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Splitting the relative assembly map, Nil-terms and involutions

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We show that the relative Farrell–Jones assembly map from the family of finite subgroups to the family of virtually cyclic subgroups for algebraic K-theory is split injective in the setting where the coefficients are additive categories with group action. This generalizes a result of Bartels for rings as coefficients. We give an explicit description of the relative term. This enables us to show that it vanishes rationally if we take coefficients in a regular ring. Moreover, it is, considered as a $\mathbb{Z}[\mathbb{Z}/2]$-module by the involution coming from taking dual modules, an extended module and in particular all its Tate cohomology groups vanish, provided that the infinite virtually cyclic subgroups of type I of G are orientable. The latter condition is for instance satisfied for torsionfree hyperbolic groups.

Introduction

0A. ***Motivation.*** The *K-theoretic Farrell–Jones conjecture* [1993, 1.6 on page 257 and 1.7 on page 262] for a group G and a ring R predicts that the *assembly map*

$$\mathrm{asmb}_n : H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_R) \to H_n^G(G/G; \boldsymbol{K}_R) = K_n(RG)$$

is an isomorphism for all $n \in \mathbb{Z}$. Here $\underline{\underline{E}}G = E_{\mathcal{VC}}(G)$ is the *classifying space for the family $\mathcal{VC}$ of virtually cyclic subgroups* and $H_n^G(-; \boldsymbol{K}_R^G)$ is the *G-homology theory* associated to a specific covariant functor $\boldsymbol{K}_R^G$ from the orbit category $\mathsf{Or}(G)$ to the category $\mathsf{Spectra}$ of spectra. It satisfies $H_n^G(G/H; \boldsymbol{K}_R^G) = \pi_n(\boldsymbol{K}^G(G/H)) = K_n(RH)$ for any subgroup $H \subseteq G$ and $n \in \mathbb{Z}$. The assembly map is induced by the projection $\underline{\underline{E}}G \to G/G$. More information about the Farrell–Jones conjecture and the classifying spaces for families can be found for instance in the survey articles [Lück 2005; Lück and Reich 2005].

Let $\underline{E}G = E_{\mathcal{F}\mathrm{in}}(G)$ be the *classifying space for the family $\mathcal{F}in$ of finite subgroups*, sometimes also called the *classifying space for proper G-actions*. The G-map $\underline{E}G \to \underline{\underline{E}}G$, which is unique up to G-homotopy, induces a so-called *relative*

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assembly map

$$\overline{\mathrm{asmb}}_n : H_n^G(\underline{E}G; \boldsymbol{K}_R) \to H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_R).$$

The main result of a paper by Bartels [2003, Theorem 1.3] says that $\overline{\mathrm{asmb}}_n$ is split injective for all $n \in \mathbb{Z}$.

In this paper we improve on this result in two different directions: First, we generalize from the context of rings R to the context of additive categories $\mathcal{A}$ with G-action. This improvement allows us to consider twisted group rings and involutions twisted by an orientation homomorphism $G \to \{\pm 1\}$; moreover one obtains better inheritance properties and gets fibered versions for free.

Secondly, we give an explicit description of the relative term in terms of so-called NK-spectra. This becomes relevant for instance in the study of the involution on the cokernel of the relative assembly map induced by an involution of $\mathcal{A}$. In more detail, we prove:

0B. ***Splitting the relative assembly map.*** Our main splitting result is:

Theorem 0.1 (splitting the K-theoretic assembly map from $\mathcal{F}\mathrm{in}$ to $\mathcal{VC}$). *Let G be a group and let $\mathcal{A}$ be an additive G-category. Let n be any integer.*

Then the relative K-theoretic assembly map

$$\overline{\mathrm{asmb}}_n : H_n^G(\underline{E}G; \boldsymbol{K}_{\mathcal{A}}^G) \to H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G)$$

is split injective. In particular we obtain a natural splitting

$$H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{E}G; \boldsymbol{K}_{\mathcal{A}}^G) \oplus H_n^G(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G).$$

Moreover, *there exists an* $\mathsf{Or}(G)$*-spectrum* $\boldsymbol{NK}_{\mathcal{A}}^G$ *and a natural isomorphism*

$$H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G).$$

Here $E_{\mathcal{VC}_I}(G)$ denotes the classifying space for the family of virtually cyclic subgroups of type I; see Section 1. The proof will appear in Section 7. The point is that, instead of considering $\boldsymbol{K}_R^G$ for a ring R, we can treat the more general setup $\boldsymbol{K}_{\mathcal{A}}^G$ for an additive G-category $\mathcal{A}$, as explained in [Bartels and Lück 2010; Bartels and Reich 2007]. (One recovers the case of a ring R if one considers for $\mathcal{A}$ the category R-FGF of finitely generated free R-modules with the trivial G-action. Notice that we tacitly always apply idempotent completion to the additive categories before taking K-theory.) Whereas in [Bartels 2003, Theorem 1.3] just a splitting is constructed, we construct explicit $\mathsf{Or}(G)$-spectra $\boldsymbol{NK}_{\mathcal{A}}^G$ and identify the relative terms. This is crucial for the following results.

0C. *Involutions and vanishing of Tate cohomology.* We will prove in Section 8C:

Theorem 0.2 (the relative term is induced). *Let G be a group and let $\mathcal{A}$ be an additive G-category with involution. Suppose that the virtually cyclic subgroups of type I of G are orientable* (see Definition 8.5).

Then the $\mathbb{Z}/2$-module $H_n(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G)$ is isomorphic to $\mathbb{Z}[\mathbb{Z}/2] \otimes_{\mathbb{Z}} A$ for some $\mathbb{Z}$-module A.

In [Farrell et al. 2016] we consider the conclusion of Theorem 0.2 that the Tate cohomology groups $\widehat{H}^i(\mathbb{Z}/2, H_n(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G))$ vanish for all $i, n \in \mathbb{Z}$ if the virtually cyclic subgroups of type I of G are orientable. In general the Tate spectrum of the involution on the Whitehead spectrum plays a role in the study of automorphisms of manifolds (see [Weiss and Williams 2000, Section 4], for example).

0D. *Rational vanishing of the relative term.*

Theorem 0.3. *Let G be a group and let R be a regular ring.*

Then the relative assembly map

$$\overline{\operatorname{asmb}}_n : H_n^G(\underline{E}G; \boldsymbol{K}_R^G) \to H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_R^G)$$

is rationally bijective for all $n \in \mathbb{Z}$.

If $R = \mathbb{Z}$ and $n \leq -1$, then, by [Farrell and Jones 1995], the relative assembly map $H_n^G(\underline{E}G; \boldsymbol{K}_{\mathbb{Z}}^G) \xrightarrow{\cong} H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_{\mathbb{Z}}^G)$ is an isomorphism.

In Section 10, we briefly discuss further computations of the relative term $H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G) \cong H_n^G(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G)$.

0E. *A fibered case.* In Section 11 we discuss a fibered situation which will be relevant for [Farrell et al. 2016] and can be handled by our general treatment for additive G-categories.

1. Virtually cyclic groups

A virtually cyclic group V is called *of type I* if it admits an epimorphism to the infinite cyclic group, and *of type II* if it admits an epimorphism onto the infinite dihedral group. The statements appearing in the next lemma are well-known; we insert a proof for the reader's convenience.

Lemma 1.1. *Let V be an infinite virtually cyclic group.*

(i) *V is either of type I or of type II.*

(ii) *The following assertions are equivalent*:

   (a) *V is of type I*;

   (b) *$H_1(V)$ is infinite*;

(c) $H_1(V)/\operatorname{tors}(V)$ *is infinite cyclic*;
(d) *the center of* V *is infinite.*

(iii) *There exists a unique maximal normal finite subgroup* $K_V \subseteq V$, *i.e.*, K_V *is a finite normal subgroup and every normal finite subgroup of* V *is contained in* K_V.

(iv) *Let* $Q_V := V/K_V$. *Then we obtain a canonical exact sequence*

$$1 \to K_V \xrightarrow{i_V} V \xrightarrow{p_V} Q_V \to 1.$$

Moreover, Q_V *is infinite cyclic if and only if* V *is of type I*, *and* Q_V *is isomorphic to the infinite dihedral group if and only if* V *is of type II.*

(v) *Let* $f : V \to Q$ *be any epimorphism onto the infinite cyclic group or onto the infinite dihedral group. Then the kernel of* f *agrees with* K_V.

(vi) *Let* $\phi : V \to W$ *be a homomorphism of infinite virtually cyclic groups with infinite image. Then* ϕ *maps* K_V *to* K_W *and we obtain the canonical commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_V & \xrightarrow{i_V} & V & \xrightarrow{p_V} & Q_V & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi_K} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_Q} & & \\
1 & \longrightarrow & K_W & \xrightarrow{i_W} & W & \xrightarrow{p_W} & Q_W & \longrightarrow & 1
\end{array}$$

with injective ϕ_Q.

Proof. (ii) If V is of type I, then we obtain epimorphisms

$$V \to H_1(V) \to H_1(V)/\operatorname{tors}(H_1(V)) \to \mathbb{Z}.$$

The kernel of $V \to \mathbb{Z}$ is finite, since for an exact sequence $1 \to \mathbb{Z} \xrightarrow{i} V \xrightarrow{q} H \to 1$ with finite H the composite of $V \to \mathbb{Z}$ with i is injective and hence the restriction of q to the kernel of $V \to \mathbb{Z}$ is injective. This implies that $H_1(V)$ is infinite and $H_1(V)/\operatorname{tors}(H_1(V))$ is infinite cyclic. If $H_1(V)/\operatorname{tors}(H_1(V))$ is infinite cyclic or if $H_1(V)$ is infinite, then $H_1(V)$ surjects onto $\mathbb{Z}$ and hence so does V. This shows (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c).

Consider the exact sequence $1 \to \operatorname{cent}(V) \to V \to V/\operatorname{cent}(V) \to 1$, where $\operatorname{cent}(V)$ is the center of V. Suppose that $\operatorname{cent}(V)$ is infinite. Then $V/\operatorname{cent}(V)$ is finite and the Lyndon–Serre spectral sequence yields an isomorphism $\operatorname{cent}(V)\otimes_{\mathbb{Z}}\mathbb{Q} \to H_1(V;\mathbb{Q})$. Hence $H_1(V)$ is infinite. This shows (d) $\Longrightarrow$ (b).

Suppose that V is of type I. Choose an exact sequence $1 \to K \to V \to \mathbb{Z} \to 1$ with finite K. Let $v \in V$ be an element which is mapped to a generator of $\mathbb{Z}$. Then conjugation with v induces an automorphism of K. Since K is finite, we can find a natural number k such that conjugation with v^k induces the identity on K. One easily checks that v^k belongs to the center of V and v has infinite order. This shows (a) $\Longrightarrow$ (d) and finishes the proof of assertion (ii).

(iii) If K_1 and K_2 are two finite normal subgroups of V, then

$$K_1 \cdot K_2 := \{v \in V \mid v = k_1 k_2 \text{ for some } k_1 \in K_1 \text{ and } k_2 \in K_2\}$$

is a finite normal subgroup of V. Hence we are left to show that V has only finitely many different finite normal subgroups.

To see this, choose an exact sequence $1 \to \mathbb{Z} \xrightarrow{i} V \xrightarrow{f} H \to 1$ for some finite group H. The map f induces a map from the finite normal subgroups of V to the normal subgroups of H; we will show that it is an injection. Let $t \in V$ be the image under i of some generator of $\mathbb{Z}$ and consider two finite normal subgroups K_1 and K_2 of V with $f(K_1) = f(K_2)$. Consider $k_1 \in K_1$. We can find $k_2 \in K_2$ and $n \in \mathbb{Z}$ with $k_2 = k_1 \cdot t^n$. Then t^n belongs to the finite normal subgroup $K_1 \cdot K_2$. This implies $n = 0$ and hence $k_1 = k_2$. This shows $K_1 \subseteq K_2$. By symmetry we get $K_1 = K_2$. Since H contains only finitely many subgroups, we conclude that there are only finitely many different finite normal subgroups in V. Now assertion (iii) follows.

(i) and (iv) Let V be an infinite virtually cyclic group. Then Q_V is an infinite virtually cyclic subgroup which does not contain a nontrivial finite normal subgroup. There exists an exact sequence $1 \to \mathbb{Z} \xrightarrow{i} Q_V \xrightarrow{f} H \to 1$ for some finite group H. There exists a subgroup of index at most two $H' \subseteq H$ such that the conjugation action of H on $\mathbb{Z}$ restricted to H' is trivial. Put $Q'_V = f^{-1}(H')$. Then the center of Q'_V contains $i(\mathbb{Z})$ and hence is infinite. By assertion (ii) we can find an exact sequence $1 \to K \to Q'_V \xrightarrow{f} \mathbb{Z} \to 1$ with finite K. The group Q'_V contains a unique maximal normal finite subgroup K' by assertion (iii). This implies that $K' \subseteq Q'_V$ is characteristic. Since Q'_V is a normal subgroup of Q_V, $K' \subseteq Q_V$ is a normal subgroup and therefore K' is trivial. Hence Q'_V contains no nontrivial finite normal subgroup. This implies that Q'_V is infinite cyclic. Since Q'_V is a normal subgroup of index 2 in Q_V and Q_V contains no nontrivial finite normal subgroup, Q_V is infinite cyclic or D_∞.

In particular we see that every infinite virtual cyclic group is of type I or of type II. It remains to show that an infinite virtually cyclic group V which is of type II cannot be of type I. If $1 \to K \to V \to D_\infty \to 1$ is an extension with finite K, then we obtain from the Lyndon–Serre spectral sequence an exact sequence $H_1(K) \otimes_{\mathbb{Z}Q} \mathbb{Z} \to H_1(V) \to H_1(D_\infty)$. Hence $H_1(V)$ is finite, since both $H_1(D_\infty)$ and $H_1(K)$ are finite. We conclude from assertion (ii) that V is not of type I. This finishes the proof of assertions (i) and (iv).

(v) Since V is virtually cyclic, the kernel of f is finite. Since Q does not contain a nontrivial finite normal subgroup, every normal finite subgroup of V is contained in the kernel of f. Hence $\ker(f)$ is the unique maximal finite normal subgroup of V.

(vi) Since K_W is finite and the image of ϕ is by assumption infinite, the composite $p_W \circ \phi : V \to Q_W$ has infinite image. Since Q_W is isomorphic to $\mathbb{Z}$ or D_∞, the same is true for the image of $p_W \circ \phi : V \to Q_W$. By assertion (v) the kernel of $p_W \circ \phi : V \to Q_W$ is K_V. Hence $\phi(K_V) \subseteq K_W$ and ϕ induces maps ϕ_K and ϕ_Q making the diagram appearing in assertion (vi) commutative. Since the image of $p_W \circ \phi : V \to Q_W$ is infinite, $\phi_Q(Q_V)$ is infinite. This implies that ϕ_Q is injective since both Q_V and Q_W are isomorphic to D_∞ or $\mathbb{Z}$. This finishes the proof of Lemma 1.1. □

2. Some categories attached to homogeneous spaces

Let G be a group and let S be a G-set, for instance a homogeneous space G/H. Let $\mathcal{G}^G(S)$ be the associated *transport groupoid*. Objects are the elements in S. The set of morphisms from s_1 to s_2 consists of those elements $g \in G$ for which $gs_1 = s_2$. Composition is given by the group multiplication in G. Obviously $\mathcal{G}^G(S)$ is a connected groupoid if G acts transitively on S. A G-map $f : S \to T$ induces a functor $\mathcal{G}^G(f) : \mathcal{G}^G(S) \to \mathcal{G}^G(T)$ by sending an object $s \in S$ to $f(s)$ and a morphism $g : s_1 \to s_2$ to the morphism $g : f(s_1) \to f(s_2)$. We mention that for two objects s_1 and s_2 in $\mathcal{G}^G(S)$ the induced map $\operatorname{mor}_{\mathcal{G}^G(S)}(s_1, s_2) \to \operatorname{mor}_{\mathcal{G}^G(T)}(f(s_1), f(s_2))$ is injective.

A functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ of categories is called an *equivalence* if there exists a functor $F' : \mathcal{C}_1 \to \mathcal{C}_0$ with the property that $F' \circ F$ is naturally equivalent to the identity functor $\mathrm{id}_{\mathcal{C}_0}$ and $F \circ F'$ is naturally equivalent to the identity functor $\mathrm{id}_{\mathcal{C}_1}$. A functor F is a natural equivalence if and only if it is *essentially surjective* (i.e., it induces a bijection on the isomorphism classes of objects) and it is *full* and *faithful*, (i.e., for any two objects c, d in $\mathcal{C}_0$ the induced map $\operatorname{mor}_{\mathcal{C}_0}(c, d) \to \operatorname{mor}_{\mathcal{C}_1}(F(c), F(d))$ is bijective).

Given a monoid M, let $\widehat{M}$ be the category with precisely one object and M as the monoid of endomorphisms of this object. For any subgroup H of G, the inclusion

$$e(G/H) : \widehat{H} \to \mathcal{G}^G(G/H), \quad g \mapsto (eH \xrightarrow{g} eH)$$

(where $e \in G$ is the unit element), is an equivalence of categories, whose inverse sends $g : g_1H \to g_2H$ to $g_2^{-1}gg_1 \in G$.

Now fix an infinite virtually cyclic subgroup $V \subseteq G$ of type I. Then Q_V is an infinite cyclic group. Let $\operatorname{gen}(Q_V)$ be the set of generators. Given a generator $\sigma \in \operatorname{gen}(Q_V)$, define $Q_V[\sigma]$ to be the submonoid of Q_V consisting of elements of the form σ^n for $n \in \mathbb{Z}, n \geq 0$. Let $V[\sigma] \subseteq V$ be the submonoid given by $p_V^{-1}(Q_V[\sigma])$. Let $\mathcal{G}^G(G/V)[\sigma]$ be the subcategory of $\mathcal{G}^G(G/V)$ whose objects are the objects in

$\mathcal{G}^G(G/V)$ and whose morphisms $g : g_1V \to g_2V$ satisfy $g_2^{-1}gg_1 \in V[\sigma]$. Notice that $\mathcal{G}^G(G/V)[\sigma]$ is not a groupoid anymore, but any two objects are isomorphic. Let $\mathcal{G}^G(G/V)_K$ be the subcategory of $\mathcal{G}^G(G/V)$ whose objects are the objects in $\mathcal{G}^G(G/V)$ and whose morphisms $g : g_1V \to g_2V$ satisfy $g_2^{-1}gg_1 \in K_V$. Obviously $\mathcal{G}^G(G/V)_K$ is a connected groupoid and a subcategory of $\mathcal{G}^G(G/V)[\sigma]$.

We obtain the commutative diagram of categories

$$\begin{tikzcd}
\mathcal{G}^G(G/V)[\sigma] \arrow[r, "j(G/V)[\sigma]"'] & \mathcal{G}^G(G/V) \\
\widehat{V[\sigma]} \arrow[u, "e(G/V)[\sigma]", "\cong"'] \arrow[r, "\widehat{j_V[\sigma]}"'] & \widehat{V} \arrow[u, "\cong", "e(G/V)"']
\end{tikzcd} \tag{2.1}$$

whose horizontal arrows are induced by the obvious inclusions and whose left vertical arrow is the restriction of $e(G/V)$ (and is also an equivalence of categories). The functor $e(G/V)$ also restricts to an equivalence of categories

$$e(G/V)_K : \widehat{K_V} \xrightarrow{\simeq} \mathcal{G}^G(G/V)_K. \tag{2.2}$$

Remark. The relation between the categories $\widehat{K_V}$, $\widehat{V[\sigma]}$ and $\widehat{V}$ and the categories $\mathcal{G}^G(G/V)_K$, $\mathcal{G}^G(G/V)[\sigma]$ and $\mathcal{G}^G(G/V)$ is analogous to the relation between the fundamental group of a path connected space and its fundamental groupoid.

Let $\overline{\sigma} \in V$ be any element which is mapped under the projection $p_V : V \to Q_V$ to the fixed generator σ. Right multiplication with $\overline{\sigma}$ induces a G-map $R_\sigma : G/K_V \to G/K_V,\ gK_V \mapsto g\overline{\sigma}K_V$. One easily checks that R_σ depends only on σ and is independent of the choice of $\overline{\sigma}$. Let $\mathrm{pr}_V : G/K_V \to G/V$ be the projection. We obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{G}^G(G/K_V) \arrow[rr, "R_\sigma"] \arrow[dr, "\mathcal{G}^G(\mathrm{pr}_V)"'] & & \mathcal{G}^G(G/K_V) \arrow[dl, "\mathcal{G}^G(\mathrm{pr}_V)"] \\
& \mathcal{G}^G(G/V) &
\end{tikzcd} \tag{2.3}$$

3. Homotopy colimits of $\mathbb{Z}$-linear and additive categories

Homotopy colimits of additive categories have been defined for instance in [Bartels and Lück 2010, Section 5]. Here we review their definition and describe some properties, first in the setting of $\mathbb{Z}$-linear categories.

Recall that a $\mathbb{Z}$-linear category is a category where all Hom-sets are provided with the structure of abelian groups and composition is bilinear. Denote by $\mathbb{Z}$-$\mathcal{C}at$ the category whose objects are $\mathbb{Z}$-linear categories and whose morphisms are additive functors between them. Given a collection of $\mathbb{Z}$-linear categories $(\mathcal{A}_i)_{i\in I}$, their coproduct $\coprod_{i\in I} \mathcal{C}_i$ in $\mathbb{Z}$-$\mathcal{C}at$ exists and has the following explicit description:

Objects are pairs (i, X) where $i \in I$ and $X \in \mathcal{A}_i$. The abelian group of morphisms $(i, X) \to (j, Y)$ is nonzero only if $i = j$, in which case it is $\mathrm{mor}_{\mathcal{A}_i}(X, Y)$.

Let $\mathcal{C}$ be a small category. Given a contravariant functor $F : \mathcal{C} \to \mathbb{Z}\text{-}\mathcal{C}\mathfrak{at}$, its *homotopy colimit* (see [Thomason 1979], for instance)

$$\int_{\mathcal{C}} F \tag{3.1}$$

is the $\mathbb{Z}$-linear category obtained from the coproduct $\coprod_{c \in \mathcal{C}} F(c)$ by adjoining morphisms

$$T_f : (d, f^*X) \to (c, X)$$

for each $(c, X) \in \coprod_{c \in \mathcal{C}} F(c)$ and each morphism $f : d \to c$ in $\mathcal{C}$. (Here we write f^*X for $F(f)(X)$.) They are subject to the relations that $T_{\mathrm{id}} = \mathrm{id}$ and that all possible diagrams

$$\begin{tikzcd}
(e, g^*f^*X) \arrow[r, "T_g"] \arrow[dr, "T_{f\circ g}"'] & (d, f^*X) \arrow[d, "T_f"] \\
& (c, X)
\end{tikzcd}
\qquad
\begin{tikzcd}
(d, f^*X) \arrow[r, "T_f"] \arrow[d, "f^*u"] & (c, X) \arrow[d, "u"] \\
(d, f^*Y) \arrow[r, "T_f"] & (c, Y)
\end{tikzcd}$$

are to be commutative.

Hence, a morphism in $\int_{\mathcal{C}} F$ from (x, A) to (y, B) can be uniquely written as a sum

$$\sum_{f \in \mathrm{mor}_{\mathcal{C}}(x,y)} T_f \circ \phi_f, \tag{3.2}$$

where $\phi_f : A \to f^*B$ is a morphism in $F(x)$ and all but finitely many of the morphisms ϕ_f are zero. The composition of two such morphisms can be determined by the distributivity law and the rule

$$(T_f \circ \phi) \circ (T_g \circ \psi) = T_{f\circ g} \circ (g^*\phi \circ \psi),$$

which just follows from the fact that both upper squares are commutative.

Using this description, it follows that the homotopy colimit has the following universal property for additive functors $\int_{\mathcal{C}} F \to \mathcal{A}$ into some other $\mathbb{Z}$-linear category $\mathcal{A}$: Suppose that we are given additive functors $j_c : F(c) \to \mathcal{A}$ for each $c \in \mathcal{C}$ and morphisms $S_f : j_d(f^*X) \to j_c(X)$ for each $X \in F(c)$ and each $f : d \to c$ in $\mathcal{C}$. If $S_{\mathrm{id}} = \mathrm{id}$ and all possible diagrams

$$\begin{tikzcd}
j_c(g^*f^*X) \arrow[r, "S_g"] \arrow[dr, "S_{f\circ g}"'] & j_d(f^*X) \arrow[d, "S_f"] \\
& j_c(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
j_d(f^*X) \arrow[r, "S_f"] \arrow[d, "j_d(f^*u)"] & j_c(X) \arrow[d, "j_c(u)"] \\
j_d(f^*Y) \arrow[r, "S_f"] & j_c(Y)
\end{tikzcd}$$

are commutative, then this data specifies an additive functor $\int_{\mathcal{C}} F \to \mathcal{A}$ by sending T_f to S_f.

The homotopy colimit is functorial in F. Namely, if $S : F_0 \to F_1$ is a natural transformation of contravariant functors $\mathcal{C} \to \mathbb{Z}\text{-}\mathsf{Cat}$, then it induces an additive functor

$$\int_{\mathcal{C}} S : \int_{\mathcal{C}} F_0 \to \int_{\mathcal{C}} F_1 \tag{3.3}$$

of $\mathbb{Z}$-linear categories. It is defined using the universal property by sending $F_0(c)$ to $F_1(c) \subset \int_{\mathcal{C}} F_1$ via S and "sending T_f to T_f". In more detail, the image of $T_f : (c, f^*(X)) \to (d, X)$ in $\int_{\mathcal{C}} F_0$ is given by $T_f : (c, f^*(S(X))) \to (d, S(X))$ in $\int_{\mathcal{C}} F_1$. Obviously we have, for $S_1 : F_0 \to F_1$ and $S_2 : F_1 \to F_2$,

$$\left(\int_{\mathcal{C}} S_2\right) \circ \left(\int_{\mathcal{C}} S_1\right) = \int_{\mathcal{C}} (S_2 \circ S_1), \tag{3.4}$$

$$\int_{\mathcal{C}} \mathrm{id}_F = \mathrm{id}_{\int_{\mathcal{C}} F}. \tag{3.5}$$

The construction is also functorial in $\mathcal{C}$. Namely, let $W : \mathcal{C}_1 \to \mathcal{C}_2$ be a covariant functor. Then we obtain a covariant functor

$$W_* : \int_{\mathcal{C}_1} F \circ W \to \int_{\mathcal{C}_2} F \tag{3.6}$$

of additive categories that is the identity on each $F(W(c))$ and "sends T_f to $T_{W(f)}$", again interpreted appropriately. For covariant functors $W_1 : \mathcal{C}_1 \to \mathcal{C}_2$, $W_2 : \mathcal{C}_2 \to \mathcal{C}_3$ and a contravariant functor $F : \mathcal{C}_3 \to \mathsf{Add}\text{-}\mathsf{Cat}$, we have

$$(W_2)_* \circ (W_1)_* = (W_2 \circ W_1)_*, \tag{3.7}$$

$$(\mathrm{id}_{\mathcal{C}})_* = \mathrm{id}_{\int_{\mathcal{C}} F}. \tag{3.8}$$

These two constructions are compatible. Namely, given a natural transformation $S : F_1 \to F_2$ of contravariant functors $\mathcal{C}_2 \to \mathbb{Z}\text{-}\mathsf{Cat}$ and a covariant functor $W : \mathcal{C}_1 \to \mathcal{C}_2$, we get

$$\left(\int_{\mathcal{C}_2} S\right) \circ W_* = W_* \circ \left(\int_{\mathcal{C}_1} (S \circ W)\right). \tag{3.9}$$

Lemma 3.10.

(i) *Let $W : \mathcal{D} \to \mathcal{C}$ be an equivalence of categories. Let $F : \mathcal{C} \to \mathbb{Z}\text{-}\mathsf{Cat}$ be a contravariant functor. Then*

$$W_* : \int_{\mathcal{D}} F \circ W \to \int_{\mathcal{C}} F$$

is an equivalence of categories.

(ii) *Let $\mathcal{C}$ be a category and let $S : F_1 \to F_2$ be a transformation of contravariant functors $\mathcal{C} \to \mathbb{Z}$-$\mathsf{Cat}$ such that, for every object c in $\mathcal{C}$, the functor $S(c) : F_0(c) \to F_1(c)$ is an equivalence of categories. Then*

$$\int_{\mathcal{C}} S : \int_{\mathcal{C}} F_1 \to \int_{\mathcal{C}} F_2$$

is an equivalence of categories.

The proof is an easy exercise. Note the general fact that, if $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between $\mathbb{Z}$-linear categories such that F is an equivalence between the underlying categories, then it follows automatically that there exists an additive inverse equivalence F' and two additive natural equivalences $F' \circ F \simeq \mathrm{id}_{\mathcal{C}}$ and $F \circ F' \simeq \mathrm{id}_{\mathcal{D}}$.

Notation 3.11. If $W : \mathcal{C}_1 \to \mathcal{C}$ is the inclusion of a subcategory, then the same is true for W_*. If no confusion is possible, we just write

$$\int_{\mathcal{C}_1} F := \int_{\mathcal{C}_1} F \circ W \subset \int_{\mathcal{C}} F.$$

Denote by Add-Cat the category whose objects are additive categories and whose morphisms are given by additive functors between them. Notice that $\int_{\mathcal{C}} F$ is not necessarily an additive category even if all the $F(c)$ are — the direct sum $(c, X) \oplus (d, Y)$ need not exist. However, any isomorphism $f : c \to d$ in $\mathcal{C}$ induces an isomorphism $T_f : (c, f^*Y) \to (d, Y)$, so that

$$(c, X) \oplus (d, Y) \cong (c, X) \oplus (c, f^*Y) \cong (c, X \oplus f^*Y).$$

Hence, if in the index category all objects are isomorphic and all the $F(c)$ are additive, then $\int_{\mathcal{C}} F$ is an additive category. Since for additive categories $\mathcal{A}$, $\mathcal{B}$ we have

$$\mathrm{mor}_{\mathbb{Z}\text{-}\mathsf{Cat}}(\mathcal{A}, \mathcal{B}) = \mathrm{mor}_{\mathsf{Add}\text{-}\mathsf{Cat}}(\mathcal{A}, \mathcal{B})$$

(in both cases the morphisms are just additive functors), the universal property for additive functors $\int_{\mathcal{C}} F \to \mathcal{A}$ into $\mathbb{Z}$-linear categories extends to a universal property for additive functors into additive categories.

In the general case of an arbitrary indexing category, the homotopy colimit in the setting of additive categories still exists. It is obtained by freely adjoining direct sums to the homotopy colimit for $\mathbb{Z}$-linear categories; the universal properties then hold in the setting of "additive categories with choice of direct sum". We will not discuss this in detail here since, in all the cases we will consider, the indexing category has the property that any two objects are isomorphic.

Notation 3.12. If the indexing category $\mathcal{C}$ has a single object and $F : \mathcal{C} \to \mathbb{Z}$-$\mathsf{Cat}$ is a contravariant functor, then we will write X instead of $(*, X)$ for a typical element of the homotopy colimit. The structural morphisms in $\int_{\mathcal{C}} F$ thus take the simple form

$$T_f : f^*X \to X$$

for f a morphism (from the single object to itself) in $\mathcal{C}$.

4. The twisted Bass–Heller–Swan theorem for additive categories

Given an additive category $\mathcal{A}$, we denote by $\boldsymbol{K}(\mathcal{A})$ the nonconnective K-theory spectrum associated to it (after idempotent completion); see [Lück and Steimle 2014; Pedersen and Weibel 1989]. Thus we obtain a covariant functor

$$\boldsymbol{K} : \mathsf{Add\text{-}Cat} \to \mathsf{Spectra}. \tag{4.1}$$

Let $\mathcal{B}$ be an additive $\mathbb{Z}$-category, i.e., an additive category with a right action of the infinite cyclic group. Fix a generator σ of the infinite cyclic group $\mathbb{Z}$. Let $\Phi : \mathcal{B} \to \mathcal{B}$ be the automorphism of additive categories given by multiplication with σ. Of course, one can recover the $\mathbb{Z}$-action from Φ. Since $\widehat{\mathbb{Z}}$ has precisely one object, we can and will identify the set of objects of $\int_{\widehat{\mathbb{Z}}} \mathcal{B}$ and $\mathcal{B}$ in the sequel. Let $i_{\mathcal{B}} : \mathcal{B} \to \int_{\widehat{\mathbb{Z}}} \mathcal{B}$ be the inclusion into the homotopy colimit.

The structural morphisms $T_\sigma : \Phi(B) \to B$ of $\int_{\widehat{\mathbb{Z}}} \mathcal{B}$ assemble to a natural isomorphism $i_{\mathcal{B}} \circ \Phi \to i_{\mathcal{B}}$ in the following diagram:

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\Phi"] \arrow[dr, "i_{\mathcal{B}}"'] & & \mathcal{B} \arrow[dl, "i_{\mathcal{B}}"] \\ & \int_{\widehat{\mathbb{Z}}} \mathcal{B} & \end{tikzcd}$$

If we apply the nonconnective K-theory spectrum to it, we obtain a diagram of spectra which commutes up to preferred homotopy:

$$\begin{tikzcd} \boldsymbol{K}(\mathcal{B}) \arrow[rr, "\boldsymbol{K}(\Phi)"] \arrow[dr, "\boldsymbol{K}(i_{\mathcal{B}})"'] & & \boldsymbol{K}(\mathcal{B}) \arrow[dl, "\boldsymbol{K}(i_{\mathcal{B}})"] \\ & \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\right) & \end{tikzcd}$$

It induces a map of spectra

$$\boldsymbol{a}_{\mathcal{B}} : T_{\boldsymbol{K}(\Phi)} \to \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\right),$$

where $T_{\boldsymbol{K}(\Phi)}$ is the mapping torus of the map of spectra $\boldsymbol{K}(\Phi) : \boldsymbol{K}(\mathcal{B}) \to \boldsymbol{K}(\mathcal{B})$, which is defined as the pushout

$$\begin{tikzcd}
\boldsymbol{K}(\mathcal{B}) \wedge \{0,1\}_+ = \boldsymbol{K}(\mathcal{B}) \vee \boldsymbol{K}(\mathcal{B}) \arrow[r, "\boldsymbol{K}(\Phi)\vee \mathrm{id}_{\boldsymbol{K}(\mathcal{B})}"] \arrow[d] & \boldsymbol{K}(\mathcal{B}) \arrow[d] \\
\boldsymbol{K}(\mathcal{B}) \wedge [0,1]_+ \arrow[r] & T_{\boldsymbol{K}(\Phi)}
\end{tikzcd}$$

Let $\mathbb{Z}[\sigma]$ be the submonoid $\{\sigma^n \mid n \in \mathbb{Z},\ n \geq 0\}$ generated by σ. Let $j[\sigma] : \mathbb{Z}[\sigma] \to \mathbb{Z}$ be the inclusion. Let $i_{\mathcal{B}}[\sigma] : \mathcal{B} \to \int_{\widehat{\mathbb{Z}]\sigma]}} \mathcal{B}$ be the inclusion induced by $i_{\mathcal{B}}$. Define a functor of additive categories

$$\mathrm{ev}_{\mathcal{B}}[\sigma] : \int_{\widehat{\mathbb{Z}[\sigma]}} \mathcal{B} \to \mathcal{B}$$

extending the identity on $\mathcal{B}$ by sending a morphism T_{σ^n} to 0 for $n > 0$. (Of course, $\sigma^0 = \mathrm{id}$ must go to the identity.) We obtain the following diagram of spectra:

$$\begin{tikzcd}
\boldsymbol{K}(\mathcal{B}) \arrow[r, "\boldsymbol{K}(i_{\mathcal{B}}[\sigma])"] \arrow[rr, bend right, "\mathrm{id}"'] & \boldsymbol{K}\big(\int_{\widehat{\mathbb{Z}[\sigma]}} \mathcal{B}\big) \arrow[r, "\boldsymbol{K}(\mathrm{ev}_{\mathcal{B}}[\sigma])"] & \boldsymbol{K}(\mathcal{B})
\end{tikzcd}$$

Define $\boldsymbol{NK}(\mathcal{B}, \sigma)$ as the homotopy fiber of the map $\boldsymbol{K}(\mathrm{ev}_{\mathcal{B}}[\sigma]) : \boldsymbol{K}\big(\int_{\widehat{\mathbb{Z}[\sigma]}} \mathcal{B}\big) \to \boldsymbol{K}(\mathcal{B})$. Let $\boldsymbol{b}_{\mathcal{B}}[\sigma]$ denote the composite

$$\boldsymbol{b}_{\mathcal{B}}[\sigma] : \boldsymbol{NK}(\mathcal{B}, \sigma) \to \boldsymbol{K}\bigg(\int_{\widehat{\mathbb{Z}[\sigma]}} \mathcal{B}\bigg) \to \boldsymbol{K}\bigg(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\bigg)$$

of the canonical map with the inclusion. Let $\mathrm{gen}(\mathbb{Z})$ be the set of generators of the infinite cyclic group $\mathbb{Z}$. Put

$$\boldsymbol{NK}(\mathcal{B}) := \bigvee_{\sigma \in \mathrm{gen}(\mathbb{Z})} \boldsymbol{NK}(\mathcal{B}, \sigma)$$

and define

$$\boldsymbol{b}_{\mathcal{B}} := \bigvee_{\sigma \in \mathrm{gen}(\mathbb{Z})} \boldsymbol{b}_{\mathcal{B}}[\sigma] : \bigvee_{\sigma \in \mathrm{gen}(\mathbb{Z})} \boldsymbol{NK}(\mathcal{B}, \sigma) \to \boldsymbol{K}\bigg(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\bigg).$$

The proof of the following result can be found in [Lück and Steimle 2016]. The case where the $\mathbb{Z}$-action on $\mathcal{B}$ is trivial and one considers only K-groups in dimensions $n \leq 1$ has already been treated by [Ranicki 1992, Chapters 10 and 11]. If R is a ring with an automorphism and one takes $\mathcal{B}$ to be the category R-FGF of finitely generated free R-modules with the induced $\mathbb{Z}$-action, Theorem 4.2 boils down for higher algebraic K-theory to the twisted Bass–Heller–Swan decomposition of [Grayson 1988, Theorems 2.1 and 2.3].

Theorem 4.2 (twisted Bass–Heller–Swan decomposition for additive categories). *The map of spectra*

$$\boldsymbol{a}_{\mathcal{B}} \vee \boldsymbol{b}_{\mathcal{B}} : T_{\boldsymbol{K}(\Phi)} \vee \boldsymbol{NK}(\mathcal{B}) \xrightarrow{\simeq} \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\right)$$

is a weak equivalence of spectra.

5. Some additive categories associated to an additive G-category

Let G be a group. Let $\mathcal{A}$ be an additive G-category, i.e., an additive category with a right G-operation by isomorphisms of additive categories. We can consider $\mathcal{A}$ as a contravariant functor $\widehat{G} \to \mathsf{Add\text{-}Cat}$. Fix a homogeneous G-space G/H. Let $\mathrm{pr}_{G/H} : \mathcal{G}^G(G/H) \to \mathcal{G}^G(G/G) = \widehat{G}$ be the projection induced by the canonical G-map $G/H \to G/G$. Then we obtain a covariant functor $\mathcal{G}^G(G/H) \to \mathsf{Add\text{-}Cat}$ by sending G/H to $\mathcal{A} \circ \mathrm{pr}_G$. Let $\int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_{G/H}$ be the additive category given by the homotopy colimit (defined in (3.1)) of this functor. A G-map $f : G/H \to G/K$ induces a functor $\mathcal{G}^G(f) : \mathcal{G}^G(G/H) \to \mathcal{G}^G(G/K)$ which is compatible with the projections to $\widehat{G}$. Hence it induces a functor of additive categories — see (3.6) —

$$\mathcal{G}^G(f)_* : \int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_{G/H} \to \int_{\mathcal{G}^G(G/K)} \mathcal{A} \circ \mathrm{pr}_{G/K}.$$

Thus we obtain a covariant functor

$$\mathsf{Or}(G) \to \mathsf{Add\text{-}Cat}, \quad G/H \mapsto \int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_{G/H}. \tag{5.1}$$

Remark 5.2. Applying Lemma 3.10(i) to the equivalence of categories $e(G/H) : \widehat{H} \to \mathcal{G}^G(G/H)$, we see that the functor (5.1), at G/H, takes the value $\int_{\widehat{H}} \mathcal{A}$, where $\mathcal{A}$ carries the restricted H-action. The more complicated description is however needed for the functoriality.

Notation 5.3. For the sake of brevity, we will just write $\mathcal{A}$ for any composite $\mathcal{A} \circ \mathrm{pr}_{G/H}$ if no confusion is possible. In this notation, (5.1) takes the form

$$G/H \mapsto \int_{\mathcal{G}^G(G/H)} \mathcal{A}.$$

Let $V \subseteq G$ be an infinite virtually cyclic subgroup of type I. In the sequel we abbreviate $K = K_V$ and $Q = Q_V$. Let $\mathrm{pr}_K : \mathcal{G}^G(G/V)_K \to \widehat{K}$ be the functor which sends a morphism $g : g_1 V \to g_2 V$ to $g_2^{-1} g g_1 \in K$.

Fixing a generator σ of the infinite cyclic group Q, the inclusions $\mathcal{G}^G(G/V)_K \subset \mathcal{G}^G(G/V)[\sigma] \subset \mathcal{G}^G(G/V)$ induce inclusions

$$\int_{\mathcal{G}^G(G/V)_K} \mathcal{A} \subset \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \subset \int_{\mathcal{G}^G(G/V)} \mathcal{A}. \tag{5.4}$$

Actually, the category into the middle retracts onto the smaller one. To see this, define a retraction

$$\operatorname{ev}(G/V)[\sigma]_K : \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \to \int_{\mathcal{G}^G(G/V)_K} \mathcal{A} \tag{5.5}$$

as follows: It is the identity on every copy of the additive category $\mathcal{A}$ inside the homotopy colimit. Let $T_g : (g_1V, g^*A) \to (g_2V, A)$ be a structural morphism in the homotopy colimit, where $g : g_1V \to g_2V$ in $\mathcal{G}^G(G/V)[\sigma]$ is a morphism in $\mathcal{G}^G(G/V)[\sigma]$ (that is, g is an element of G satisfying $g_2^{-1}gg_1 \in V[\sigma]$). If

$$g_2^{-1}gg_1 \in K \subset V[\sigma],$$

then g is by definition a morphism in $\mathcal{G}^G(G/V)_K \subset \mathcal{G}^G(G/V)[\sigma]$ and we may let

$$\operatorname{ev}(G/V)[\sigma]_K(T_g) = T_g.$$

Otherwise we send the morphism T_g to 0. This is well-defined, since for two elements $h_1, h_2 \in V[\sigma]$ we have $h_1h_2 \in K$ if and only if both $h_1 \in K$ and $h_2 \in K$ hold.

Similarly the inclusion $\int_{\widehat{K}} \mathcal{A} \subset \int_{\widehat{V[\sigma]}} \mathcal{A}$ is split by a retraction

$$\operatorname{ev}_V[\sigma] : \int_{\widehat{V[\sigma]}} \mathcal{A} \to \int_{\widehat{K}} \mathcal{A}$$

defined as follows: On the copy of $\mathcal{A}$ inside $\int_{\widehat{V[\sigma]}} \mathcal{A}$, the functor is defined to be the identity. A structural morphism $T_g : g^*A \to A$ is sent to itself if $g \in K$, and to zero otherwise. One easily checks that the following diagram commutes (where the unlabelled arrows are inclusions) and has equivalences of additive categories as vertical maps:

$$\begin{tikzcd}
\int_{\mathcal{G}^G(G/V)_K} \mathcal{A} \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \arrow[r, "\operatorname{ev}(G/V)[\sigma]"] & \int_{\mathcal{G}^G(G/V)_K} \mathcal{A} \\
\int_{\widehat{K}} \mathcal{A} \arrow[u, "(e(G/V)_K)_*", "\cong"'] \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \int_{\widehat{V[\sigma]}} \mathcal{A} \arrow[u, "e(G/V)[\sigma]_*", "\cong"'] \arrow[r, "\operatorname{ev}_V[\sigma]"] & \int_{\widehat{K}} \mathcal{A} \arrow[u, "(e(G/V)_K)_*", "\cong"']
\end{tikzcd} \tag{5.6}$$

We obtain from (2.1) and Lemma 3.10(i) the following commutative diagram of additive categories with equivalences of additive categories as vertical maps:

$$\begin{tikzcd} \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \arrow[r] & \int_{\mathcal{G}^G(G/V)} \mathcal{A} \\ \int_{\widehat{V[\sigma]}} \mathcal{A} \arrow[r] \arrow[u, "e(G/V)[\sigma]_*", "\cong"'] & \int_{\widehat{V}} \mathcal{A} \arrow[u, "\cong", "e(G/V)_*"'] \end{tikzcd} \tag{5.7}$$

(where again the unlabelled arrows are the inclusions).

Now we abbreviate $\mathcal{B} = \int_{\widehat{K}} \mathcal{A}$. Next we define a right Q-action on $\mathcal{B}$ which will depend on a choice of an element $\overline{\sigma} \in V$ such that $p_V : V \to Q$ sends $\overline{\sigma}$ to σ. Such an element induces a section of the projection $G \to Q$ by which any action of G induces an action of Q. In short, the action of Q on $\mathcal{B}$ is given by the action of G onto $\mathcal{A} \subset \mathcal{B}$, and by the conjugation action of Q on the indexing category $\widehat{K}$. In more detail, the action of $\sigma \in Q$ is specified by the automorphism

$$\Phi : \int_{\widehat{K}} \mathcal{A} \to \int_{\widehat{K}} \mathcal{A}$$

defined as follows: A morphism $\varphi : A \to B$ in $\mathcal{A}$ is sent to $\overline{\sigma}^*\varphi : \overline{\sigma}^*A \to \overline{\sigma}^*B$, and a structural morphism $T_g : g^*A \to A$ is sent to the morphism

$$T_{\overline{\sigma}^{-1}g\overline{\sigma}} : \overline{\sigma}^*g^*A = (\overline{\sigma}^{-1}g\overline{\sigma})^*\overline{\sigma}^*A \to \overline{\sigma}^*A.$$

With this notation we obtain an additive functor

$$\Psi : \int_{\widehat{Q}} \mathcal{B} \to \int_{\widehat{V}} \mathcal{A}$$

defined to extend the inclusion $\mathcal{B} = \int_{\widehat{K}} \mathcal{A} \to \int_{\widehat{V}} \mathcal{A}$ and such that a structural morphism $T_\sigma : \Phi(A) \to A$ is sent to $T_{\overline{\sigma}} : \Phi(A) = \overline{\sigma}^*A \to A$.

In more detail, a morphism in $\int_{\widehat{Q}} \mathcal{B}$ can be uniquely written as a finite sum

$$\sum_{n\in\mathbb{Z}} T_{\overline{\sigma}^n} \circ \left(\sum_{k\in K} T_k \circ \phi_{k,n} \right) = \sum_{n,k} T_{\overline{\sigma}^n \cdot k} \circ \phi_{k,n}.$$

Since any element in V is uniquely a product $\overline{\sigma}^n \cdot k$ with $k \in K$, the functor Ψ is fully faithful. As it is the identity on objects, Ψ is an isomorphism of categories. It also restricts to an isomorphism of categories

$$\Psi[\sigma] : \int_{\widehat{Q[\sigma]}} \mathcal{B} \to \int_{\widehat{V[\sigma]}} \mathcal{A}.$$

Define a functor

$$\mathrm{ev}_{\mathcal{B}}[\sigma] : \int_{\widehat{Q[\sigma]}} \mathcal{B} \to \mathcal{B}$$

as follows. It is the identity functor on $\mathcal{B}$, and a nonidentity structural morphism $T_q : q^*B \to B$ is sent to 0. One easily checks using (5.6) and (5.7) that the following diagram of additive categories commutes (with unlabelled arrows given by inclusions) and that all vertical arrows are equivalences of additive categories:

$$
\begin{tikzcd}
\int_{\mathcal{G}^G(G/V)_K} \mathcal{A} & & \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \arrow[ll, "\mathrm{ev}(G/V)[\sigma]_K"'] \arrow[rr] & & \int_{\mathcal{G}^G(G/V)} \mathcal{A} \\
\int_{\widehat{K}} \mathcal{A} \arrow[u, "(e(G/V)_K)_*", "\wr"'] & & \int_{\widehat{V[\sigma]}} \mathcal{A} \arrow[ll, "\mathrm{ev}_V[\sigma]"'] \arrow[rr] \arrow[u, "e(G/V)[\sigma]_*", "\wr"'] & & \int_{\widehat{V}} \mathcal{A} \arrow[u, "e(G/V)_*", "\wr"'] \\
\mathcal{B} \arrow[u, "\mathrm{id}", "\cong"'] & & \int_{\widehat{Q[\sigma]}} \mathcal{B} \arrow[ll, "\mathrm{ev}_{\mathcal{B}}[\sigma]"'] \arrow[rr] \arrow[u, "\Psi[\sigma]", "\cong"'] & & \int_{\widehat{Q}} \mathcal{B} \arrow[u, "\Psi", "\cong"']
\end{tikzcd}
\tag{5.8}
$$

Recall from Section 2 that $q_V : G/K \to G/V$ is the projection and that R_σ is the automorphism of $\int_{\mathcal{G}^G(G/K)} \mathcal{A}$ induced by right multiplication with $\overline{\sigma}$.

We have the following (not necessarily commutative) diagram of additive categories, all of whose vertical arrows are equivalences of additive categories, and the unlabelled arrows are the inclusions:

$$
\begin{tikzcd}
\int_{\mathcal{G}^G(G/K)} \mathcal{A} \arrow[rr, "R_\sigma"] \arrow[dr, "\mathcal{G}^G(\mathrm{pr}_V)_*"'] & & \int_{\mathcal{G}^G(G/K)} \mathcal{A} \arrow[dl, "\mathcal{G}^G(\mathrm{pr}_V)_*"] \\
& \int_{\mathcal{G}^G(G/V)} \mathcal{A} & \\
\int_{\widehat{K}} \mathcal{A} \arrow[uu, "(e(G/V)_K)_*", "\wr"'] \arrow[rr, "\Phi"] \arrow[dr] & & \int_{\widehat{K}} \mathcal{A} \arrow[uu, "\wr", "(e(G/V)_K)_*"'] \arrow[dl] \\
& \int_{\widehat{V}} \mathcal{A} \arrow[uu, "e(G/V)_*", "\wr"'] & \\
\mathcal{B} \arrow[uu, "\mathrm{id}", "\cong"'] \arrow[rr, "\Phi"] \arrow[dr, "i_{\mathcal{B}}"'] & & \mathcal{B} \arrow[uu, "\cong", "\mathrm{id}"'] \arrow[dl, "i_{\mathcal{B}}"] \\
& \int_{\widehat{Q}} \mathcal{B} \arrow[uu, "\Psi", "\wr"'] &
\end{tikzcd}
\tag{5.9}
$$

The lowest triangle commutes up to a preferred natural isomorphism $T : i_{\mathcal{B}} \circ \Phi \xrightarrow{\cong} i_{\mathcal{B}}$, which is part of the structural data of the homotopy colimit. We equip the middle triangle with the natural isomorphism $\Psi \circ T$. Explicitly it is just given by the structural morphisms $T_{\overline{\sigma}} : \overline{\sigma}^* A \to A$.

The three squares ranging from the middle to the lower level commute and the two natural equivalences above are compatible with these squares. The top triangle commutes. The back upper square commutes up to a preferred natural isomorphism

$S : (e(G/V)_K)_* \circ \Phi \xrightarrow{\cong} R_\sigma \circ (e(G/V)_K)_*$. It assigns to an object $A \in \mathcal{A}$, which is the same as an object in $\int_{\widehat{K}} \mathcal{A}$, the structural isomorphism

$$S(A) := T_{\overline{\sigma}} : (eK, \overline{\sigma}^* A) \to (\overline{\sigma} K, A).$$

The other two squares joining the upper to the middle level commute. From the explicit description of the natural isomorphisms it becomes apparent that the preferred natural isomorphism for the middle triangle defined above and the preferred natural isomorphism for the upper back square are compatible, in the sense that $e(G/V)[\sigma]_* \circ \Psi \circ T = \mathcal{G}^G(\mathrm{pr}_V)_* \circ S$.

6. Some K-theory-spectra over the orbit category

In this section we introduce various K-theory spectra. For a detailed introduction to spaces, spectra and modules over a category and some constructions of K-theory spectra, we refer to [Davis and Lück 1998].

Given an additive G-category $\mathcal{A}$, we obtain a covariant $\mathsf{Or}(G)$-spectrum

$$\boldsymbol{K}_{\mathcal{A}}^G : \mathsf{Or}(G) \to \mathsf{Spectra}, \quad G/H \mapsto \boldsymbol{K}\left(\int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_{G/H}\right), \tag{6.1}$$

by the composite of the two functors (4.1) and (5.1). It is naturally equivalent to the covariant $\mathsf{Or}(G)$-spectrum, which is written in the same way and constructed in [Bartels and Reich 2007, Definition 3.1].

We again adopt Notation 5.3, abbreviating an expression such as $\mathcal{A} \circ \mathrm{pr}_{G/H}$ just by $\mathcal{A}$. Given a virtually cyclic subgroup $V \subseteq G$, we obtain the following map of spectra induced by the functors $j(G/V)[\sigma]_*$ of (5.4) and $\mathrm{ev}(G/V)[\sigma]$ of (5.5):

$$\boldsymbol{K}\left(\int_{\mathcal{G}^G(G/V)_K} \mathcal{A}\right) \xleftarrow{\boldsymbol{K}(\mathrm{ev}(G/V)[\sigma])} \boldsymbol{K}\left(\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A}\right) \xrightarrow{\boldsymbol{K}(j(G/V)[\sigma]_*)} \boldsymbol{K}\left(\int_{\mathcal{G}^G(G/V)} \mathcal{A}\right).$$

Notation 6.2. Let $\boldsymbol{NK}(G/V; \mathcal{A}, \sigma)$ be the spectrum given by the homotopy fiber of $\boldsymbol{K}\big(\mathrm{ev}(G/V)[\sigma]_*\big) : \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A}\big) \to \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)_K} \mathcal{A}\big)$.

Let $\boldsymbol{l} : \boldsymbol{NK}(G/V; \mathcal{A}, \sigma) \to \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A}\big)$ be the canonical map of spectra. Define the map of spectra

$$\boldsymbol{j}(G/V; \mathcal{A}, \sigma) : \boldsymbol{NK}(G/V; \mathcal{A}, \sigma) \to \boldsymbol{K}\left(\int_{\mathcal{G}^G(G/V)} \mathcal{A}\right)$$

to be the composite $\boldsymbol{K}(j(G/V)[\sigma]_*) \circ \boldsymbol{l}$.

Consider a G-map $f : G/V \to G/W$, where V and W are virtually cyclic groups of type I. It induces a functor $\mathcal{G}^G(f) : \mathcal{G}^G(G/V) \to \mathcal{G}^G(G/W)$.

It also induces a bijection

$$\mathrm{gen}(f) : \mathrm{gen}(Q_V) \to \mathrm{gen}(Q_W) \tag{6.3}$$

as follows. Fix an element $g \in G$ such that $f(eV) = gW$. Then $g^{-1}Vg \subseteq W$. The injective group homomorphism $c(g) : V \to W,\ v \mapsto g^{-1}vg$, induces an injective group homomorphism $Q_{c(g)} : Q_V \to Q_W$ by Lemma 1.1(vi). For $\sigma \in \mathrm{gen}(Q_V)$ let $\mathrm{gen}(f)(\sigma) \in \mathrm{gen}(Q_W)$ be uniquely determined by the property that $Q_{c(g)}(\sigma) = \mathrm{gen}(f)(\sigma)^n$ for some $n \geq 1$. One easily checks that this is independent of the choice of $g \in G$ with $f(eV) = gW$ since, for $w \in W$, the conjugation homomorphism $c(w) : W \to W$ induces the identity on Q_W. Using Lemma 1.1(vi) it follows that $\mathcal{G}^G(f) : \mathcal{G}^G(G/V) \to \mathcal{G}^G(G/W)$ induces functors

$$\begin{aligned} \mathcal{G}^G(f)[\sigma] &: \mathcal{G}^G(G/V)[\sigma] \to \mathcal{G}^G(G/W)[\mathrm{gen}(f)(\sigma)], \\ \mathcal{G}^G(f)_K &: \mathcal{G}^G(G/V)_K \to \mathcal{G}^G(G/W)_K. \end{aligned}$$

Hence we obtain a commutative diagram of maps of spectra

$$\begin{tikzcd}
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)_K} \mathcal{A}\big) \arrow[r, "\boldsymbol{K}((\mathcal{G}^G(f)_K)_*)"] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/W)_K} \mathcal{A}\big) \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A}\big) \arrow[r, "\boldsymbol{K}(\mathcal{G}^G(f)[\sigma]_*)"] \arrow[u, "\boldsymbol{K}(\mathrm{ev}(G/V)[\sigma])"] \arrow[d, "\boldsymbol{K}(j(G/V)[\sigma]_*)"'] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/W)[\mathrm{gen}(f)(\sigma)]} \mathcal{A}\big) \arrow[u, "\boldsymbol{K}(\mathrm{ev}(G/W)[\mathrm{gen}(f)(\sigma)])"'] \arrow[d, "\boldsymbol{K}(j(G/W)[\mathrm{gen}(f)(\sigma)]_*)"] \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)} \mathcal{A}\big) \arrow[r, "\boldsymbol{K}(\mathcal{G}^G(f)_*)"] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/W)} \mathcal{A}\big)
\end{tikzcd}$$

Thus we obtain a map of spectra

$$\boldsymbol{NK}(f; \mathcal{A}, \sigma) : \boldsymbol{NK}(G/V; \mathcal{A}, \sigma) \to \boldsymbol{NK}(G/W; \mathcal{A}, \mathrm{gen}(f)(\sigma))$$

such that the following diagram of spectra commutes:

$$\begin{tikzcd}
\boldsymbol{NK}(G/V; \mathcal{A}, \sigma) \arrow[r, "\boldsymbol{NK}(f;\mathcal{A},\sigma)"] \arrow[d, "\boldsymbol{j}(G/V;\mathcal{A},\sigma)"'] & \boldsymbol{NK}(G/W; \mathcal{A}, \mathrm{gen}(f)(\sigma)) \arrow[d, "\boldsymbol{j}(G/W;\mathcal{A},\mathrm{gen}(f)(\sigma))"] \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)} \mathcal{A}\big) \arrow[r, "\boldsymbol{K}(\mathcal{G}^G(f)_*)"'] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/W)} \mathcal{A}\big)
\end{tikzcd}$$

Let $\mathcal{VC}_I$ be the family of subgroups of G which consists of all finite groups and all virtually cyclic subgroups of type I. Let $\mathsf{Or}_{\mathcal{VC}_I}(G)$ be the full subcategory of the orbit category $\mathsf{Or}(G)$ consisting of objects G/V for which V belongs to $\mathcal{VC}_I$. Define a functor

$$\boldsymbol{NK}^G_{\mathcal{A}} : \mathsf{Or}_{\mathcal{VC}_I}(G) \to \mathsf{Spectra}$$

as follows: It sends G/H for a finite subgroup H to the trivial spectrum and G/V for a virtually cyclic subgroup V of type I to $\bigvee_{\sigma \in \mathrm{gen}(Q_V)} \boldsymbol{NK}(G/V; \mathcal{A}, \sigma)$. Consider a map $f : G/V \to G/W$. If V or W is finite, it is sent to the trivial map.

Suppose that both V and W are infinite virtually cyclic subgroups of type I. Then it is sent to the wedge of the two maps

$$\begin{aligned} \boldsymbol{NK}(f;\mathcal{A},\sigma_1) &: \boldsymbol{NK}(G/V;\mathcal{A},\sigma_1) \to \boldsymbol{NK}(G/W;\mathcal{A},\operatorname{gen}(f)(\sigma_1)), \\ \boldsymbol{NK}(f;\mathcal{A},\sigma_2) &: \boldsymbol{NK}(G/V;\mathcal{A},\sigma_2) \to \boldsymbol{NK}(G/W;\mathcal{A},\operatorname{gen}(f)(\sigma_2)), \end{aligned}$$

for $\operatorname{gen}(Q_V) = \{\sigma_1, \sigma_2\}$.

The restriction of the covariant $\mathsf{Or}(G)$-spectrum $\boldsymbol{K}^G_{\mathcal{A}} : \mathsf{Or}(G) \to \mathsf{Spectra}$ to $\mathsf{Or}_{\mathcal{VC}_I}(G)$ will be denoted by the same symbol

$$\boldsymbol{K}^G_{\mathcal{A}} : \mathsf{Or}_{\mathcal{VC}_I}(G) \to \mathsf{Spectra}.$$

The wedge of the maps $\boldsymbol{j}(G/V;\mathcal{A},\sigma_1)$ and $\boldsymbol{j}(G/V;\mathcal{A},\sigma_2)$ for V a virtually cyclic subgroup of G of type I yields a map of spectra $\boldsymbol{NK}^G_{\mathcal{A}}(G/V) \to \boldsymbol{K}^G_{\mathcal{A}}(G/V)$. Thus we obtain a transformation of functors from $\mathsf{Or}_{\mathcal{VC}_I}(G)$ to $\mathsf{Spectra}$,

$$\boldsymbol{b}^G_{\mathcal{A}} : \boldsymbol{NK}^G_{\mathcal{A}} \to \boldsymbol{K}^G_{\mathcal{A}}. \tag{6.4}$$

7. Splitting the relative assembly map and identifying the relative term

Let X be a G-space. It defines a contravariant $\mathsf{Or}(G)$-space $O^G(X)$, i.e., a contravariant functor from $\mathsf{Or}(G)$ to the category of spaces, by sending G/H to the H-fixed point set $\operatorname{map}_G(G/H, X) = X^H$. Let $O^G(X)_+$ be the pointed $\mathsf{Or}(G)$-space, where $O^G(X)_+(G/H)$ is obtained from $O^G(X)(G/H)$ by adding an extra base point. If $f : X \to Y$ is a G-map, we obtain a natural transformation $O^G(f)_+ : O^G(X)_+ \to O^G(Y)_+$.

Let $\boldsymbol{E}$ be a covariant $\mathsf{Or}(G)$-spectrum, i.e., a covariant functor from $\mathsf{Or}(G)$ to the category of spectra. Fix a G-space Z. Define the covariant $\mathsf{Or}(G)$-spectrum

$$\boldsymbol{E}_Z : \mathsf{Or}(G) \to \mathsf{Spectra}$$

as follows. It sends an object G/H to the spectrum $O^G(G/H \times Z)_+ \wedge_{\mathsf{Or}(G)} \boldsymbol{E}$, where $\wedge_{\mathsf{Or}(G)}$ is the wedge product of a pointed space and a spectrum over a category (see [Davis and Lück 1998, Section 1], where $\wedge_{\mathsf{Or}(G)}$ is denoted by $\otimes_{\mathsf{Or}(G)}$). The obvious identification of $O^G(G/H)_+(?) \wedge_{\mathsf{Or}(G)} \boldsymbol{E}(?)$ with $\boldsymbol{E}(G/H)$ and the projection $G/H \times Z \to G/H$ yields a natural transformation of covariant functors $\mathsf{Or}(G) \to \mathsf{Spectra}$,

$$\boldsymbol{a} : \boldsymbol{E}_Z \to \boldsymbol{E}. \tag{7.1}$$

Lemma 7.2. *Given a G-space X, there exists an isomorphism of spectra*

$$\boldsymbol{u}^G(X) : O^G(X \times Z)_+ \wedge_{\mathsf{Or}(G)} \boldsymbol{E} \xrightarrow{\cong} O^G(X)_+ \wedge_{\mathsf{Or}(G)} \boldsymbol{E}_Z,$$

which is natural in X and Z.

Proof. The smash product $\wedge_{\mathsf{Or}(G)}$ is associative, i.e., there is a natural isomorphism of spectra

$$\begin{aligned}\big(O^G(X)_+(?_1)\wedge_{\mathsf{Or}(G)} O^G(?_2\times Z)_+(?_1)\big)\wedge_{\mathsf{Or}(G)} \boldsymbol{E}(?_2)\\ \xrightarrow{\cong} O^G(X)_+(?_1)\wedge_{\mathsf{Or}(G)}\big(O^G(?_2\times Z)_+(?_1)\wedge_{\mathsf{Or}(G)}\boldsymbol{E}(?_2)\big).\end{aligned}$$

There is a natural isomorphism of covariant $\mathsf{Or}(G)$-spaces

$$O^G(X\times Z)_+ \xrightarrow{\cong} O^G(X)_+(?)\wedge_{\mathsf{Or}(G)} O^G(?\times Z)_+$$

which, evaluated at G/H, sends $\alpha : G/H\to X\times Z$ to $(\mathrm{pr}_1\circ\alpha)\wedge(\mathrm{id}_{G/H}\times(\mathrm{pr}_2\circ\alpha))$ if pr_i is the projection onto the i-th factor of $X\times Z$. The inverse evaluated at G/H sends $(\beta_1 : G/K\to X)\wedge(\beta_2 : G/H\to G/K\times Z)$ to $(\beta_1\times\mathrm{id}_Z)\circ\beta_2$. The composite of these two isomorphisms yield the desired isomorphism $\boldsymbol{u}^G(X)$. □

If $\mathcal{F}$ is a family of subgroups of the group G, e.g., $\mathcal{VC}_I$ or the family $\mathcal{F}\mathrm{in}$ of finite subgroups, then we denote by $E_{\mathcal{F}}(G)$ the classifying space of $\mathcal{F}$. (For a survey on these spaces we refer for instance to [Lück 2005].) Let $\underline{E}G$ denote the classifying space for proper G-actions, or in other words, a model for $E_{\mathcal{F}\mathrm{in}}(G)$. If we restrict a covariant $\mathsf{Or}(G)$ spectrum $\boldsymbol{E}$ to $\mathsf{Or}_{\mathcal{VC}_I}(G)$, we will denote it by the same symbol $\boldsymbol{E}$ and analogously for $O^G(X)$.

Lemma 7.3. *Let $\mathcal{F}$ be a family of subgroups. Let X be a G-CW-complex whose isotropy groups belong to $\mathcal{F}$. Let $\mathbf{E}$ be a covariant $\mathsf{Or}(G)$-spectrum. Then there is a natural homeomorphism of spectra*

$$O^G(X)_+\wedge_{\mathsf{Or}_{\mathcal{F}}(G)}\boldsymbol{E}\xrightarrow{\cong} O^G(X)_+\wedge_{\mathsf{Or}(G)}\boldsymbol{E}.$$

Proof. Let $I : \mathsf{Or}_{\mathcal{F}}(G)\to\mathsf{Or}(G)$ be the inclusion. The claim follows from the adjunction of the induction I_* and restriction I^* — see [Davis and Lück 1998, Lemma 1.9] — and the fact that for the $\mathsf{Or}(G)$-space $O^G(X)$ the canonical map $I_*I^*O^G(X)\to O^G(X)$ is a homeomorphism of $\mathsf{Or}(G)$-spaces. □

In the sequel we will abbreviate $\boldsymbol{E}_{\underline{E}G}$ by $\underline{\boldsymbol{E}}$.

Lemma 7.4. *Let $\boldsymbol{E}$ be a covariant $\mathsf{Or}(G)$-spectrum. Let $f : \underline{E}G\to E_{\mathcal{VC}_I}(G)$ be a G-map. (It is unique up to G-homotopy.) Then there is an up-to-homotopy commutative diagram of spectra whose upper horizontal map is a weak equivalence*

$$\begin{tikzcd} O^G(E_{\mathcal{VC}_I}(G))\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\underline{\boldsymbol{E}} \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{id}\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\boldsymbol{a}"'] & & O^G(\underline{E}G)\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\boldsymbol{E} \arrow[dl, "O^G(f)\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\mathrm{id}"] \\ & O^G(E_{\mathcal{VC}_I}(G))\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\boldsymbol{E} & \end{tikzcd}$$

Proof. From Lemma 7.2 we obtain a commutative diagram with an isomorphism as horizontal map

$$\begin{tikzcd} O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \underline{\boldsymbol{E}} \arrow[rr, "\cong"] \arrow[dr, "\mathrm{id}\wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)}\boldsymbol{a}"'] & & O^G(E_{\mathcal{VC}_I}(G) \times \underline{E}G) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} \arrow[dl, "O^G(\mathrm{pr}_1)\wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)}\mathrm{id}"] \\ & O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} & \end{tikzcd}$$

where $\mathrm{pr}_1 : E_{\mathcal{VC}_I}(G) \times \underline{E}G \to E_{\mathcal{VC}_I}(G)$ is the obvious projection. The projection $\mathrm{pr}_2 : E_{\mathcal{VC}_I}(G) \times \underline{E}G \to \underline{E}G$ is a G-homotopy equivalence and its composite with $f : \underline{E}G \to E_{\mathcal{VC}_I}(G)$ is G-homotopic to pr_1. Hence the following diagram of spectra commutes up to G-homotopy and has a weak equivalence as horizontal map:

$$\begin{tikzcd} O^G(E_{\mathcal{VC}_I}(G) \times \underline{E}G) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} \arrow[rr, "\simeq", "O^G(\mathrm{pr}_2)\wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)}\mathrm{id}"'] \arrow[dr, "O^G(\mathrm{pr}_1)\wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)}\mathrm{id}"'] & & O^G(\underline{E}G) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} \arrow[dl, "O^G(f)\wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)}\mathrm{id}"] \\ & O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} & \end{tikzcd}$$

Putting these two diagrams together finishes the proof of Lemma 7.4 □

If $\boldsymbol{E}$ is the functor $\boldsymbol{K}_{\mathcal{A}}^G$ defined in (6.1) and $Z = \underline{E}G$, we will write $\underline{\boldsymbol{K}}_{\mathcal{A}}^G$ for $\underline{\boldsymbol{E}} = \boldsymbol{E}_{\underline{E}G}$.

Lemma 7.5. *Let H be a finite group or an infinite virtually cyclic group of type I. Then the map of spectra* (see (6.4) and (7.1))

$$\boldsymbol{a}(G/H) \vee \boldsymbol{b}(G/H) : \underline{\boldsymbol{K}}_{\mathcal{A}}^G(G/H) \vee \boldsymbol{NK}_{\mathcal{A}}^G(G/H) \to \boldsymbol{K}_{\mathcal{A}}^G(G/H)$$

is a weak equivalence.

Proof. Given an infinite cyclic subgroup $V \subseteq G$ of type I, we next construct an up-to-homotopy commutative diagram (on the next page) of spectra whose vertical arrows are all weak homotopy equivalences for $K = K_V$ and $Q = Q_V$. Let $i_V : V \to G$ be the inclusion and $p_V : V \to Q_V := V/K_V$ be the projection.

We first explain the vertical arrows, starting at the top. The first one is the identity by definition. The second one comes from the G-homeomorphism $G/V \times \underline{E}G \xrightarrow{\cong} (i_V)_*(i_V)^*\underline{E}G = G \times_V \underline{E}G$ sending (gV, x) to $(g, g^{-1}x)$. The third one comes from the adjunction of the induction $(i_V)_*$ and restriction i_V^*; see [Davis and Lück 1998, Lemma 1.9]. The fourth one comes from the fact that $p_V^* EQ$ and $i_V^*\underline{E}G$ are both models for $\underline{E}V$ and hence are V-homotopy equivalent. The fifth one comes from the adjunction of the restriction p_V^* with the coinduction $(p_V)_!$; see [Davis and Lück 1998, Lemma 1.9]. The sixth one comes from the fact that EQ is a free Q-CW-complex and Lemma 7.3 applied to the family consisting of one subgroup, namely the trivial subgroup. The seventh one comes from the identification

$(p_V)_!(i_V)^*\boldsymbol{K}^G_{\mathcal{A}}(Q_V/1) = (i_V)^*\boldsymbol{K}^G_{\mathcal{A}}(V/K) = \boldsymbol{K}^G_{\mathcal{A}}(G/K)$. The last one comes from the obvious homeomorphism if we use for EQ_V the standard model with $\mathbb{R}$ as the underlying $Q_V{=}\mathbb{Z}$-space. The arrow $\boldsymbol{a}'(G/V)$ is induced by the upper triangle in (5.9), which commutes (strictly). One easily checks that the diagram commutes:

$$
\begin{tikzcd}
\underline{\boldsymbol{K}}^G_{\mathcal{A}}(G/V) \arrow[d, "(1)"', "\mathrm{id}"] \arrow[rdddd, bend left, "\boldsymbol{a}(G/V)"] & \\
O^G(G/V \times \underline{E}G)_+ \wedge_{\mathrm{Or}(G)} \boldsymbol{K}^G_{\mathcal{A}} \arrow[d, "(2)"', "\cong"] & \\
O^G((i_V)_*(i_V)^*\underline{E}G)_+ \wedge_{\mathrm{Or}(G)} \boldsymbol{K}^G_{\mathcal{A}} \arrow[d, "(3)"', "\cong"] & \\
O^V((i_V)^*\underline{E}G)_+ \wedge_{\mathrm{Or}(V)} (i_V)^*\boldsymbol{K}^G_{\mathcal{A}} \arrow[d, "(4)"', "\simeq"] & \\
O^V((p_V)^*EQ_V)_+ \wedge_{\mathrm{Or}(V)} (i_V)^*\boldsymbol{K}^G_{\mathcal{A}} \arrow[d, "(5)"', "\cong"] & \boldsymbol{K}^G_{\mathcal{A}}(G/V) \\
O^{Q_V}(EQ_V)_+ \wedge_{\mathrm{Or}(Q_V)} (p_V)_!(i_V)^*\boldsymbol{K}^G_{\mathcal{A}} \arrow[d, "(6)"', "\cong"] & \\
(EQ_V)_+ \wedge_{Q_V} (p_V)_!(i_V)^*\boldsymbol{K}^G_{\mathcal{A}}(Q_V/1) \arrow[d, "(7)"', "\cong"] & \\
(EQ_V)_+ \wedge_{Q_V} \boldsymbol{K}^G_{\mathcal{A}}(G/K) \arrow[d, "(8)"', "\cong"] & \\
T_{\boldsymbol{K}(R_\sigma)\colon \boldsymbol{K}^G_{\mathcal{A}}(G/K)\to \boldsymbol{K}^G_{\mathcal{A}}(G/K)} \arrow[ruuuu, bend right, "\boldsymbol{a}'(G/V)"'] &
\end{tikzcd}
$$

Here is a short explanation of the diagram above. The map $\boldsymbol{a}(G/V)$ is basically given by the projection $G/V \times \underline{E}G \to G/V$. Following the equivalences (1) through (5), this corresponds to projecting EQ_V to a point. On the domain of the equivalence (8), this corresponds to projecting EQ_V to a point and taking the inclusion-induced map $\boldsymbol{K}^G_{\mathcal{A}}(G/K) \to \boldsymbol{K}^G_{\mathcal{A}}(G/V)$ on the other factor. But this is precisely the definition of the map $\boldsymbol{a}'(G/V)$.

From the diagram (5.9) (including the preferred equivalences and the fact that a natural isomorphism of functors induces a preferred homotopy after applying the

K-theory spectrum) we obtain the following diagram of spectra, which commutes up to homotopy and has weak homotopy equivalences as vertical arrows:

$$\begin{tikzcd}
T_{\boldsymbol{K}(R_\sigma):\boldsymbol{K}^G_{\mathcal{A}}(G/K)\to\boldsymbol{K}^G_{\mathcal{A}}(G/K)} \arrow[r, "\boldsymbol{a}'(G/V)"] & \boldsymbol{K}^G_{\mathcal{A}}(G/V) \\
T_{\boldsymbol{K}(\Phi):\boldsymbol{K}\left(\int_{\widehat{K}}\mathcal{A}\right)\to\boldsymbol{K}\left(\int_{\widehat{K}}\mathcal{A}\right)} \arrow[r, "\boldsymbol{a}''(G/V)"] \arrow[u, "\simeq"] & \boldsymbol{K}\left(\int_{\widehat{V}}\mathcal{A}\right) \arrow[u, "\simeq"'] \\
T_{\boldsymbol{K}(\phi):\boldsymbol{K}(\mathcal{B})\to\boldsymbol{K}(\mathcal{B})} \arrow[r, "\boldsymbol{a}_{\mathcal{B}}"] \arrow[u, "\mathrm{id}"] & \boldsymbol{K}\left(\int_{\widehat{Q}}\mathcal{B}\right) \arrow[u, "\cong"']
\end{tikzcd}$$

We obtain from the diagram (5.8) the following commutative diagram of spectra with weak homotopy equivalences as vertical arrows:

$$\begin{tikzcd}
\boldsymbol{NK}^G_{\mathcal{A}}(G/V) \arrow[r, "\boldsymbol{b}(G/V)"] & \boldsymbol{K}^G_{\mathcal{A}}(G/V) \\
\boldsymbol{NK}(\mathcal{B}) \arrow[r, "\boldsymbol{b}_{\mathcal{B}}"] \arrow[u, "\simeq"] & \boldsymbol{K}(\int_{\widehat{Q_V}}\mathcal{B}) \arrow[u, "\simeq"']
\end{tikzcd}$$

We conclude from the three diagrams of spectra above that

$$\boldsymbol{a}(G/V)\vee\boldsymbol{b}(G/V):\underline{\boldsymbol{K}}^G_{\mathcal{A}}(G/V)\vee\boldsymbol{NK}^G_{\mathcal{A}}(G/V)\to\boldsymbol{K}^G_{\mathcal{A}}(G/V)$$

is a weak homotopy of spectra if and only if

$$\boldsymbol{a}_{\mathcal{B}}\vee\boldsymbol{b}_{\mathcal{B}}:T_{\boldsymbol{K}(\phi):\boldsymbol{K}(\mathcal{B})\to\boldsymbol{K}(\mathcal{B})}\vee\boldsymbol{NK}(\mathcal{B})\to\boldsymbol{K}\left(\int_{\widehat{Q_V}}\mathcal{B}\right)$$

is a weak homotopy equivalence. Since this is just the assertion of Theorem 4.2, the claim of Lemma 7.5 follows in the case where H is an infinite virtually cyclic group of type I.

It remains to consider the case where H is finite. Then $\boldsymbol{NK}^G_{\mathcal{A}}(G/V)$ is, by definition, the trivial spectrum. Hence it remains to show for a finite subgroup H of G that $\boldsymbol{a}(G/H):\underline{\boldsymbol{K}}^G_{\mathcal{A}}(G/H)\to\boldsymbol{K}^G_{\mathcal{A}}(G/H)$ is a weak homotopy equivalence. This follows from the fact that the projection $G/H\times\underline{E}G\to G/H$ is a G-homotopy equivalence for finite H. □

Recall that any covariant $\mathsf{Or}(G)$-spectrum $\boldsymbol{E}$ determines a G-homology theory $H^G_*(-;\boldsymbol{E})$ satisfying $H^G_n(G/H;\boldsymbol{E})=\pi_n(\boldsymbol{E}(G/H))$, namely (see [Davis and Lück 1998]) put

$$H^G_*(X;\boldsymbol{E}):=\pi_*(O^G(X)\wedge_{\mathsf{Or}(G)}\boldsymbol{E}). \tag{7.6}$$

In the sequel we often follow the convention in the literature to abbreviate $\underline{\underline{E}}G := E_{\mathcal{VC}}(G)$ for the family $\mathcal{VC}$ of virtually cyclic subgroups. Recall that for two families of subgroups $\mathcal{F}_1$ and $\mathcal{F}_2$ with $\mathcal{F}_1 \subseteq \mathcal{F}_2$ there is, up to G-homotopy, one G-map $f : E_{\mathcal{F}_1}(G) \to E_{\mathcal{F}_2}(G)$. We will define $H_n(E_{\mathcal{F}_1}(G) \to E_{\mathcal{F}_2}(G); \boldsymbol{K}_R^G) := H_n(\mathrm{cyl}(f), E_{\mathcal{F}_1}(G); \boldsymbol{K}_R^G)$, where $(\mathrm{cyl}(f), E_{\mathcal{F}_1}(G))$ is the G-pair coming from the mapping cylinder of f.

Notice that $\boldsymbol{NK}_{\mathcal{A}}^G$ is defined only over $\mathrm{Or}_{VCyc_I}(G)$. It can be extended to a spectrum over $\mathrm{Or}(G)$ by applying the coinduction functor — see [Davis and Lück 1998, Definition 1.8] — associated to the inclusion $\mathrm{Or}_{\mathcal{VC}_I}(G) \to \mathrm{Or}(G)$, so that the G-homology theory $H_n^G(-; \boldsymbol{NK}_{\mathcal{A}}^G)$ makes sense for all pairs of G-CW-complexes (X, A). Moreover, $H_n^G(X; \boldsymbol{NK}_{\mathcal{A}}^G)$ can be identified with $\pi_n(O^G(X) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G)$ for all G-CW-complexes X.

The remainder of this section is devoted to the proof of Theorem 0.1. Its proof will need the following result, taken from [Davis et al. 2011, Remark 1.6]:

Theorem 7.7 (passage from $\mathcal{VC}_I$ to $\mathcal{VC}$ in K-theory). *The relative assembly map*

$$H_n^G(E_{\mathcal{VC}_I}(G); \boldsymbol{K}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G)$$

is bijective for all $n \in \mathbb{Z}$.

Hence, in the proof of Theorem 0.1 we only have to deal with the passage from $\mathcal{F}$in to $\mathcal{VC}_I$.

Proof of Theorem 0.1. From Lemma 7.5 and [Davis and Lück 1998, Lemma 4.6], we obtain a weak equivalence of spectra

$$\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} (\boldsymbol{a} \vee \boldsymbol{b}) : O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} (\underline{\boldsymbol{K}}_{\mathcal{A}}^G \vee \boldsymbol{NK}_{\mathcal{A}}^G) \\ \to O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{K}_{\mathcal{A}}^G.$$

Hence we obtain a weak equivalence of spectra

$$(\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{a}) \vee (\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{b}) : \\ \big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \underline{\boldsymbol{K}}_{\mathcal{A}}^G\big) \vee \big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G\big) \\ \to O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{K}_{\mathcal{A}}^G.$$

If we combine this with Lemma 7.4 we obtain a weak equivalence of spectra

$$(f \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \mathrm{id}) \vee (\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{b}) : \\ (O^G(\underline{E}G) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{K}_{\mathcal{A}}^G) \vee (O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G) \\ \to O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{K}_{\mathcal{A}}^G.$$

Using Lemma 7.3 this yields a natural weak equivalence of spectra

$$(f \wedge_{\mathrm{Or}(G)} \mathrm{id}) \vee \boldsymbol{b}' : (O^G(\underline{E}G) \wedge_{\mathrm{Or}(G)} \boldsymbol{K}_{\mathcal{A}}^G) \vee \big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G\big) \\ \to O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}(G)} \boldsymbol{K}_{\mathcal{A}}^G,$$

where $\boldsymbol{b}'$ comes from $\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{b}$. If we take homotopy groups, we obtain for every $n \in \mathbb{Z}$ an isomorphism

$$H_n^G(f; \boldsymbol{K}_{\mathcal{A}}^G) \oplus \pi_n(\boldsymbol{b}') : H_n^G(\underline{E}G; \boldsymbol{K}_{\mathcal{A}}^G) \oplus \pi_n\big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G\big) \\ \xrightarrow{\cong} H_n(E_{\mathcal{VC}_I}(G); \boldsymbol{K}_{\mathcal{A}}^G).$$

We have already explained above that $H_n^G(E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G)$ can be identified with $\pi_n\big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G\big)$. Since, by construction, $\boldsymbol{NK}_{\mathcal{A}}^G(G/H)$ is the trivial spectrum for finite H and all isotropy groups of $\underline{E}G$ are finite, we conclude $H_n^G(\underline{E}G; \boldsymbol{NK}_{\mathcal{A}}^G) = 0$ for all $n \in \mathbb{Z}$ from Lemma 7.3. We derive from the long exact sequence of $f : \underline{E}(G) \to E_{\mathcal{VC}_I}(G)$ that the canonical map

$$H_n^G(E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G)$$

is bijective for all $n \in \mathbb{Z}$. Hence we obtain for all $n \in \mathbb{Z}$ a natural isomorphism

$$H_n^G(f; \boldsymbol{K}_{\mathcal{A}}^G) \oplus b_n : H_n^G(\underline{E}G; \boldsymbol{K}_{\mathcal{A}}^G) \oplus H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G) \\ \xrightarrow{\cong} H_n(E_{\mathcal{VC}_I}(G); \boldsymbol{K}_{\mathcal{A}}^G).$$

From the long exact homology sequence associated to $f : \underline{E}G \to E_{\mathcal{VC}_I}(G)$, we conclude that the map

$$H_n^G(f; \boldsymbol{K}_{\mathcal{A}}^G) : H_n^G(\underline{E}G; \boldsymbol{K}_{\mathcal{A}}^G) \to H_n^G(E_{\mathcal{VC}_I}(G); \boldsymbol{K}_{\mathcal{A}}^G)$$

is split injective, there is a natural splitting

$$H_n^G(E_{\mathcal{VC}_I}(G); \boldsymbol{K}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{E}G; \boldsymbol{K}_{\mathcal{A}}^G) \oplus H_n(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}_{\mathcal{A}}^G),$$

and there exists a natural isomorphism, which is induced by the natural transformation $\boldsymbol{b} : \boldsymbol{NK}_{\mathcal{A}}^G \to \boldsymbol{K}_{\mathcal{A}}^G$ of spectra over $\mathrm{Or}_{\mathcal{VC}_I}(G)$,

$$H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}_{\mathcal{A}}^G).$$

Now Theorem 0.1 follows from Theorem 7.7. □

8. Involutions and vanishing of Tate cohomology

8A. ***Involutions on K-theory spectra.*** Let $\mathcal{A} = (\mathcal{A}, I)$ be *an additive G-category with involution*, i.e., an additive G-category $\mathcal{A}$ together with a contravariant functor

$I\colon \mathcal{A} \to \mathcal{A}$ satisfying $I \circ I = \mathrm{id}_{\mathcal{A}}$ and $I \circ R_g = R_g \circ I$ for all $g \in G$. Examples coming from twisted group rings, or more generally crossed product rings equipped with involutions twisted by orientation homomorphisms, are discussed in [Bartels and Lück 2010, Section 8].

In the sequel for a category $\mathcal{C}$ we denote its *opposite* category by $\mathcal{C}^{\mathrm{op}}$. It has the same objects as $\mathcal{C}$. A morphism in $\mathcal{C}^{\mathrm{op}}$ from x to y is a morphism $y \to x$ in $\mathcal{C}$. Obviously we can and will identify $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

Next we define a covariant functor

$$I(G/H)\colon \int_{\mathcal{G}^G(G/H)} \mathcal{A} \to \left(\int_{\mathcal{G}^G(G/H)} \mathcal{A}\right)^{\mathrm{op}}. \tag{8.1}$$

It is defined to extend the involution

$$\coprod_{x \in \mathcal{G}^G(G/H)} I\colon \coprod_{x \in \mathcal{G}^G(G/H)} \mathcal{A} \to \left(\coprod_{x \in \mathcal{G}^G(G/H)} \mathcal{A}\right)^{\mathrm{op}}$$

and to send a structural morphism $T_g\colon (g_1H, A\cdot g) \to (g_2H, A)$ to the morphism $T_{g^{-1}}\colon (g_2H, I(A)) \to (g_1H, I(A)\cdot g)$. One easily checks $I(G/H) \circ I(G/H) = \mathrm{id}$.

Notice that there is a canonical identification $\boldsymbol{K}(\mathcal{B}^{\mathrm{op}}) = \boldsymbol{K}(\mathcal{B})$ for every additive category $\mathcal{B}$. Hence $I(G/H)$ induces a map of spectra

$$\boldsymbol{i}(G/H) = \boldsymbol{K}(I(G/H))\colon \boldsymbol{K}\left(\int_{\mathcal{G}^G(G/H)} \mathcal{A}\right) \to \boldsymbol{K}\left(\int_{\mathcal{G}^G(G/H)} \mathcal{A}\right)$$

such that $\boldsymbol{i}(G/H) \circ \boldsymbol{i}(G/H) = \mathrm{id}$. Let $\mathbb{Z}/2\text{-}\mathsf{Spectra}$ be the category of spectra with a (strict) $\mathbb{Z}/2$-operation. Thus the functor $\boldsymbol{K}_R^G$ becomes a functor

$$\boldsymbol{K}_R^G\colon \mathsf{Or}(G) \to \mathbb{Z}/2\text{-}\mathsf{Spectra}. \tag{8.2}$$

Consider an infinite virtually cyclic subgroup $V \subseteq G$ and a fixed generator $\sigma \in Q_V$. The functor $I(G/V)$ of (8.1) induces functors

$$I(G/H)[\sigma]\colon \int_{\mathcal{G}^G(G/H)[\sigma]} \mathcal{A} \to \left(\int_{\mathcal{G}^G(G/H)[\sigma^{-1}]} \mathcal{A}\right)^{\mathrm{op}},$$
$$I(G/H)_K\colon \int_{\mathcal{G}^G(G/H)_K} \mathcal{A} \to \left(\int_{\mathcal{G}^G(G/H)_K} \mathcal{A}\right)^{\mathrm{op}}.$$

Since $\mathrm{ev}(G/V)[\sigma^{-1}]_* \circ I(G/V)[\sigma] = I(G/V)_K \circ \mathrm{ev}(G/V)[\sigma]$ and

$$j(G/V)[\sigma^{-1}]_* \circ I(G/V)[\sigma] = I(G/V) \circ j(G/V)[\sigma]_*,$$

we obtain a commutative diagram of spectra

$$\begin{tikzcd}
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)_K}\mathcal{A}\big) \arrow[r, "\boldsymbol{K}(I(G/V)_K)"] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)_K}\mathcal{A}\big) \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma]}\mathcal{A}\big) \arrow[r, "\boldsymbol{K}(I(G/V))[\sigma]"] \arrow[u, "\boldsymbol{K}(\mathrm{ev}(G/V)[\sigma]_*)"] \arrow[d, "\boldsymbol{K}(j(G/V)[\sigma]_*)"'] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma^{-1}]}\mathcal{A}\big) \arrow[u, "\boldsymbol{K}(\mathrm{ev}(G/V)[\sigma^{-1}]_*)"'] \arrow[d, "\boldsymbol{K}(j(G/V)[\sigma^{-1}]_*)"] \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)}\mathcal{A}\big) \arrow[r, "\boldsymbol{K}(I(G/V))"] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)}\mathcal{A}\big)
\end{tikzcd}$$

Since $I(G/H)[\sigma^{-1}]\circ I(G/H)[\sigma]=\mathrm{id}$ and $I(G/H)_K\circ I(G/H)_K=\mathrm{id}$, we obtain a $\mathbb{Z}/2$-operation on $\boldsymbol{NK}^G_{\mathcal{A}}$ and hence a functor

$$\boldsymbol{NK}^G_{\mathcal{A}}:\mathsf{Or}(G)\to\mathbb{Z}/2\text{-}\mathsf{Spectra},\tag{8.3}$$

and we conclude:

Lemma 8.4. *The transformation $\boldsymbol{b}:\boldsymbol{NK}^G_{\mathcal{A}}\to\boldsymbol{K}^G_{\mathcal{A}}$ of $\mathsf{Or}_{\mathcal{VC}_I}(G)$-spectra is compatible with the $\mathbb{Z}/2$-actions.*

8B. *Orientable virtually cyclic subgroups of type I.*

Definition 8.5 (orientable virtually cyclic subgroups of type I). Given a group G, we say that *the infinite virtually cyclic subgroups of type I of G are orientable* if there is, for every virtually cyclic subgroup V of type I, a choice σ_V of a generator of the infinite cyclic group Q_V with the following property: whenever V and V' are infinite virtually cyclic subgroups of type I, and $f:V\to V'$ is an inclusion or a conjugation by some element of G, then the map $Q_f:Q_V\to Q_W$ sends σ_V to a positive multiple of σ_W. Such a choice of elements $\{\sigma_V\mid V\in\mathcal{VC}_I\}$ is called an *orientation*.

Lemma 8.6. *Suppose that the virtually cyclic subgroups of type I of G are orientable. Then all infinite virtually cyclic subgroups of G are of type I, and the fundamental group $\mathbb{Z}\rtimes\mathbb{Z}$ of the Klein bottle is not a subgroup of G.*

Proof. Suppose that G contains an infinite virtually cyclic subgroup V of type II. Then Q_V is the infinite dihedral group. Its commutator $[Q_V,Q_V]$ is infinite cyclic. Let W be the preimage of the commutator $[Q_V,Q_V]$ under the canonical projection $p_V:V\to Q_V$. There exists an element $y\in Q_V$ such that conjugation by y induces $-\mathrm{id}$ on $[Q_V,Q_V]$. Obviously W is an infinite virtually cyclic group of type I, and the restriction of p_V to W is the canonical map $p_W:W\to Q_W=[Q_V,Q_V]$. Choose an element $x\in V$ with $p_V(x)=y$. Conjugation by x induces an automorphism of W which induces $-\mathrm{id}$ on Q_W. Hence the virtually cyclic subgroups of type I of G are not orientable.

The statement about the Klein bottle is obvious. $\square$

For the notions of a CAT(0)-group and of a hyperbolic group we refer for instance to [Bridson and Haefliger 1999; Ghys and de la Harpe 1990; Gromov 1987]. The fundamental group of a closed Riemannian manifold is hyperbolic if the sectional curvature is strictly negative, and is a CAT(0)-group if the sectional curvature is nonpositive.

Lemma 8.7. *Let G be a hyperbolic group. Then the infinite virtually cyclic subgroups of type I of G are orientable if and only if all infinite virtually cyclic subgroups of G are of type I.*

Proof. The "only if" statement follows from Lemma 8.6. To prove the "if" statement, assume that all infinite virtually cyclic subgroups of G are of type I.

By [Lück and Weiermann 2012, Example 3.6], every hyperbolic group satisfies the condition $(NM_{\mathcal{F}\text{in}\subseteq\mathcal{VC}_I})$, i.e., every infinite virtually cyclic subgroup V is contained in a unique maximal one $V_{\max}$ and the normalizer of $V_{\max}$ satisfies $NV_{\max} = V_{\max}$. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups. Since by assumption $V \in \mathcal{M}$ is of type I, we can fix a generator $\sigma_V \in Q_V$ for each $V \in \mathcal{M}$.

Consider any infinite virtually cyclic subgroup W of G type I. Choose $g \in G$ and $V \in \mathcal{M}$ such that $gWg^{-1} \subseteq V$. Then conjugation with g induces an injection $Q_{c(g)} : Q_W \to Q_V$ by Lemma 1.1(vi). We equip W with the generator $\sigma_W \in Q_W$ for which there exists an integer $n \geq 1$ with $Q_{c(g)}(\sigma_W) = (\sigma_V)^n$. This is independent of the choice of g and V: for every $g \in G$ and $V \in \mathcal{M}$ with $|gVg^{-1} \cap V| = \infty$, the condition $(NM_{\mathcal{F}\text{in}\subseteq\mathcal{VC}_I})$ implies that g belongs to V and conjugation with an element $g \in V$ induces the identity on Q_V. □

Lemma 8.8. *Let G be a* CAT(0)*-group. Then the infinite virtually cyclic subgroups of type I of G are orientable if and only if all infinite virtually cyclic subgroups of G are of type I and the fundamental group $\mathbb{Z} \rtimes \mathbb{Z}$ of the Klein bottle is not a subgroup of G.*

Proof. Because of Lemma 8.6 it suffices to construct for a CAT(0)-group an orientation for its infinite virtually cyclic subgroups of type I, provided that all infinite virtually cyclic subgroups of G are of type I and the fundamental group $\mathbb{Z} \rtimes \mathbb{Z}$ of the Klein bottle is not a subgroup of G.

Consider on the set of infinite virtually cyclic subgroups of type I of G the relation $\sim$, where we put $V_1 \sim V_2$ if and only if there exists an element $g \in G$ with $|gV_1g^{-1} \cap V_2| = \infty$. This is an equivalence relation since, for any infinite virtually cyclic group V and elements $v_1, v_2 \in V$ of infinite order, we can find integers n_1, n_2 with $v_1^{n_1} = v_2^{n_2}$, $n_1 \neq 0$ and $n_2 \neq 0$. Choose a complete system of representatives $\mathcal{S}$ for the classes under $\sim$. For each element $V \in \mathcal{S}$ we choose an orientation $\sigma_V \in Q_V$.

Given any infinite virtually cyclic subgroup $W \subseteq G$ of type I we define a preferred generator $\sigma_W \in Q_W$ as follows: Choose $g \in G$ and $V \in \mathcal{S}$ with $|gWg^{-1} \cap V| = \infty$. Let $i_1 : gWg^{-1} \cap V \to W$ be the injection sending v to $g^{-1}vg$ and $i_2 : gWg^{-1} \cap V \to V$ be the inclusion. By Lemma 1.1(vi) we obtain injections of infinite cyclic groups $Q_{i_1} : Q_{gWg^{-1}\cap V} \to Q_W$ and $Q_{i_2} : Q_{gWg^{-1}\cap V} \to Q_V$. Equip Q_W with the generator σ_W for which there exist integers $n_1,\ n_2 \geq 1$ and $\sigma \in Q_{gWg^{-1}\cap V}$ with $Q_{i_1}(\sigma) = (\sigma_W)^{n_1}$ and $Q_{i_2}(\sigma) = (\sigma_V)^{n_2}$.

We have to show that this is well-defined. Obviously it is independent of the choice of σ, n_1 and n_2. It remains to show that the choice of g does not matter. For this purpose we have to consider the special case $W = V$ and have to show that the new generator σ_W agrees with the given one σ_V. We conclude from [Lück 2009, Lemma 4.2] and the argument about the validity of condition (C) appearing in the proof of [Lück 2009, Theorem 1.1(ii)] that there exists an infinite cyclic subgroup $C \subseteq gVg^{-1} \cap V$ such that g belongs to the normalizer $N_G C$. It suffices to show that conjugation with g induces the identity on C. Let $H \subseteq G$ be the subgroup generated by g and C. We obtain a short exact sequence $1 \to C \to H \xrightarrow{\mathrm{pr}} H/C \to 1$, where H/C is the cyclic subgroup generated by $\mathrm{pr}(g)$. Suppose that H/C is finite. Then H is an infinite virtually cyclic subgroup of G which must, by assumption, be of type I. Since the center of H must be infinite by Lemma 1.1(ii) and hence the intersection of the center of H with C is infinite cyclic, the conjugation action of g on C must be trivial. Suppose that H/C is infinite. Then H is the fundamental group of the Klein bottle if the conjugation action of g on C is nontrivial. Since the fundamental group of the Klein bottle is not a subgroup of G by assumption, the conjugation action of g on C is trivial also in this case. □

8C. *Proof of Theorem 0.2.* Let $\mathrm{Or}_{\mathcal{VC}_I \setminus \mathcal{F}\mathrm{in}}(G)$ be the full subcategory of the orbit category $\mathrm{Or}(G)$ consisting of those objects G/V for which V is an infinite virtually cyclic subgroup of type I. We obtain a functor

$$\mathrm{gen}(Q_?) : \mathrm{Or}_{\mathcal{VC}_I \setminus \mathcal{F}\mathrm{in}}(G) \to \mathbb{Z}/2\text{-}\mathsf{Sets}$$

sending G/V to $\mathrm{gen}(Q_V)$, and a G-map $f : G/V \to G/W$ to $\mathrm{gen}(f)$ as defined in (6.3). The $\mathbb{Z}/2$-action on $\mathrm{gen}(Q_V)$ is given by taking the inverse of a generator. The condition that the virtually cyclic subgroups of type I of G are orientable (see Definition 8.5) is equivalent to the condition that the functor $\mathrm{gen}(Q_?)$ is isomorphic to the constant functor sending G/V to $\mathbb{Z}/2$. A choice of an orientation corresponds to a choice of such an isomorphism.

Proof of Theorem 0.2. Because of Theorem 0.1 and Lemma 8.4 it suffices to show that the $\mathbb{Z}[\mathbb{Z}/2]$-module $H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G)$ is isomorphic to $\mathbb{Z}[\mathbb{Z}/2] \otimes_{\mathbb{Z}} A$ for some $\mathbb{Z}$-module A.

Fix an orientation $\{\sigma_V \mid V \in \mathcal{VC}_I\}$ in the sense of Definition 8.5. We have the $\mathrm{Or}_{\mathcal{VC}_I}(G)$-spectrum

$$\boldsymbol{NK}_R^{G'} : \mathrm{Or}_{\mathcal{VC}_I}(G) \to \mathtt{Spectra},$$

which sends G/V to the trivial spectrum if V is finite and to $\boldsymbol{NK}(G/V; \mathcal{A}, \sigma_V)$ if V is infinite virtually cyclic of type I. This is well-defined by the orientability assumption. Now there is an obvious natural isomorphism of functors from $\mathrm{Or}_{\mathcal{VC}_I}(G)$ to the category of $\mathbb{Z}/2$-spectra

$$\boldsymbol{NK}_{\mathcal{A}}^{G'} \wedge (\mathbb{Z}/2)_+ \xrightarrow{\cong} \boldsymbol{NK}_{\mathcal{A}}^{G},$$

which is a weak equivalence of $\mathrm{Or}_{\mathcal{VC}_I}(G)$-spectra. It induces a $\mathbb{Z}[\mathbb{Z}/2]$-isomorphism

$$H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^{G'}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^{G}).$$

This finishes the proof of Theorem 0.2. □

9. Rational vanishing of the relative term

This section is devoted to the proof of Theorem 0.3.

Consider the following diagram of groups, where the vertical maps are inclusions of subgroups of finite index and the horizontal arrows are automorphisms:

$$\begin{tikzcd} H \arrow[r, "\phi"] \arrow[d, "i"] & H \arrow[d, "i"] \\ K \arrow[r, "\psi"] & K \end{tikzcd}$$

We obtain a commutative diagram

$$\begin{tikzcd} K_n(RH_\phi[t]) \arrow[r, "{i[t]_*}"] \arrow[d, "(\mathrm{ev}_H)_*"] & K_n(RK_\psi[t]) \arrow[r, "{i[t]^*}"] \arrow[d, "(\mathrm{ev}_K)_*"] & K_n(RH_\phi[t]) \arrow[d, "(\mathrm{ev}_H)_*"] \\ K_n(RH) \arrow[r, "i_*"] & K_n(RK) \arrow[r, "i^*"] & K_n(RH) \end{tikzcd} \tag{9.1}$$

as follows: i_* and i^* are the maps induced by induction and restriction with the ring homomorphism $Ri : RH \to RK$; $i[t]_*$ and $i[t]^*$ are the maps induced by induction and restriction with the ring homomorphism $Ri[t] : RH_\phi[t] \to RK_\psi[t]$; $\mathrm{ev}_H : RH_\phi[t] \to RH$ and $\mathrm{ev}_K : RK_\psi[t] \to RK$ are the ring homomorphisms given by putting $t = 0$.

The left square is obviously well-defined and commutative. The right square is well-defined since the restriction of RK to RH by Ri is a finitely generated free RH-module and the restriction of $RK_\psi[t]$ to $RH_\phi[t]$ by $Ri[t]$ is a finitely generated free RH_ϕ-module by the following argument.

Put $l := [K : H]$. Choose a subset $\{k_1, k_2, \ldots, k_l\}$ of K such that K/H can be written as $\{k_1 H, k_2 H, \ldots, k_l H\}$. The map

$$\alpha : \bigoplus_{i=1}^{l} RH \xrightarrow{\cong} i^*RK, \quad (x_1, x_2, \ldots, x_l) \mapsto \sum_{i=1}^{l} x_i \cdot k_i,$$

is an homomorphism of RH-modules and the map

$$\beta : \bigoplus_{i=1}^{l} RH_\phi[t] \xrightarrow{\cong} i[t]^*RK_\psi[t], \quad (y_1, y_2, \ldots, y_l) \mapsto \sum_{i=1}^{l} y_i \cdot k_i,$$

is a homomorphism of $RH_\phi[t]$-modules. Obviously α is bijective. The map β is bijective since for any integer m we get $K/H = \{\psi^m(k_1)H, \psi^m(k_2)H, \ldots, \psi^m(k_i)H\}$.

To show that the right square commutes we have to define for every finitely generated projective $RK_\psi[t]$-module P a natural RH-isomorphism

$$T(P) : (\mathrm{ev}_H)_* i[t]^* P \xrightarrow{\cong} i^*(\mathrm{ev}_K)_* P.$$

First we define $T(P)$. By the adjunction of induction and restriction it suffices to construct a natural map $T'(P) : i_*(\mathrm{ev}_H)_* i[t]^* P \to (\mathrm{ev}_K)_* P$. Since $i \circ \mathrm{ev}_H = \mathrm{ev}_K \circ i[t]$ we have to construct a natural map $T''(P) : i[t]_* i[t]^* P \to P$, since then we define $T'(P)$ to be $(\mathrm{ev}_K)_*(T''(P))$. Now define $T''(P)$ to be the adjoint of the identity $\mathrm{id} : i[t]^* P \to i[t]^* P$. Explicitly $T(P)$ sends an element $h \otimes x$ in $(\mathrm{ev}_H)_* i[t]^* P = RH \otimes_{\mathrm{ev}_H} i[t]^* P$ to the element $i(h) \otimes x$ in $i^*(\mathrm{ev}_K)_* P = RK \otimes_{\mathrm{ev}_K} P$.

Obviously $T(P)$ is natural in P and compatible with direct sums. Hence, in order to show that $T(P)$ is bijective for all finitely generated projective $RK_\psi[t]$-modules P, it suffices to do that for $P = RK_\psi[t]$. Now the claim follows since the following diagram of RH-modules commutes:

$$\begin{tikzcd}
RH \otimes_{\mathrm{ev}_H} i[t]^*RK_\psi[t] \arrow[rr, "T(RK_\psi{[t]})"] & & i^*(RK \otimes_{\mathrm{ev}_K} RK_\psi[t]) \arrow[d, "\cong"] \\
RH \otimes_{\mathrm{ev}_H} \left(\bigoplus_{i=1}^{l} RH_\phi[t]\right) \arrow[u, "\mathrm{id} \otimes_{\mathrm{ev}_H} \beta"', "\cong"] & & i^*RK \\
\bigoplus_{i=1}^{l} RH \otimes_{\mathrm{ev}_H} RH_\phi[t] \arrow[u, "\cong"] \arrow[rr, "\cong"] & & \bigoplus_{i=1}^{l} RH \arrow[u, "\alpha"]
\end{tikzcd}$$

where the isomorphisms α and β have been defined above and all other arrows marked with $\cong$ are the obvious isomorphisms. Recall that $NK_n(RH, R\phi)$ is by definition the kernel of $(\mathrm{ev}_H)_* : K_n(RH_\phi[t]) \to K_n(RH)$ and the analogous statement holds for $NK_n(RK, R\psi)$.

The diagram (9.1) induces homomorphisms

$$\begin{aligned} i_* &: NK_n(RH, R\phi) \to NK_n(RK, R\psi), \\ i^* &: NK_n(RK, R\psi) \to NK_n(RH, R\phi). \end{aligned}$$

Since both composites

$$K_n(RH_\phi[t]) \xrightarrow{i[t]^* \circ i[t]_*} K_n(RH_\phi[t]) \quad \text{and} \quad K_n(RH) \xrightarrow{i^* \circ i_*} K_n(RH)$$

are multiplication with l, we conclude:

Lemma 9.2. *The composite $i^* \circ i_* : NK_n(RH, R\phi) \to NK_n(RH, R\phi)$ is multiplication with l for all $n \in \mathbb{Z}$.*

Lemma 9.3. *Let $\phi : K \to K$ be an inner automorphism of the group K. Then there is, for all $n \in \mathbb{Z}$, an isomorphism*

$$NK_n(RK, R\phi) \xrightarrow{\cong} NK_n(RK).$$

Proof. Let k be an element such that ϕ is given by conjugation with k. We obtain a ring isomorphism

$$\eta : RK_{R\phi}[t] \xrightarrow{\cong} RK[t], \qquad \sum_i \lambda_i t^i \mapsto \lambda_i k^i t^i.$$

Let $\mathrm{ev}_{RK,\phi} : RK_\phi[t] \to RK$ and $\mathrm{ev}_{RK} : RK[t] \to RK$ be the ring homomorphisms given by putting $t = 0$. Then we obtain a commutative diagram with isomorphisms as vertical arrows

$$\begin{tikzcd} K_n(RK_{R\phi}[t]) \arrow[r, "\cong", "\eta"'] \arrow[d, "\mathrm{ev}_{RK,\phi}"] & K_n(RK[t]) \arrow[d, "\mathrm{ev}_{RK}"] \\ K_n(RK) \arrow[r, "\cong", "\mathrm{id}"'] & K_n(RK) \end{tikzcd}$$

It induces the desired isomorphism $NK_n(RK, R\phi) \xrightarrow{\cong} NK_n(RK)$. □

Remark. As the referee has pointed out, this results holds more generally (with identical proof) for the twisted Bass group $NF(S, \phi)$ of any functor F from rings to abelian groups and any inner ring automorphism $\phi : S \to S$.

Theorem 9.4. *Let R be a regular ring. Let K be a finite group of order r and let $\phi : K \xrightarrow{\cong} K$ be an automorphism of order s. Then $NK_n(RK, R\phi)[1/rs] = 0$ for all $n \in \mathbb{Z}$. In particular, $NK_n(RK, R\phi) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for all $n \in \mathbb{Z}$.*

Proof. Let t be a generator of the cyclic group $\mathbb{Z}/s$ of order s. Consider the semidirect product $K \rtimes_\phi \mathbb{Z}/s$. Let $i : K \to K \rtimes_\phi \mathbb{Z}/s$ be the canonical inclusion.

Let ψ be the inner automorphism of $K \rtimes_\phi \mathbb{Z}/s$ given by conjugation with t. Then $[K \rtimes_\phi \mathbb{Z}/s : K] = s$ and the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, "i"'] & K \arrow[d, "i"] \\ K \rtimes_\phi \mathbb{Z}/s \arrow[r, "\psi"'] & K \rtimes_\phi \mathbb{Z}/s \end{tikzcd}$$

Lemmas 9.2 and 9.3 yield maps $i_* : NK_n(RK, \phi) \to NK_n(R[K \rtimes_\phi \mathbb{Z}/s])$ and $i^* : NK_n(R[K \rtimes_\phi \mathbb{Z}/s]) \to NK_n(RK, \phi)$ such that $i^* \circ i_* = s \cdot \mathrm{id}$. This implies that $NK_n(RK, \phi)[1/s]$ is a direct summand in $NK_n(R[K \rtimes_\phi \mathbb{Z}/s])[1/s]$. Since R is regular by assumption and hence $NK_n(R)$ vanishes for all $n \in \mathbb{Z}$, we conclude from [Hambleton and Lück 2012, Theorem A] that

$$NK_n(R[K \rtimes_\phi \mathbb{Z}/s])[1/rs] = 0.$$

(For $R = \mathbb{Z}$ and some related rings, this has already been proved by Weibel [1981, (6.5), p. 490].) This implies $NK_n(RK, \phi)[1/rs] = 0$. □

Theorem 9.4 has already been proved for $R = \mathbb{Z}$ in [Grunewald 2008, Theorem 5.11].

Now we are ready to give the proof of Theorem 0.3.

Proof of Theorem 0.3. Because of Theorem 0.1 it suffices to prove, for all $n \in \mathbb{Z}$,

$$H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_R^G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \{0\}.$$

There is a spectral sequence converging to $H_{p+q}^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_R^G)$ whose E^2-term is the Bredon homology

$$E_{p,q}^2 = H_p^{\mathbb{Z}\mathrm{Or}_{\mathcal{VC}_I}(G)}(\underline{E}G \to E_{\mathcal{VC}_I}(G); \pi_q(\boldsymbol{NK}_R^G))$$

with coefficients in the covariant functor from $\mathrm{Or}_{\mathcal{VC}_I}(G)$ to the category of $\mathbb{Z}$-modules coming from composing $\boldsymbol{NK}_R^G : \mathrm{Or}_{\mathcal{VC}_I}(G) \to \mathsf{Spectra}$ with the functor taking the q-homotopy group; see [Davis and Lück 1998, Theorems 4.7 and 7.4]. Since $\mathbb{Q}$ is flat over $\mathbb{Z}$, it suffices to show, for all $V \in \mathcal{VC}_I$,

$$\pi_q(\boldsymbol{NK}_R^G(G/V)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

If V is finite, $\boldsymbol{NK}_R^G(G/V)$ is by construction the trivial spectrum and the claim is obviously true. If V is a virtually cyclic group of type I, then we conclude from the diagram (5.6) that

$$\pi_n(\boldsymbol{NK}_R^G(G/V)) \cong NK_n(RK_V, R\phi) \oplus NK_n(RK_V, R\phi^{-1}).$$

Now the claim follows from Theorem 9.4. □

10. On the computation of the relative term

In this section we give some further information about the computation of the relative term $H_n^G(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_R^G) \cong H_n^G(\underline{E}G \to E_{\mathcal{VC}}(G); \boldsymbol{NK}_R^G)$.

Lück and Weiermann [2012] give a systematic analysis of how the space $E_{\mathcal{VC}_I}(G)$ is obtained from $\underline{E}G$. We say that G satisfies the condition $(M_{\mathcal{F}\text{in}\subseteq\mathcal{VC}_I})$ if any virtually cyclic subgroup of type I is contained in a unique maximal infinite cyclic subgroup of type I. We say that G satisfies the condition $(NM_{\mathcal{F}\text{in}\subseteq\mathcal{VC}_I})$ if it satisfies $(M_{\mathcal{F}\text{in}\subseteq\mathcal{VC}_I})$ and, for any maximal virtually cyclic subgroup V of type I, its normalizer N_GV agrees with V. Every word hyperbolic group satisfies $(NM_{\mathcal{F}\text{in}\subseteq\mathcal{VC}_I})$; see [Lück and Weiermann 2012, Example 3.6].

Suppose that G satisfies $(M_{\mathcal{F}\text{in}\subseteq\mathcal{VC}_I})$. Let $\mathcal{M}$ be a complete system of representatives V of the conjugacy classes of maximal virtually cyclic subgroups of type I. Then we conclude from [Lück and Weiermann 2012, Corollary 2.8] that there exists a G-pushout of G-CW-complexes with inclusions as horizontal maps

$$\begin{tikzcd}
\coprod_{V\in\mathcal{M}} G\times_{N_GV}\underline{E}N_GV \arrow[r, "i"] \arrow[d, "\coprod_{V\in\mathcal{M}}\mathrm{id}_G\times f_V"] & \underline{E}G \arrow[d, "f"] \\
\coprod_{V\in\mathcal{M}} G\times_{N_GV} E_{\mathcal{VC}_I}(N_GV) \arrow[r] & E_{\mathcal{VC}_I}(G)
\end{tikzcd}$$

This yields for all $n\in\mathbb{Z}$ an isomorphism, using the induction structure in the sense of [Lück 2002, Section 1],

$$\bigoplus_{V\in\mathcal{M}} H_n^{N_GV}(\underline{E}N_GV \to E_{\mathcal{VC}_I}(N_GV); \boldsymbol{K}_R^{N_GV}) \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}_R^G).$$

Combining this with Theorem 0.1 yields the isomorphism

$$\bigoplus_{V\in\mathcal{M}} H_n^{N_GV}(\underline{E}N_GV \to E_{\mathcal{VC}_I}(N_GV); \boldsymbol{NK}_R^{N_GV}) \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}_R^G).$$

Suppose now that G satisfies $(NM_{\mathcal{F}\text{in}\subseteq\mathcal{VC}_I})$ and recall that $NK_R^G(V/H)=0$ for finite H, by definition. Then the isomorphism above reduces to the isomorphism

$$\bigoplus_{V\in\mathcal{M}} \pi_n(\boldsymbol{NK}_R^V(V/V)) \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}_R^G),$$

and $\pi_n(\boldsymbol{NK}_R^V(V/V))$ is the Nil-term $NK_n(RK_V, R\phi)\oplus NK_n(RK_V; R\phi^{-1})$ appearing in the twisted version of the Bass–Heller–Swan decomposition of RV (see [Grayson 1988, Theorems 2.1 and 2.3]) if we write $V\cong K_V\rtimes_\phi\mathbb{Z}$.

11. Fibered version

We illustrate in this section, by an example which will be crucial in [Farrell et al. 2016], that we do get information from our setting also in a fibered situation.

Let $p : X \to B$ be a map of path connected spaces. We will assume that it is π_1-surjective, i.e., induces an epimorphism on fundamental groups. Suppose that B admits a universal covering $q : \tilde{B} \to B$.

Choose base points $x_0 \in X$, $b_0 \in B$ and $\tilde{b}_0 \in \tilde{B}$ satisfying $p(x_0) = b_0 = q(\tilde{b}_0)$. We will abbreviate $\Gamma = \pi_1(X, x_0)$ and $G = \pi_1(B, b_0)$. Recall that we have a free right proper G-action on $\tilde{B}$ and q induces a homeomorphism $\tilde{B}/G \xrightarrow{\cong} B$. For a subgroup $H \subseteq G$ denote by $q(G/H) : \tilde{B} \times_G G/H = \tilde{B}/H \to B$ the obvious covering induced by q. The pullback construction yields a commutative square of spaces

$$\begin{tikzcd}
X(G/H) \arrow[r, "\bar{q}(G/H)"] \arrow[d, "\bar{p}(G/H)"'] & X \arrow[d, "p"] \\
\tilde{B} \times_G G/H \arrow[r, "q(G/H)"'] & B
\end{tikzcd}$$

where $\bar{q}(G/H)$ is again a covering. This yields covariant functors from the orbit category of G to the category of topological spaces,

$$\underline{B} : \mathsf{Or}(G) \to \mathsf{Spaces}, \quad G/H \mapsto \tilde{B} \times_G G/H,$$
$$\underline{X} : \mathsf{Or}(G) \to \mathsf{Spaces}, \quad G/H \mapsto X(G/H).$$

The assumption that p is π_1-surjective ensures that $X(G/H)$ is path connected for all $H \subseteq G$.

By composition with the fundamental groupoid functor we obtain a functor

$$\underline{\Pi(X)} : \mathsf{Or}(G) \to \mathsf{Groupoids}, \quad G/H \mapsto \Pi(X(G/H)).$$

Let R-FGF be the additive category whose set of objects is $\{R^n \mid n = 0, 1, 2, \dots\}$ and whose morphisms are R-linear maps. In the sequel it will always be equipped with the trivial G- or Γ-action or considered as constant functor $\mathcal{G} \to \mathsf{Add\text{-}Cat}$. Consider the functor

$$\xi : \mathsf{Groupoids} \to \mathsf{Spectra}, \quad \mathcal{G} \mapsto \boldsymbol{K}\left(\int_{\mathcal{G}} R\text{-}\mathsf{FGF}\right).$$

The composite of the last two functors yields a functor

$$\boldsymbol{K}(p) := \xi \circ \underline{\Pi(X)} : \mathsf{Or}(G) \to \mathsf{Spectra}.$$

Associated to this functor there is — see [Davis and Lück 1998] — a G-homology theory $H_*^G(-; \boldsymbol{K}(p)) := \pi_n(O^G(-) \wedge_{\mathsf{Or}(G)} \boldsymbol{K}(p))$. We will be interested in the associated assembly map induced by the projection $\underline{\underline{E}}G \to G/G$,

$$H_n^G(\underline{\underline{E}}G;\boldsymbol{K}(p)) \to H_n^G(G/G;\boldsymbol{K}(p)) \cong K_n(R\Gamma). \tag{11.1}$$

The goal of this section is to identify this assembly map with the assembly map

$$H_n^G(\underline{\underline{E}}G;\boldsymbol{K}_{\mathcal{A}}) \to H_n^G(G/G;\boldsymbol{K}_{\mathcal{A}}) = K_n(R\Gamma)$$

for a suitable additive category with G-action $\mathcal{A}$. Thus the results of this paper apply also in the fibered setup.

Consider the functor

$$\underline{\mathcal{G}^\Gamma} : \mathsf{Or}(G) \to \mathsf{Groupoids}, \quad G/H \mapsto \mathcal{G}^\Gamma(G/H),$$

where we consider G/H as a Γ-set by restriction along the group homomorphism $\Gamma \to G$ induced by p.

Lemma 11.2. *There is a natural equivalence*

$$T : \underline{\mathcal{G}^\Gamma} \to \underline{\Pi(X)}$$

of covariant functors $\mathsf{Or}(G) \to \mathsf{Groupoids}$.

Proof. Given an object G/H in $\mathsf{Or}(G)$, we have to specify an equivalence of groupoids $T(G/H) : \mathcal{G}^\Gamma(G/H) \to \Pi(X(G/H))$. For an object in $\mathcal{G}^\Gamma(G/H)$ which is given by an element $wH \in G/H$, define $T(wH)$ to be the point in $X(G/H)$ which is determined by $(\tilde{b}_0, wH) \in \tilde{B} \times_G G/H$ and $x_0 \in X$. This makes sense since $q(G/H)((\tilde{b}_0, wH)) = b_0 = q(x_0)$.

Let $\gamma : w_0H \to w_1H$ be a morphism in $\mathcal{G}^\Gamma(G/H)$. Choose a loop u_X in X at $x_0 \in X$ which represents γ. Let u_B be the loop $p \circ u_X$ in B at $b_0 \in B$. There is precisely one path $u_{\tilde{B}}$ in $\tilde{B}$ which starts at $\tilde{b}_0$ and satisfies $q \circ u_{\tilde{B}} = u_B$. Let $[u_B] \in G$ be the class of u_B, or, equivalently, the image of γ under $\pi_1(p, x_0) : \Gamma \to G$. By definition of the right G-action on $\tilde{B}$ we have $\tilde{b}_0 \cdot [u_B] = u_B(1)$. Define a path $u_{\tilde{B}/H}$ in $\tilde{B} \times_G G/H$ from $(\tilde{b}_0, w_0H)$ to $(\tilde{b}_0, w_1H)$ by $t \mapsto (u_B(t), w_0H)$. This is indeed a path ending at $(\tilde{b}_0, w_1H)$ since $(\tilde{b}_0 \cdot [u_B], w_0H) = (\tilde{b}_0, [u_B] \cdot w_0H) = (\tilde{b}_0, w_1H)$ holds in $\tilde{B} \times_G G/H$. Obviously the composite of $u_{\tilde{B}/H}$ with $q(G/H) : \tilde{B} \times_G G/H \to B$ is u_B. Hence $u_{\tilde{B}/H}$ and u_X determine a path in $X(G/H)$ from $T(w_0H) \to T(w_1H)$ and hence a morphism $T(w_0H) \to T(w_1H)$ in $\Pi(X(G/H))$. One easily checks that the homotopy class (relative to the endpoints) of u depends only on γ. Thus we obtain the desired functor $T(G/H) : \mathcal{G}^\Gamma(G/H) \to \Pi(X(G/H))$. One easily checks that they fit together, so that we obtain a natural transformation $T : \underline{\mathcal{G}^\Gamma} \to \underline{\Pi(X)}$.

At a homogeneous space G/H, the value of $\underline{\mathcal{G}^\Gamma}$ is a groupoid equivalent to the group $\pi_1(p, x_0)^{-1}(H)$, while the value of $\underline{\Pi(X)}$ is a groupoid equivalent to the fundamental group of $X(G/H)$. Up to this equivalence, the functor T, at G/H, is the standard identification of these two groupoids. Hence T is a natural equivalence. $\square$

We obtain a covariant functor

$$\boldsymbol{K}(p)' : \mathsf{Or}(G) \to \mathsf{Spectra}, \quad G/H \mapsto \boldsymbol{K}\Big(\int_{\mathcal{G}^{\Gamma}(G/H)} R\text{-}\mathsf{FGF}\Big).$$

Lemma 11.2 implies that the following diagram commutes, where the vertical arrow is the isomorphism induced by T:

$$\begin{tikzcd}
H_n^G(\underline{\underline{E}}G;\boldsymbol{K}(p)) \arrow[dr, "H_n^G(\mathrm{pr};\boldsymbol{K}(p))"] & \\
& H_n^G(G/G;\boldsymbol{K}(p)) = K_n(R\Gamma) \\
H_n^G(\underline{\underline{E}}G;\boldsymbol{K}(p)') \arrow[uu, "T_*", "\cong"'] \arrow[ur, "H_n^G(\mathrm{pr};\boldsymbol{K}(p)')"'] &
\end{tikzcd}$$

Now the functor $\boldsymbol{K}(p)'$ is, up to natural equivalence, of the form $\boldsymbol{K}_{\mathcal{A}}^G$ for some additive G-category, namely for $\mathcal{A} = \mathrm{ind}_{q:\Gamma\to G} R\text{-}\mathsf{FGF}$; see [Bartels and Lück 2010, (11.5) and Lemma 11.6]. We conclude:

Lemma 11.3. *The assembly map (11.1) is an isomorphism for all $n \in \mathbb{Z}$ if the K-theoretic Farrell–Jones conjecture for additive categories holds for G.*

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