 1$. But this follows because $\mathrm{AJ}^{\mathrm{LW}}_X$ (and also $\mathrm{AJ}^d_{\widetilde{X}}$) is an isomorphism on the n-torsion subgroups by [Biswas and Srinivas 1999, Theorem 1.1], finishing the proof of the theorem. □

Remark 8.7. For arbitrary $d \geq 1$, the map $\mathrm{AJ}^{d,Q}_X$ in (8.5) is known to be an isomorphism only up to multiplication by $(d-1)!$. This prevents us from extending Theorem 8.4 to higher dimensions if X has singularities in codimension one. We also warn the reader that unlike $\mathrm{AJ}^{d,Q}_X$ in (8.5), the map $\mathrm{AJ}^{\mathrm{LW}}_X$ in (8.6) is not defined via the Chern class map on $K_0(X)$. These maps coincide only up to multiplication by $(d-1)!$.

8.8. ***Injectivity of the cycle class map.*** Like the case of smooth schemes, the Roitman torsion theorem for singular schemes has many potential applications. Here, we use this to prove our next main result of this section. It was shown by Levine [1987, Theorem 3.2] that for a smooth projective scheme X of dimension d over $\mathbb{C}$, the cycle class map $H^{2d}(X, \mathbb{Z}(d)) \to K_0(X)$ (see (5.11)) is injective. We generalize this to singular schemes as follows.

Theorem 8.9. *Let X be a projective scheme of dimension d over $\mathbb{C}$. Assume that either $d \leq 2$ or X is regular in codimension one. Then the cycle class map $\mathrm{cyc}_0 : H^{2d}(X, \mathbb{Z}(d)) \to KH_0(X)$ is injective.*

Proof. We note that $\mathrm{cyc}_0 : H^{2d}(X, \mathbb{Z}(d)) \to KH_0(X)$ is induced by the spectral sequences (4.28) and (5.11), both of which degenerate with rational coefficients. In particular, $\mathrm{Ker}(\mathrm{cyc}_0)$ is a torsion group.

On the other hand, if $\dim(X) \le 2$, (7.7) and Theorem 8.4 tell us that the composite map $\tilde{c}_X^d : H^{2d}(X, \mathbb{Z}(d)) \xrightarrow{\mathrm{cyc}_0} KH_0(X) \xrightarrow{c_{X,0,d}} H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d))$ is an isomorphism on the torsion subgroups. We must therefore have $\mathrm{Ker}(\mathrm{cyc}_0) = 0$.

If X is regular in codimension one, we let $\widetilde{X} \to X$ be a resolution of singularities and consider the commutative diagram

$$\begin{array}{ccc} H^{2d}(X, \mathbb{Z}(d)) & \xrightarrow{\mathrm{cyc}_{X,0}} & KH_0(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^{2d}(\widetilde{X}, \mathbb{Z}(d)) & \xrightarrow{\mathrm{cyc}_{\widetilde{X},0}} & K_0(\widetilde{X}) \end{array}$$

We have shown in the proof of Theorem 8.4 that the left vertical arrow is an isomorphism on the torsion subgroups. The bottom horizontal arrow is injective by [Levine 1987, Theorem 3.2]. It follows that $\mathrm{cyc}_{X,0}$ is injective on the torsion subgroup. We must therefore have $\mathrm{Ker}(\mathrm{cyc}_{X,0}) = 0$. This finishes the proof. □

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