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ON SUMMABILITY OF MULTILINEAR OPERATORS AND APPLICATIONS

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Abstract. This article has two clear motivations, one technical and one practical. The technical motivation unifies in a single formulation a huge family of inequalities that have been produced separately over the last ninety years in different contexts. But we do not just join inequalities; our method also creates a family of inequalities that were invisible by previous approaches. The practical motivation is to show that our new approach has the strength to attack various problems. We provide new applications of our family of inequalities, continuing recent work by Maia, Nogueira, and Pellegrino.

1. Introduction

Absolutely summing linear operators (see [13]) can be generalized to the nonlinear framework by several different approaches. There is a vast recent literature in this line and also some works attempting to unify different approaches (see, e.g., [9], [20], [25]).

The following notion, conceived by Popa and, independently, by Bayart, Pellegrino, and Rueda, is perhaps the most general approach to absolutely summing multilinear operators. Let $m \geq 1$, let $E_1, \ldots, E_m, F$ be Banach spaces, and let $T : E_1 \times \cdots \times E_m \to F$ be an m-linear operator. Let also $\Lambda \subset \mathbb{N}^m$. For $r \in (0, \infty)$ and $p \geq 1$, we say that T is Λ-(r, p)-*summing* if there exists a constant $C > 0$

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such that, for all sequences $x(j) \subset E_j^{\mathbb{N}}$, $1 \le j \le m$,

$$\Big(\sum_{\mathbf{i}\in\Lambda}\big\|T(x_{\mathbf{i}})\big\|^r\Big)^{\frac{1}{r}} \le C\big\|x(1)\big\|_{w,p}\cdots\big\|x(m)\big\|_{w,p},$$

where $T(x_{\mathbf{i}})$ stands for $T(x_{i_1}(1),\ldots,x_{i_m}(m))$ and $\|x\|_{w,p}$ stands for the weak ℓ_p-norm of x defined by

$$\|x\|_{w,p} = \sup_{\|x^*\|\le 1}\Big(\sum_{i=1}^{\infty}\big|x^*(x_i)\big|^p\Big)^{\frac{1}{p}}.$$

When $\Lambda = \{(n,\ldots,n) : n\in\mathbb{N}\}$, we get the definition of an (r,p)-absolutely summing map which was introduced in [2]. When $\Lambda = \mathbb{N}^m$, we recover the notion of an (r,p)-multiple summing map introduced in [8] and [19] (see also [5] for recent advances in the theory). In this article, we investigate intermediary situations, that is, the cases of sets Λ strictly located between $\{(n,\ldots,n) : n\in\mathbb{N}\}$ and $\mathbb{N}^m$.

For $p\in[1,\infty]$, as usual, we consider the Banach spaces of weakly p-summable sequences

$$\ell_p^w(E) := \big\{(x_j)_{j=1}^{\infty}\subset E : \big\|(x_j)_{j=1}^{\infty}\big\|_{w,p}<\infty\big\}$$

and strongly p-summable sequences

$$\ell_p(E) := \Big\{(x_j)_{j=1}^{\infty}\subset E : \big\|(x_j)_{j=1}^{\infty}\big\|_p := \Big(\sum_{j=1}^{\infty}\|x_j\|^p\Big)^{\frac{1}{p}}<\infty\Big\}.$$

Throughout this article, the topological dual of E is denoted by E^* and the conjugate of $1\le p\le\infty$ is represented by p^*; that is, $\frac{1}{p}+\frac{1}{p^*}=1$. As usual, the e_j's are canonical vectors and

$$\|T\| := \sup_{\|x_1\|,\ldots,\|x_m\|\le 1}\big\|T(x_1,\ldots,x_m)\big\|$$

for any continuous m-linear mapping $T : E_1\times\cdots\times E_m \to F$. Henceforth $\mathcal{L}(E_1,\ldots,E_m;F)$ stands for the Banach space of all bounded m-linear operators from $E_1\times\cdots\times E_m$ to F endowed with this sup norm.

The canonical isometric isomorphisms (see [13, Proposition 2.2]) $\mathcal{L}(\ell_{p^*},E) = \ell_p^w(E)$ and $\mathcal{L}(c_0,E)=\ell_1^w(E)$ tell us that certain cases of summability of multilinear operators are equivalent to investigating

$$\Big(\sum_{\mathbf{i}\in\Lambda}\big\|T(e_{\mathbf{i}})\big\|^r\Big)^{\frac{1}{r}} \le C\|T\|,$$

for $T:\ell_p\times\cdots\times\ell_p\to F$ or $T:c_0\times\cdots\times c_0\to F$, and this is precisely when the theory of Hardy–Littlewood inequalities meets the theory of absolutely summing multilinear operators.

Results related to summability of multilinear operators date back, at least, to the 1930s, when Littlewood proved his seminal 4/3 inequality. Since then, several different related results and approaches have appeared. For example, the Bohnenblust–Hille [7] and Hardy–Littlewood [16] inequalities can be considered two keystones in the theory of multilinear operators. In the last thirty years, several multilinear variants of these classical inequalities have emerged. Let us classify them depending on whether the involved sum is done in one or all indices.

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, let m be a positive integer, and let $1 \leq p_1, \ldots, p_m \leq \infty$. From now on, for $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty]^m$, let

$$\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

We will also denote $X_p := \ell_p$ for $1 \leq p < \infty$, and $X_\infty := c_0$.

(I) Sums in one index ($\Lambda = \{(n, \ldots, n) : n \in \mathbb{N}\}$).

- Aron and Globevnik [4]: For every continuous m-linear form $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$,

$$\sum_{i=1}^{\infty} \left|T(e_i, \ldots, e_i)\right| \leq \|T\|. \tag{1.1}$$

- Zalduendo [26]: Let $|\frac{1}{\mathbf{p}}| < 1$. For every continuous m-linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$,

$$\left(\sum_{i=1}^{\infty} \left|T(e_i, \ldots, e_i)\right|^{\frac{1}{1-|\frac{1}{\mathbf{p}}|}}\right)^{1-|\frac{1}{\mathbf{p}}|} \leq \|T\|. \tag{1.2}$$

(II) Sums in all indices ($\Lambda = \mathbb{N}^m$).

- Bohnenblust–Hille inequality (see [7]): There exists a constant $C^{\mathbb{K}}_{m,\infty} \geq 1$ such that, for every continuous m-linear form $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$,

$$\left(\sum_{i_1, \ldots, i_m = 1}^{\infty} \left|T(e_{i_1}, \ldots, e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq C^{\mathbb{K}}_{m,\infty} \|T\|. \tag{1.3}$$

- Hardy–Littlewood [16] and Praciano-Pereira [22]: Let $|\frac{1}{\mathbf{p}}| \leq \frac{1}{2}$. There exists a constant $C^{\mathbb{K}}_{m,\mathbf{p}} \geq 1$ such that, for every continuous m-linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$,

$$\left(\sum_{i_1, \ldots, i_m = 1}^{\infty} \left|T(e_{i_1}, \ldots, e_{i_m})\right|^{\frac{2m}{m+1-2|\frac{1}{\mathbf{p}}|}}\right)^{\frac{m+1-2|\frac{1}{\mathbf{p}}|}{2m}} \leq C^{\mathbb{K}}_{m,\mathbf{p}} \|T\|. \tag{1.4}$$

- Hardy–Littlewood [16] and Dimant–Sevilla-Peris [14]: Let $\frac{1}{2} \leq |\frac{1}{\mathbf{p}}| < 1$. There exists a constant $D^{\mathbb{K}}_{m,\mathbf{p}} \geq 1$ such that

$$\left(\sum_{i_1, \ldots, i_m = 1}^{\infty} \left|T(e_{i_1}, \ldots, e_{i_m})\right|^{\frac{1}{1-|\frac{1}{\mathbf{p}}|}}\right)^{1-|\frac{1}{\mathbf{p}}|} \leq D^{\mathbb{K}}_{m,\mathbf{p}} \|T\| \tag{1.5}$$

  for every continuous m-linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$.

All exponents involved in the previous inequalities are sharp. An extended version of the Hardy–Littlewood/Praciano-Pereira inequality was presented in [1] (see also [24] for a slightly general version).

- Albuquerque, Bayart, Pellegrino, and Seoane-Sepúlveda [1]: Let $|\frac{1}{\mathbf{p}}| \leq \frac{1}{2}$ and $\mathbf{q} := (q_1, \ldots, q_m) \in [(1 - |\frac{1}{\mathbf{p}}|)^{-1}, 2]^m$. There is a constant $C^{\mathbb{K}}_{m,\mathbf{p},\mathbf{q}} \geq 1$ such that
$$\Bigg(\sum_{i_1=1}^{\infty}\Bigg(\cdots\Bigg(\sum_{i_m=1}^{\infty}\big|T(e_{i_1}, \ldots, e_{i_m})\big|^{q_m}\Bigg)^{\frac{q_{m-1}}{q_m}}\cdots\Bigg)^{\frac{q_1}{q_2}}\Bigg)^{\frac{1}{q_1}} \leq C^{\mathbb{K}}_{m,\mathbf{p},\mathbf{q}}\|T\| \tag{1.6}$$
for every continuous m-linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$ if and only if
$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2} - \Big|\frac{1}{\mathbf{p}}\Big|.$$

Remark 1.1. Throughout the article, the optimal constants of each of the above inequalities will be denoted exactly as they were previously stated.

We note the following.

(a) Zalduendo's theorem, for $p_1 = \cdots = p_m = \infty$, recovers Aron and Globevnik's theorem.
(b) The Hardy–Littlewood/Praciano-Pereira inequality, when $p_1 = \cdots = p_m = \infty$, recovers the Bohnenblust–Hille inequality.
(c) If $q_1 = \cdots = q_m = \frac{2m}{m+1-2|\frac{1}{\mathbf{p}}|}$ in (1.6), then we recover the Hardy–Littlewood/Praciano-Pereira inequality and we will denote
$$C^{\mathbb{K}}_{m,\mathbf{p},\left(\frac{2m}{m+1-2|\frac{1}{\mathbf{p}}|},\ldots,\frac{2m}{m+1-2|\frac{1}{\mathbf{p}}|}\right)}$$
by $C^{\mathbb{K}}_{m,\mathbf{p}}$. Moreover, if $p_1 = \cdots = p_m = p$, then we will denote $C^{\mathbb{K}}_{m,\mathbf{p}}$ by $C^{\mathbb{K}}_{m,p}$.

The first main objective of this article is to combine—in a single formulation—all the above inequalities that were produced separately and in different contexts and that apparently did not match. We do not do this only for the mathematical beauty of unifying theories that were treated in completely different ways, but because this also provides subtle bits of information that were not previously accessible, such as, for example, giving a definitive answer to a problem initially considered by Carando, Defant, and Sevilla-Peris [10] (this improvement was recently made by Maia, Nogueira, and Pellegrino [17] using our main theorem). This and some other findings were only possible at the time when the theories were no longer seen separately. Despite their importance in several fields of mathematics (e.g., quantum information theory, Dirichlet series, and so forth), the optimal constants of the m-linear Hardy–Littlewood inequalities are still unknown. For the real case of the Bohnenblust–Hille inequality, it is known that the optimal constants are not contractive and, very recently, a computational approach to calculate the optimal constants of the Bohnenblust–Hille inequality was successfully implemented using the Wolfram Language (see [11]). As an application of our unified approach, we can analyze under what conditions we can improve such inequalities in order to have contractive constants. In fact, in Section 3 we will study how the consideration of the blocks in the Bohnenblust–Hille inequalities can make the new inequalities become contractive.

Let n be a positive integer, and from now on e_i^n denotes the n-tuple $(e_i, \ldots, e_i)$. Furthermore, if $n_1, \ldots, n_k \geq 1$ are such that $n_1 + \cdots + n_k = m$, then $(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k})$ represents the m-tuple:

$$(e_{i_1}, \overset{n_1 \text{times}}{\ldots}, e_{i_1}, \ldots, e_{i_k}, \overset{n_k \text{times}}{\ldots}, e_{i_k}).$$

The main result of this article (Theorem 2.4) extends and unifies (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6) by considering intermediary setups for Λ. Theorem 2.4 provides the following particular case whenever $p_1 = \cdots = p_m = p$, which has a more friendly statement.

Theorem 1.2. *Let $m \geq k \geq 1$, let $m < p \leq \infty$, and let $n_1, \ldots, n_k \geq 1$ be such that $n_1 + \cdots + n_k = m$. Then for every continuous m-linear form $T : X_p \times \cdots \times X_p \to \mathbb{K}$, there is a constant $M_{k,m,p}^{\mathbb{K}} \geq 1$ such that*

$$\Big(\sum_{i_1, \ldots, i_k = 1}^{\infty} \big| T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}) \big|^{\rho} \Big)^{\frac{1}{\rho}} \leq M_{k,m,p}^{\mathbb{K}} \|T\|,$$

with

$$\rho = \frac{p}{p-m} \quad \text{for } m < p \leq 2m \qquad \text{and} \qquad M_{k,m,p}^{\mathbb{K}} \leq D_{k,(\frac{p}{n_1}, \ldots, \frac{p}{n_k})}^{\mathbb{K}}$$

and

$$\rho = \frac{2kp}{kp + p - 2m} \quad \text{for } p \geq 2m \qquad \text{and} \qquad M_{k,m,p}^{\mathbb{K}} \leq C_{k,(\frac{p}{n_1}, \ldots, \frac{p}{n_k})}^{\mathbb{K}}. \tag{1.7}$$

Above, $C_{k,(\frac{p}{n_1}, \ldots, \frac{p}{n_k})}^{\mathbb{K}}$ and $D_{k,(\frac{p}{n_1}, \ldots, \frac{p}{n_k})}^{\mathbb{K}}$ are the constants from (1.4) and (1.5), respectively. Moreover, in both cases, the exponent ρ is optimal.

Remark 1.3. It is interesting to stress that the optimal exponent for the case $p > 2m$ is not the exponent of the k-linear case. It is a kind of combination of the cases of k-linear and m-linear forms, as can be seen in (1.7). In general, we have the following.

- If $m < p < 2m$, then the optimal exponent depends only on m.
- If $p = 2m$, then the optimal exponent does not depend on m or k.
- If $2m < p < \infty$, then the optimal exponent depends on m and k.
- If $p = \infty$, then the optimal exponent depends only on k.

The proof of the main result combines two different tools based on tensor products. First, we prove a k-linearization method for n-linear operators ($n \geq k$) which is an inductive refinement of the well-known linearization method. Second, we use the description of the diagonal of the tensor product of ℓ_p-spaces based on [3, Theorem 1.3] and [23, Example 2.23(b)]. It is worth mentioning that the Zalduendo and Aron–Globevnik inequalities can be proved in a straightforward way by means of this technique (see Remark 2.5).

The search for optimal constants for the Bohnenblust–Hille inequality is an active research area nowadays (see, e.g., [1], [6], [12], [18], [21] and the references therein). Very recently, our main theorem (Theorem 2.4) was applied in [17] to show that the asymptotic constants of the Bohnenblust–Hille inequality for complex m-homogeneous polynomials whose monomials have a uniformly bounded number of variables do not depend on m. This is a striking result since the prior work [10], using a completely different technique, just obtained constants growing polynomially with m. Section 3 provides applications of our main result (Theorem 2.4) in the analysis of the contractivity of the constants appearing in the inequalities when considering special sets Λ. We will prove that the Bohnenblust–Hille inequalities are "almost" contractive. More precisely, if $m, k, n_1, \ldots, n_k \geq 1$ are positive integers such that $n_1 + \cdots + n_k = m$, by considering sums over the index set $\Lambda \subset \mathbb{N}^m$ that gathers all m-tuples (note that Λ is composed of k "blocks")

$$(i_1, \overset{n_1 \text{ times}}{\ldots}, i_1, \ldots, i_k, \overset{n_k \text{ times}}{\ldots}, i_k), \quad i_1, \ldots, i_k \in \mathbb{N},$$

and if $k = k(m)$ is such that

$$\lim_{m \to \infty} \frac{k \log k}{m} = 0,$$

then Theorem 3.1 will provide the contractivity of the Bohnenblust–Hille inequality.

2. Bohnenblust–Hille and Hardy–Littlewood for block-type sets Λ

Besides motivating the introduction of a new approach to the theory of summability of multilinear operators, the main purpose of this section is to present a unified version of the Bohnenblust–Hille and Hardy–Littlewood inequalities with partial sums (i.e., we will consider sums allowed to run over a set Λ with fewer indices) which also recovers Zalduendo's and Aron–Globevnik's inequalities. A tensorial perspective will present an important role in this matter, establishing an intrinsic relationship between the exponents and constants involved and the number of indices taken on the sums.

We need to introduce some other terminologies. The product

$$\widehat{\bigotimes_{j \in \{1,\ldots,n\}}}^{\pi} E_j = E_1 \widehat{\otimes}^{\pi} \cdots \widehat{\otimes}^{\pi} E_n$$

denotes the completed projective n-fold tensor product of $E_1, \ldots, E_n$. The tensor $x_1 \otimes \cdots \otimes x_n$ is denoted by $\bigotimes_{j \in \{1,\ldots,n\}} x_j$ for short, whereas $\bigotimes_n x$ denotes the tensor $x \otimes \cdots \otimes x$. In a similar way, $\bigtimes_{j \in \{1,\ldots,n\}} E_j$ denotes the product space $E_1 \times \cdots \times E_n$.

Recall that $X_p = \ell_p$ if $1 \leq p < \infty$ and that $X_p = c_0$ if $p = \infty$. Let n be a positive integer, and let $1 \leq p_1, \ldots, p_n \leq \infty$ be such that $\frac{1}{p_1} + \cdots + \frac{1}{p_n} < 1$. From now on in this section, r, s are defined by $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_n}$ and $\frac{1}{s} + \frac{1}{r} = 1$. Let $D_r \subset X_{p_1} \widehat{\otimes}^{\pi} \cdots \widehat{\otimes}^{\pi} X_{p_n}$ be the linear span of the tensors $\bigotimes_n e_i$, and let $\overline{D}_r$ be its closure. Additionally, we will use the following notation. For Banach spaces $E_1, \ldots, E_m$ and an element $x_j \in E_j$, for some $j \in \{1, \ldots, m\}$, the symbol $x_j \cdot e_j$ represents the vector $x_j \cdot e_j \in E_1 \times \cdots \times E_m$ such that its jth coordinate is $x_j \in E_j$, and 0 otherwise.

The following lemma, although known for $1 \leq p_1, \ldots, p_n < \infty$ (see [3, Theorem 1.3]), is the key to Theorem 2.4 and so we give a constructive proof inspired by [23, Example 2.23(b)].

Lemma 2.1. *The map* $u_r : X_r \to \overline{D_r}$, *given by* $u_r(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} a_i \bigotimes_n e_i$, *is an isometric isomorphism that is surjective.*

Proof. For the sake of simplicity, we will show only the case $1 \leq p_1, \ldots, p_n < \infty$. In all the other cases, that is, when one or more X_i's are c_0, the proof can be easily adapted.

Let $\theta = \sum_{i=1}^{k} a_i \bigotimes_n e_i$. Using the orthogonality of the Rademacher system, we get

$$\theta = \int_{[0,1]^{n-1}} \bigotimes_{j=1}^{n-1} \Big(\sum_{i=1}^{k} |a_i|^{\frac{r}{p_j}} r_i(t_j) e_i \Big) \otimes \Big(\sum_{i=1}^{k} \operatorname{sgn}(a_i) |a_i|^{\frac{r}{p_n}} r_i(t_1) \cdots r_i(t_{n-1}) e_i \Big) \, dt,$$

where $dt = dt_1 \cdots dt_{n-1}$ and r_i are the Rademacher functions, and $\operatorname{sgn}(a)$ is the scalar of modulus 1 such that $\operatorname{sgn}(a)a = |a|$. Hence,

$$\begin{aligned} \pi(\theta) &\leq \sum_{\substack{0 \leq t_j \leq 1 \\ 1 \leq j \leq n-1}} \Big[\prod_{j=1}^{n-1} \Big\| \sum_{i=1}^{k} |a_i|^{\frac{r}{p_j}} r_i(t_j) e_i \Big\|_{p_j} \Big] \Big\| \sum_{i=1}^{k} r_i(t_1) \cdots r_i(t_{n-1}) \operatorname{sgn}(a_i) |a_i|^{\frac{r}{p_n}} e_i \Big\|_{p_n} \\ &= \big\| (a_i)_{i=1}^{k} \big\|_r . \end{aligned}$$

To prove $\|(a_i)_{i=1}^{k}\|_r \leq \pi(\theta)$, consider the n-linear form on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ given by

$$B(x^{(1)}, \ldots, x^{(n)}) := \sum_{i=1}^{k} b_i x_i^{(1)} \cdots x_i^{(n)},$$

where $b_i = \operatorname{sgn}(a_i) \frac{|a_i|^{\frac{r}{s}}}{\|(a_i)_{i=1}^{k}\|_r^{\frac{r}{s}}}$. By Hölder's inequality,

$$\|B\| = \sup_{\substack{x^{(j)} \in B_{\ell_{p_j}} \\ 1 \leq j \leq n}} \Big| \sum_{i=1}^{k} b_i x_i^{(1)} \cdots x_i^{(n)} \Big| \leq \sup_{\substack{x^{(j)} \in B_{\ell_{p_j}} \\ 1 \leq j \leq n}} \big\| (b_i)_{i=1}^{k} \big\|_s \|x^{(1)}\|_{p_1} \cdots \|x^{(n)}\|_{p_n} = 1.$$

Therefore,

$$\pi(\theta) \geq \big| \langle \theta, B \rangle \big| = \Big| \sum_{i=1}^{k} a_i B(e_i, \ldots, e_i) \Big| = \Big| \sum_{i=1}^{k} a_i b_i \Big| = \Big(\sum_{i=1}^{k} |a_i|^r \Big)^{\frac{1}{r}}$$

and thus $\pi(\theta) = \|(a_i)_{i=1}^{k}\|_r$. By extending the isometric isomorphism to the completions, we get that $\overline{D}_r$ is isometrically isomorphic to ℓ_r. □

Using the isometry between $\overline{D_r}$ and ℓ_r provided in the preceding lemma, we get the following.

Lemma 2.2. *The sequence* $(\bigotimes_n e_i)_{i \in \mathbb{N}}$ *belongs to* $\ell_s^w(X_{p_1} \widehat{\otimes}^{\pi} \cdots \widehat{\otimes}^{\pi} X_{p_n})$ *and*

$$\Big\| \Big(\bigotimes_n e_i \Big)_{i \in \mathbb{N}} \Big\|_{w,s} = 1.$$

Proof. Observe that

$$\Big\|\Big(\bigotimes_n e_i\Big)_{i\in\mathbb{N}}\Big\|_{w,s} = \sup_{\varphi\in B_{(X_{p_1}\widehat{\otimes}^\pi\cdots\widehat{\otimes}^\pi X_{p_n})^*}} \Big(\sum_{i=1}^\infty \Big|\varphi\Big(\bigotimes_n e_i\Big)\Big|^s\Big)^{\frac{1}{s}}$$
$$= \sup_{\varphi\in B_{(\overline{D_r})^*}} \Big(\sum_{i=1}^\infty \Big|\varphi\Big(\bigotimes_n e_i\Big)\Big|^s\Big)^{\frac{1}{s}}$$
$$= \sup_{\varphi\in B_{\ell_s}} \Big(\sum_{i=1}^\infty |\varphi(e_i)|^s\Big)^{\frac{1}{s}} = 1.$$
□

The following result is a kind of k-"linearization" of a given m-linear operator and will be used in the proof of our main result.

Proposition 2.3. *Let m be a positive integer, and let $E_1,\dots,E_m,F$ be Banach spaces. Let $1\le k\le m$ and $I_1,\dots,I_k$ be pairwise disjoint nonvoid subsets of $\{1,\dots,m\}$ such that $\bigcup_{j=1}^k I_j=\{1,\dots,m\}$. Then, given $T\in\mathcal{L}(E_1,\dots,E_m;F)$, there is a unique $\widehat{T}\in\mathcal{L}(\widehat{\bigotimes}^\pi_{j\in I_1}E_j,\dots,\widehat{\bigotimes}^\pi_{j\in I_k}E_j;F)$ such that*

$$\widehat{T}\Big(\bigotimes_{j\in I_1}x_j,\dots,\bigotimes_{j\in I_k}x_j\Big)=T(x_1,\dots,x_m)$$

and $\|\widehat{T}\|=\|T\|$. The correspondence $T\leftrightarrow\widehat{T}$ determines an isometric isomorphism between the spaces $\mathcal{L}(E_1,\dots,E_m;F)$ and $\mathcal{L}(\widehat{\bigotimes}^\pi_{j\in I_1}E_j,\dots,\widehat{\bigotimes}^\pi_{j\in I_k}E_j;F)$.

Proof. We will proceed by transfinite induction on m. Note that for $m=1$ or $m=2$, there is nothing to be proved ($\widehat{T}$ is just the linearization of T whenever $m=2$ and $k=1$). Assume that the result is true for any positive integer less than m, and let $T\in\mathcal{L}(E_1,\dots,E_m;F)$ and $I_1,\dots,I_k$ be as in the statement. Assume that $|I_k|=m_k$, and fix $x_j\in E_j$, for any $j\in I_k$. Fix $\sum_{j\in I_k}x_j\cdot e_j\in \bigtimes_{j\in I_k}E_j$. Consider the continuous $(m-m_k)$-linear mapping given by

$$T_{(\sum_{j\in I_k}x_j\cdot e_j)}\Big(\sum_{i\in I_1}x_i\cdot e_i+\cdots+\sum_{i\in I_{k-1}}x_i\cdot e_i\Big):=T(x_1,\dots,x_m).$$

By the induction hypothesis, there exists a unique

$$\widetilde{T}\Big(\sum_{j\in I_k}x_j\cdot e_j\Big)\in\mathcal{L}\Big(\widehat{\bigotimes_{j\in I_1}}^\pi E_j,\dots,\widehat{\bigotimes_{j\in I_{k-1}}}^\pi E_j;F\Big)$$

such that

$$\widetilde{T}\Big(\sum_{j\in I_k}x_j\cdot e_j\Big)\Big(\bigotimes_{i\in I_1}x_i,\dots,\bigotimes_{i\in I_{k-1}}x_i\Big)$$
$$=T_{(\sum_{j\in I_k}x_j\cdot e_j)}\Big(\sum_{i\in I_1}x_i\cdot e_i+\cdots+\sum_{i\in I_{k-1}}x_i\cdot e_i\Big)$$
$$=T(x_1,\dots,x_m)$$

and

$$\Big\|\widetilde{T}\Big(\sum_{j\in I_k} x_j\cdot e_j\Big)\Big\| = \|T_{(\sum_{j\in I_k} x_j\cdot e_j)}\|.$$

Define now the m_k-linear mapping

$$A : \bigtimes_{j\in I_k} E_j \to \mathcal{L}\Big(\widehat{\bigotimes_{j\in I_1}}^{\pi} E_j, \dots, \widehat{\bigotimes_{j\in I_{k-1}}}^{\pi} E_j; F\Big)$$

given by

$$A\Big(\sum_{i\in I_k} y_i\cdot e_i\Big) := \widetilde{T}\Big(\sum_{i\in I_k} y_i\cdot e_i\Big),$$

and let $A_L \in \mathcal{L}(\widehat{\bigotimes}^{\pi}_{j\in I_k} E_j; \mathcal{L}(\widehat{\bigotimes}^{\pi}_{j\in I_1} E_j, \dots, \widehat{\bigotimes}^{\pi}_{j\in I_{k-1}} E_j; F))$ be its linearization, that is, the unique linear map from $\widehat{\bigotimes}^{\pi}_{j\in I_k} E_j$ into $\mathcal{L}(\widehat{\bigotimes}^{\pi}_{j\in I_1} E_j, \dots, \widehat{\bigotimes}^{\pi}_{j\in I_{k-1}} E_j; F)$ such that $A_L(\bigotimes_{j\in I_k} y_j) = A(\sum_{j\in I_k} y_j\cdot e_j)$. Finally, $\widehat{T} : \widehat{\bigotimes}^{\pi}_{j\in I_1} E_j \times\cdots\times \widehat{\bigotimes}^{\pi}_{j\in I_k} E_j \to F$ defined by

$$\widehat{T}(\theta_1,\dots,\theta_k) := A_L(\theta_k)(\theta_1,\dots,\theta_{k-1})$$

is k-linear, continuous, and satisfies

$$\begin{aligned}
\widehat{T}\Big(\bigotimes_{j\in I_1} x_j, \dots, \bigotimes_{j\in I_k} x_j\Big) &= A_L\Big(\bigotimes_{j\in I_k} x_j\Big)\Big(\bigotimes_{j\in I_1} x_j, \dots, \bigotimes_{j\in I_{k-1}} x_j\Big)\\
&= \widetilde{T}\Big(\sum_{i\in I_k} x_i\cdot e_i\Big)\Big(\bigotimes_{j\in I_1} x_j, \dots, \bigotimes_{j\in I_{k-1}} x_j\Big)\\
&= T(x_1,\dots,x_m)
\end{aligned}$$

and

$$\begin{aligned}
\|\widehat{T}\| &= \sup_{\substack{\theta_j\in B_{\widehat{\otimes}^{\pi}_{i\in I_j} E_i}\\ j=1,\dots,k}} \big\|A_L(\theta_k)(\theta_1,\dots,\theta_{k-1})\big\|\\
&= \|A_L\| = \|A\|\\
&= \sup_{\substack{y_i\in E_i\\ i\in I_k}} \Big\|\tilde{T}\Big(\sum_{i\in I_k} y_i\cdot e_i\Big)\Big\|\\
&= \sup_{\substack{y_i\in E_k\\ i\in I_k}} \|T_{(\sum_{i\in I_k} y_i\cdot e_i)}\| = \|T\|.
\end{aligned}$$

□

Now we prove our main result, which unifies (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6).

Theorem 2.4. *Let $1 \le k \le m$ and $n_1,\dots,n_k \ge 1$ be positive integers such that $n_1+\cdots+n_k = m$, and assume that*

$$\mathbf{p} := (p_1^{(1)}, \overset{n_1\, times}{\dots}, p_{n_1}^{(1)}, \dots, p_1^{(k)}, \overset{n_k\, times}{\dots}, p_{n_k}^{(k)}) \in [1,\infty]^m$$

is such that $0 \leq |\frac{1}{\mathbf{p}}| < 1$*. Let* $\mathbf{r} := (r_1, \ldots, r_k)$ *with* r_i *given by* $\frac{1}{r_i} = \frac{1}{p_1^{(i)}} + \cdots + \frac{1}{p_{n_i}^{(i)}}$*,* $i = 1, \ldots, k$*. Then the following hold.*

(1) *If* $0 \leq |\frac{1}{\mathbf{p}}| \leq \frac{1}{2}$ *and* $\mathbf{q} := (q_1, \ldots, q_k) \in [(1 - |\frac{1}{\mathbf{p}}|)^{-1}, 2]^k$*, then for every continuous* m*-linear form*

$$T : \Big(\bigtimes_{1 \leq i \leq n_1} X_{p_i^{(1)}}\Big) \times \cdots \times \Big(\bigtimes_{1 \leq i \leq n_k} X_{p_i^{(k)}}\Big) \to \mathbb{K},$$

$$\Big(\sum_{i_1=1}^{\infty}\Big(\cdots\Big(\sum_{i_k=1}^{\infty}\big|T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k})\big|^{q_k}\Big)^{\frac{q_{k-1}}{q_k}} \cdots\Big)^{\frac{q_1}{q_2}}\Big)^{\frac{1}{q_1}} \leq C_{k,\mathbf{r},\mathbf{q}}^{\mathbb{K}} \|T\| \tag{2.1}$$

if and only if $|\frac{1}{\mathbf{q}}| \leq \frac{k+1}{2} - |\frac{1}{\mathbf{p}}|$*. In other words, the exponents are optimal.*

(2) *If* $\frac{1}{2} \leq |\frac{1}{\mathbf{p}}| < 1$*, then for every continuous* m*-linear form*

$$T : \Big(\bigtimes_{1 \leq i \leq n_1} X_{p_i^{(1)}}\Big) \times \cdots \times \Big(\bigtimes_{1 \leq i \leq n_k} X_{p_i^{(k)}}\Big) \to \mathbb{K},$$

$$\Big(\sum_{i_1, \ldots, i_k=1}^{\infty} \big|T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k})\big|^{\frac{1}{1-|\frac{1}{\mathbf{p}}|}}\Big)^{1-|\frac{1}{\mathbf{p}}|} \leq D_{k,\mathbf{r}}^{\mathbb{K}} \|T\|. \tag{2.2}$$

Moreover, the exponent in (2.2) is optimal.

Proof. (1) Assume that $|\frac{1}{\mathbf{q}}| \leq \frac{k+1}{2} - |\frac{1}{\mathbf{p}}|$. We will use the notation

$$(p_1^{(1)}, \ldots, p_{n_1}^{(1)}, \ldots, p_1^{(k)}, \ldots, p_{n_k}^{(k)}) = (p_1, \ldots, p_m).$$

We take the k-linear mapping given in Proposition 2.3,

$$\widehat{T} : \overbrace{\bigotimes_{1 \leq i \leq n_1}}^{\pi} X_{p_i^{(1)}} \times \cdots \times \overbrace{\bigotimes_{1 \leq i \leq n_k}}^{\pi} X_{p_i^{(k)}} \to \mathbb{K},$$

that satisfies

$$\widehat{T}\Big(\bigotimes_{1 \leq i \leq n_1} x_i^{(1)}, \ldots, \bigotimes_{1 \leq i \leq n_k} x_i^{(k)}\Big) = T(x_1^{(1)}, \ldots, x_{n_1}^{(1)}, \ldots, x_1^{(k)}, \ldots, x_{n_k}^{(k)}).$$

Then

$$\widehat{T}\Big(\bigotimes_{n_1} e_{i_1}, \ldots, \bigotimes_{n_k} e_{i_k}\Big) = T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k})$$

and $\|\widehat{T}\| = \|T\|$. Thus

$$\begin{aligned}
&\Big(\sum_{i_1=1}^{\infty}\Big(\cdots\Big(\sum_{i_k=1}^{\infty}\big|T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k})\big|^{q_k}\Big)^{\frac{q_{k-1}}{q_k}} \cdots\Big)^{\frac{q_1}{q_2}}\Big)^{\frac{1}{q_1}} \\
&\quad = \Big(\sum_{i_1=1}^{\infty}\Big(\cdots\Big(\sum_{i_k=1}^{\infty}\Big|\widehat{T}\Big(\bigotimes_{n_1} e_{i_1}, \ldots, \bigotimes_{n_k} e_{i_k}\Big)\Big|^{q_k}\Big)^{\frac{q_{k-1}}{q_k}} \cdots\Big)^{\frac{q_1}{q_2}}\Big)^{\frac{1}{q_1}}.
\end{aligned}$$

For each $j = 1, \ldots, k$, we take $u_j : X_{r_j} \to \overline{D_{r_j}}$ defined by

$$u_j\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = \sum_{i=1}^{\infty} a_i \bigotimes_{n_j} e_i.$$

Lemma 2.2 will give

$$\|u_j\| = \Big\|\Big(\bigotimes_{n} e_i\Big)_{i\in\mathbb{N}}\Big\|_{w,r_j^*} = 1.$$

Finally, it is sufficient to deal with the k-linear operator $S : X_{r_1} \times \cdots \times X_{r_k} \to \mathbb{K}$ defined by

$$S(z_1, \ldots, z_k) := \widehat{T}\big(u_1(z_1), \ldots, u_k(z_k)\big),$$

which is bounded and fulfills $\|S\| \le \|\widehat{T}\|$. Combining this with (1.6) and observing that

$$\frac{1}{r_1} + \cdots + \frac{1}{r_k} = \Big|\frac{1}{\mathbf{p}}\Big|,$$

the result follows. To show that the inequality (2.1) forces the exponent to be $|\frac{1}{\mathbf{q}}| \le \frac{k+1}{2} - |\frac{1}{\mathbf{p}}|$, it suffices to prove by (1.6) that

$$\Big(\sum_{j_1=1}^{\infty}\Big(\cdots\Big(\sum_{j_k=1}^{\infty}\big|A(e_{j_1}, \ldots, e_{j_k})\big|^{q_k}\Big)^{\frac{q_{k-1}}{q_k}}\cdots\Big)^{\frac{q_1}{q_2}}\Big)^{\frac{1}{q_1}} \le C^{\mathbb{K}}_{k,\mathbf{r},\mathbf{q}}\|A\|,$$

for all continuous k-linear forms $A : X_{r_1} \times \cdots \times X_{r_k} \to \mathbb{K}$ whenever (2.1) is fulfilled by all bounded m-linear forms

$$T : \Big(\bigtimes_{1\le i\le n_1} X_{p_i^{(1)}}\Big) \times \cdots \times \Big(\bigtimes_{1\le i\le n_k} X_{p_i^{(k)}}\Big) \to \mathbb{K}.$$

Let $A : X_{r_1} \times \cdots \times X_{r_k} \to \mathbb{K}$ be a bounded k-linear form. For each $i = 1, \ldots, k$, the diagonal space $\overline{D}_{r_i}$ is complemented in $X_{p_1^{(i)}}\widehat{\otimes}^{\pi}\cdots\widehat{\otimes}^{\pi}X_{p_{n_i}^{(i)}}$ (see [3]). Consider the diagonal projection d_{r_i} from $X_{p_1^{(i)}}\widehat{\otimes}^{\pi}\cdots\widehat{\otimes}^{\pi}X_{p_{n_i}^{(i)}}$ onto $\overline{D}_{r_i}$ such that $d_{r_i}(\sum_{j_1,\ldots,j_{n_i}} a_{(j_1,\ldots,j_{n_i})} e_{j_1} \otimes \cdots \otimes e_{j_{n_i}})$ is equal to $\sum_{j_1,\ldots,j_{n_i}} a_{(j_1,\ldots,j_{n_i})} e_{j_1} \otimes \cdots \otimes e_{j_{n_i}}$ if $j_1 = \cdots = j_{n_i}$ and to 0 otherwise. Define the m-linear map $T_A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$ by

$$\begin{aligned} &T_A(x_1^{(1)}, \ldots, x_{n_1}^{(1)}, \ldots, x_1^{(k)}, \ldots, x_{n_k}^{(k)}) \\ &\quad := A\big(u_{r_1}^{-1} \circ d_{r_1}(x_1^{(1)} \otimes \cdots \otimes x_{n_1}^{(1)}), \ldots, u_{r_k}^{-1} \circ d_{r_k}(x_1^{(k)} \otimes \cdots \otimes x_{n_k}^{(k)})\big). \end{aligned}$$

The following equalities give the result:

$$\begin{aligned} T_A(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}) &= A\Big(u_{r_1}^{-1} \circ d_{r_1}\Big(\bigotimes_{n_1} e_{i_1}\Big), \ldots, u_{r_k}^{-1} \circ d_{r_k}\Big(\bigotimes_{n_k} e_{i_k}\Big)\Big) \\ &= A\Big(u_{r_1}^{-1}\Big(\bigotimes_{n_1} e_{i_1}\Big), \ldots, u_{r_k}^{-1}\Big(\bigotimes_{n_k} e_{i_k}\Big)\Big) \\ &= A(e_{i_1}, \ldots, e_{i_k}). \end{aligned}$$

(2) While the argument is similar to that of the case $0 \leq |\frac{1}{\mathbf{p}}| \leq \frac{1}{2}$, we just need to use (1.5) instead of (1.6). $\square$

An immediate and illustrative corollary is the case $p_1 = \cdots = p_m = p$ which can be stated in a cleaner form (see Theorem 1.2).

The previous theorem unifies (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6). To realize this, we just need to proceed as follows. If $k = 1$ in the first item of Theorem 2.4, then we recover (1.2) for $0 \leq |\frac{1}{\mathbf{p}}| \leq \frac{1}{2}$; $k = 1$ in Theorem 2.4(2) provides (1.2) for $\frac{1}{2} \leq |\frac{1}{\mathbf{p}}| < 1$. These items together give us Zalduendo's result (1.2), which for $p_1 = \cdots = p_m = \infty$ recovers Aron–Globevnik's theorem (1.1). If $k = m$ in Theorem 2.4(1), then we obtain (1.6). On the other hand, (1.6) implies (1.4) if $q_1 = \cdots = q_m = \frac{2m}{m+1-2|\frac{1}{\mathbf{p}}|}$ and (1.4) implies (1.3) if $p_1 = \cdots = p_m = \infty$. To obtain the Hardy–Littlewood/Dimant–Sevilla-Peris result (1.5), we just need to consider $k = m$ in the second item of Theorem 2.4.

Remark 2.5. Looking at the proof of Theorem 2.4 and choosing $k = 1$ and $n_1 = m$, we not only recover Zalduendo's and Aron–Globevnik's theorems but we also provide an alternative proof for them. In fact, for the sake of simplicity let us choose $p_1 = \cdots = p_m = p$. Let $T : X_p \times \cdots \times X_p \to \mathbb{K}$ be a continuous m-linear form, and let $p > m$. Denoting by T_L the linearization of T and, as usual, letting $\frac{p}{p-m} = 1$ when $p = \infty$, we have

$$\begin{aligned}\left(\sum_{j=1}^{\infty}\left|T(e_j, \dots, e_j)\right|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}} &= \left(\sum_{j=1}^{\infty}\left|T_L\Big(\bigotimes_m^{\pi} e_j\Big)\right|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \\ &\leq \|T_L\|\left\|\Big(\bigotimes_m^{\pi} e_j\Big)_{j=1}^{\infty}\right\|_{w,\frac{p}{p-m}}.\end{aligned}$$

But, from Lemma 2.2 we know that $\|(\widehat{\bigotimes}_m^{\pi} e_j)_{j=1}^{\infty}\|_{w,\frac{p}{p-m}} = 1$ and since $\|T_L\| = \|T\|$, the proof is done. Concerning the optimality of the exponents, it can be easily proved using an idea borrowed from [14]. In fact, consider $T_n : X_p \times \cdots \times X_p \to \mathbb{K}$ given by

$$T_n(x^{(1)}, \dots, x^{(m)}) = \sum_{j=1}^{n} x_j^{(1)} \cdots x_j^{(m)}.$$

Then, since $\|T_n\| = n^{1-\frac{m}{p}}$ and

$$\left(\sum_{j=1}^{n}\left|T_n(e_j, \dots, e_j)\right|^{r}\right)^{\frac{1}{r}} = n^{\frac{1}{r}},$$

we conclude that

$$r \geq \frac{p}{p-m}.$$

Remark 2.6. Using the canonical isometric isomorphisms for the spaces of weakly summable sequences ($\mathcal{L}(\ell_p; E) = \ell_{p^*}^{w}(E)$, $1 < p < \infty$, and $\mathcal{L}(c_0; E) = \ell_1^{w}(E)$), all the aforementioned inequalities can be translated to the theory of absolutely summing operators, motivating a general approach that encompasses the notions of absolutely summing and multiple summing operators.

3. Applications: Constants associated to special choices of Λ

For real scalars, from [15] we know that in (1.3) we have

$$C_{m,\infty}^{\mathbb{R}} \geq 2^{1-\frac{1}{m}},$$

so the Bohnenblust–Hille inequality for real scalars is obviously noncontractive. In this section, as a consequence of the main result of this article, we show that the Bohnenblust–Hille inequality is, however, somewhat "almost" contractive. More precisely, we consider sums in certain sets Λ, that is,

$$\Big(\sum_{i_1,\ldots,i_k=1}^{\infty} \big| T(e_{i_1}^{n_1},\ldots,e_{i_k}^{n_k}) \big|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq N_{k,m,\infty}^{\mathbb{K}} \|T\|,$$

and we show that if the set Λ is composed by a certain number of "blocks" $k := k(m)$ such that

$$\lim_{m\to\infty} \frac{k \log k}{m} = 0,$$

then

$$\lim_{m\to\infty} N_{k,m,\infty}^{\mathbb{K}} = 1.$$

A somewhat similar job can be done for the Hardy–Littlewood inequalities, but we omit the technical details.

It is well known that (for both real and complex scalars)

$$\Big(\sum_{i_1,\ldots,i_m=1}^{\infty} \big| T(e_{i_1},\ldots,e_{i_m}) \big|^2 \Big)^{\frac{1}{2}} \leq \|T\| \tag{3.1}$$

for all continuous m-linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$. In fact, for every positive integer n, by the Khinchin inequality for multiple sums (since the constant of the Khinchin inequality in this case is 1) we have

$$\begin{aligned}
&\Big(\sum_{i_1,\ldots,i_m=1}^{n} \big| T(e_{i_1},\ldots,e_{i_m}) \big|^2 \Big)^{1/2} \\
&\quad \leq \Big(\int_0^1 \cdots \int_0^1 \Big| \sum_{i_1,\ldots,i_m=1}^{n} r_{i_1}(t_1)\cdots r_{i_m}(t_m) T(e_{i_1},\ldots,e_{i_m}) \Big|^2 dt_1 \cdots dt_m \Big)^{1/2} \\
&\quad = \Big(\int_0^1 \cdots \int_0^1 \Big| T\Big(\sum_{i_1=1}^{n} r_{i_1}(t_1) e_{i_1}, \ldots, \sum_{i_m=1}^{n} r_{i_m}(t_m) e_{i_m} \Big) \Big|^2 dt_1 \cdots dt_m \Big)^{1/2} \\
&\quad \leq \|T\|.
\end{aligned}$$

The next theorem can be understood as a refinement of (1.3) and shows when inequalities of the Bohnenblust–Hille-type have contractive constants as the number of variables m increases. It is worth mentioning that if m increases, the number of "blocks" k can be maintained constant or increased as a function of m. By

$k = k(m)$, we mean that k can vary as a function of m. This trivially includes the case when k is kept constant.

Theorem 3.1. *Let m, k be positive integers with $k \leq m$, and let $n_1, \ldots, n_k \in \{0, 1, \ldots, m\}$ with $n_1 + \cdots + n_k = m$. Then*

$$\left(\sum_{i_1,\ldots,i_k=1}^{\infty} \left| T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq (C_{k,\infty}^{\mathbb{K}})^{\frac{k}{m}} \|T\|$$

for all continuous m-linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$. Besides, if $k = k(m)$ is such that

$$\lim_{m\to\infty} \frac{k \log k}{m} = 0,$$

then

$$\lim_{m\to\infty} (C_{k,\infty}^{\mathbb{K}})^{\frac{k}{m}} = 1.$$

Proof. We know from Theorem 2.4 that

$$\left(\sum_{i_1,\ldots,i_k=1}^{\infty} \left| T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}) \right|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \leq C_{k,\infty}^{\mathbb{K}} \|T\| \tag{3.2}$$

for all continuous m-linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$. Since

$$\frac{1}{\frac{2m}{m+1}} = \frac{\theta}{\frac{2k}{k+1}} + \frac{1-\theta}{2}$$

with

$$\theta = \frac{k}{m},$$

by (a corollary of) the Hölder inequality, and using (3.1) and (3.2), we have

$$\begin{aligned}
&\left(\sum_{i_1,\ldots,i_k=1}^{\infty} \left| T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \\
&\leq \left[\left(\sum_{i_1,\ldots,i_k=1}^{\infty} \left| T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}) \right|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \right]^{\frac{k}{m}} \left[\left(\sum_{i_1,\ldots,i_k=1}^{\infty} \left| T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}) \right|^{2} \right)^{\frac{1}{2}} \right]^{1-\frac{k}{m}} \\
&\leq \left[\left(\sum_{i_1,\ldots,i_k=1} \left| T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}) \right|^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \right]^{\frac{k}{m}} \|T\| \\
&\leq (C_{k,\infty}^{\mathbb{K}})^{\frac{k}{m}} \|T\|,
\end{aligned}$$

and the inequality is proved.

Besides, using the best-known estimates for $C_{k,\infty}^{\mathbb{K}}$ (see [6, Corollary 3.2]), we have

$$(C_{k,\infty}^{\mathbb{K}})^{\frac{k}{m}} \leq (\alpha k^{\beta})^{\frac{k}{m}}$$

for suitable $\alpha, \beta > 0$. Note that

$$\lim_{m\to\infty} (\alpha k^{\beta})^{\frac{k}{m}} = 1$$

if and only if

$$\lim_{m\to\infty} \log(\alpha k^\beta)^{\frac{k}{m}} = 0,$$

if and only if

$$\lim_{m\to\infty} \frac{k}{m}(\log\alpha + \beta\log k) = 0.$$

This last equality is valid because

$$\lim_{m\to\infty} \frac{k\log k}{m} = 0$$

implies that

$$\lim_{m\to\infty} \frac{k}{m} = 0. \qquad \square$$

Example 3.2. It is interesting to verify that our hypotheses hold for

$$k = \left\lfloor \frac{m}{(\log m)^{1+\frac{1}{\log\log\log m}}} \right\rfloor \text{ and } k = \lfloor m^{1-\frac{1}{\log\log m}} \rfloor.$$

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