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ON OPERATORS WITH CLOSED NUMERICAL RANGES

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ABSTRACT. In this article we investigate the numerical ranges of several classes of operators. It is shown that, if we let T be a hyponormal operator and let $\varepsilon > 0$, then there exists a compact operator K with norm less than ε such that $T + K$ is hyponormal and has a closed numerical range. Moreover we prove that the statement of the above type holds for other operator classes, including weighted shifts, normaloid operators, triangular operators, and block-diagonal operators.

1. INTRODUCTION

In this paper, let $\mathcal{H}$ be a complex separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$ and the ideal of compact operators on $\mathcal{H}$, respectively. The *numerical range* of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined as

$$W(T) = \big\{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \big\}.$$

Clearly, $W(T)$ is a nonempty bounded subset of $\mathbb{C}$.

The classical Toeplitz–Hausdorff theorem asserts that the numerical range $W(T)$ of T is always convex (see [9], [14]). Moreover it is well known that $\operatorname{conv} \sigma(T) \subseteq \overline{W(T)}$ (see [8, p. 115]), where $\sigma(T)$ denotes the spectrum of T, $\operatorname{conv} \sigma(T)$ denotes the convex hull of $\sigma(T)$, and $\overline{W(T)}$ denotes the closure of $W(T)$. Let $w(T)$ and $r(T)$ denote the *numerical radius* and the *spectral radius*

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of T, that is, $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$, and $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. It is known that $r(T) \leq w(T) \leq \|T\|$. It is also well known that the numerical range of a direct sum is the convex hull of the numerical ranges of the summands. Some other properties of numerical range can be found in [7], [8].

Despite the conceptual simplicity of the definition of numerical range, numerical range and its generalizations have been studied extensively because of their connections and applications to many different areas, including operator theory, C^*-algebras, quantum theory, dilation theory, Krein space operators, and unitary similarity (see [2], [7], [8], [12]). It is also well known that the numerical range $W(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is always closed when $\mathcal{H}$ is finite-dimensional; however, when $\mathcal{H}$ is infinite-dimensional, the numerical range of an operator in $\mathcal{B}(\mathcal{H})$ may be open. For instance, the numerical range of the unilateral shift operator is an open disk centered at the origin (see [8, p. 317]).

One natural question is whether operators with closed numerical ranges are dense with respect to the uniform (norm) topology. In 2003, Bourin ([3, Proposition 1.3]) proved that, if $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, then there must exist a compact operator K with $\|K\| < \varepsilon$ such that $W(T + K)$ is closed. In his proof, he even used a finite-rank operator to achieve this conclusion. In 2015, S. Zhu in [15] strengthened Bourin's result by proving that, if T is normal (transaloid) and if $\varepsilon > 0$, then there exists a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $W(T+K)$ is closed, and $T + K$ is still normal (transaloid, resp.).

The main aim of this paper is to investigate which classes of operators have the above-mentioned property. For convenience, we say that an operator class $\mathcal{A}$ is *strongly numerically closed*, if, for any $T \in \mathcal{A}$ and any $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that $T + K \in \mathcal{A}$ and $W(T + K)$ is closed.

Thus the class of normal operators and the class of transaloid operators are strongly numerically closed. We are interested in determining which special classes of operators are strongly numerically closed. For some classes of Hilbert space operators, including weighted shift operators, normaloid operators, hyponormal operators, triangular operators, and block-diagonal operators, we give an affirmative answer. Our results strengthen Bourin's and Zhu's results mentioned above. We also prove that the class of pure quasinormal operators is not strongly numerically closed (see Proposition 4.12).

The rest of this paper is organized as follows. In Section 2, we show that the class of unilateral weighted shifts and the class of bilateral weighted shifts are both strongly numerically closed. In Section 3, we prove that the class of triangular operators and the class of block-diagonal operators are both strongly numerically closed. Section 4 is devoted to showing that the class of hyponormal operators and the class of normaloid operators are both strongly numerically closed.

2. Weighted shift operators

Let $A \in \mathcal{B}(\mathcal{H})$. If there is an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ and a sequence $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ such that $Ae_n = a_n e_{n+1}$ for all $n \geq 1$, then A is called a *unilateral* *weighted shift* with weights $\{a_n\}_{n=1}^{\infty}$. The main result of this section is the following theorem.

Theorem 2.1. *The class of unilateral weighted shifts is strongly numerically closed.*

To give the proof of Theorem 2.1, we give some useful results concerning numerical ranges of unilateral weighted shifts.

Lemma 2.2 ([12, Proposition 1])**.** *If A is a unilateral weighted shift with weights $\{a_n\}_{n=1}^{\infty}$, then A is unitarily equivalent to the unilateral weighted shift $\widetilde{A}$ with weights $\{|a_n|\}_{n=1}^{\infty}$.*

By Lemma 2.2 and the definition of numerical range, it is easy to verify that $W(A) = W(\widetilde{A})$; hence we can assume that, from time to time, the weights of unilateral weighted shifts are nonnegative.

The following lemma is clear. For the reader's convenience we give its proof.

Lemma 2.3. *If A is a unilateral weighted shift, then $W(A)$ is a disk centered at the origin.*

Proof. By Lemma 2.2, A is unitarily equivalent to cA for any $c \in \mathbb{C}$ with $|c| = 1$. Thus $W(A) = W(cA) = cW(A)$ for any $c \in \mathbb{C}$ when $|c| = 1$. This implies that $W(A)$ has circular symmetry. Note that $W(A)$ is always convex. We deduce that $W(A)$ is a disk centered at the origin, either open or closed. □

Using the same method as that used in Lemma 2.3, we can easily obtain the following corollary.

Corollary 2.4. *Let $A \in \mathcal{B}(\mathbb{C}^n)$, and assume that A can be written as*

$$A = \begin{bmatrix} 0 & & & & \\ a_1 & 0 & & 0 & \\ & a_2 & 0 & & \\ & & \ddots & \ddots & \\ & 0 & & a_{n-1} & 0 \end{bmatrix}$$

relative to some orthonormal basis of $\mathbb{C}^n$. Thus $W(A)$ is a closed disk centered at the origin.

Recall that the *essential numerical range* of $T \in \mathcal{B}(\mathcal{H})$ is the nonempty set

$$W_e(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \overline{W(T+K)}.$$

According to the definition of $W_e(T)$, it is apparent that $W_e(T)$ is convex, closed, and invariant under compact perturbation. And it is well known that $\sigma_e(T) \subseteq W_e(T)$, where $\sigma_e(T)$ denotes the essential spectrum of T. (For references on the theory of essential numerical range, see [1] and [6].)

In his paper, Lancaster [11] described the following relationship between the numerical range and the essential numerical range of an operator.

Lemma 2.5 ([11, Theorem 1])**.** *If $T \in \mathcal{B}(\mathcal{H})$, then $\overline{W(T)} = \operatorname{conv}\{W(T) \cup W_e(T)\}$.*

Lemma 2.5 gives a necessary and sufficient condition for the numerical range of an operator to be closed. That is, $W(T)$ is closed if and only if $W_e(T) \subseteq W(T)$.

The following two lemmas describe the relationship between the numerical range of an operator and the numerical range of its compression.

Lemma 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$, and let M be a closed subspace of $\mathcal{H}$. If we denote by P the orthogonal projection onto M, then $W(PT|_M) \subseteq W(T)$.*

Proof. Since $M \subseteq \mathcal{H}$, we have

$$\langle PT|_M x, x\rangle = \langle T|_M x, Px\rangle = \langle Tx, x\rangle \in W(T)$$

for any $x \in M$ with $\|x\| = 1$; hence $W(PT|_M) \subseteq W(T)$. $\square$

Throughout this paper, we denote by $\mathcal{PF}(\mathcal{H})$ the set of all finite-rank orthogonal projections in $\mathcal{B}(\mathcal{H})$.

Because the following result is obvious, its proof is omitted.

Lemma 2.7. *Let $\{P_n\}_{n=1}^\infty$ be a (not necessarily increasing) sequence in $\mathcal{PF}(\mathcal{H})$ such that $P_n \to I$ in the strong operator topology. Given $A \in \mathcal{B}(\mathcal{H})$, denote $A_n = P_n A|_{P_n\mathcal{H}}$ for each $n \geq 1$. Then $\overline{W(A)} = \overline{\bigcup_{n=1}^\infty W(A_n)}$, and $w(A_n) \to w(A)$.*

Lemma 2.8. *Let $\{e_n\}_{n=1}^\infty$ be an othonormal basis of $\mathcal{H}$, and let $A \in \mathcal{B}(\mathcal{H})$ with $Ae_n = a_n e_{n+1}$ for $n \geq 1$. If $a_n \geq 0$ for all $n \geq 1$, then $\overline{W(A)} \subsetneq W(A + \varepsilon S)$ for every $\varepsilon > 0$, where S is the canonical unilateral shift defined as $Se_n = e_{n+1}$ for all $n \geq 1$.*

Proof. Without loss of generality, we assume that $w(A) = 1$. By Lemma 2.3, both $W(A)$ and $W(A + \varepsilon S)$ are disks centered at the origin for each $\varepsilon > 0$. So it suffices to show that $w(A + \varepsilon S) > 1 = w(A)$ for every $\varepsilon > 0$. Since $w(A) = 1$, then, for any fixed $\varepsilon > 0$, there exists a unit vector $x_0 = (\eta_1, \eta_2, \ldots)$ such that $|\langle Ax_0, x_0\rangle - 1| < \delta\varepsilon$, where $\delta = \frac{1}{\|A\|+\varepsilon}$. Accordingly,

$$\begin{aligned} 1 - \delta\varepsilon < \big|\langle Ax_0, x_0\rangle\big| &= \Big|\sum_{k=1}^\infty a_k \eta_k \overline{\eta_{k+1}}\Big| \\ &\leq \sum_{k=1}^\infty a_k |\eta_k||\eta_{k+1}| \\ &= \langle Ay_0, y_0\rangle, \end{aligned}$$

where $y_0 = (|\eta_1|, |\eta_2|, \ldots)$. Namely, $\langle Ay_0, y_0\rangle > 1 - \delta\varepsilon$. On the other hand, note that, if $0 \leq a_k \leq \|A\|$ for all $k \geq 1$, then

$$\begin{aligned} 1 - \delta\varepsilon < \big|\langle Ax_0, x_0\rangle\big| &\leq \sum_{k=1}^\infty a_k |\eta_k||\eta_{k+1}| \\ &\leq \|A\| \sum_{k=1}^\infty |\eta_k||\eta_{k+1}| \\ &= \|A\|\langle Sy_0, y_0\rangle; \end{aligned}$$

hence

$$\langle \varepsilon S y_0, y_0 \rangle > \frac{\varepsilon(1-\delta\varepsilon)}{\|A\|}.$$

We then have

$$\begin{aligned}\big\langle (A+\varepsilon S)y_0, y_0 \big\rangle &= \langle Ay_0, y_0\rangle + \langle \varepsilon S y_0, y_0\rangle \\ &> 1 - \delta\varepsilon + \frac{\varepsilon(1-\delta\varepsilon)}{\|A\|} \\ &= 1.\end{aligned}$$

This implies that $w(A+\varepsilon S) > 1 = w(A)$. The proof is complete. □

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Suppose that A is a unilateral weighted shift such that $Ae_n = a_n e_{n+1}$ for $n \geq 1$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis. Without loss of generality we assume that $a_n \geq 0$ for all $n \geq 1$.

If we denote by P_n the orthogonal projection onto $M_n = \mathrm{span}\{e_k : 1 \leq k \leq n\}$, then $\{P_n\}_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{PF}(\mathcal{H})$ with $P_n \to I$ in the strong operator topology. For any fixed $\varepsilon > 0$, by Lemma 2.8, we have

$$\overline{W(A)} \subsetneq W\Big(A + \frac{\varepsilon}{2}S\Big) \qquad \text{and} \qquad w(A) < w\Big(A + \frac{\varepsilon}{2}S\Big).$$

Set $T_n = P_n(A + \frac{\varepsilon}{2}S)|_{P_n\mathcal{H}}$ for each $n \geq 1$. According to Lemma 2.7, $w(T_n) \to w(A + \frac{\varepsilon}{2}S)$; thus there exists a sufficiently large n_0 such that $w(A) < w(T_{n_0})$. Moreover, according to Lemma 2.3 and Corollary 2.4, we know that both $W(A)$ and $W(T_{n_0})$ are disks centered at the origin. From the argument above, we then have $\overline{W(A)} \subsetneq W(T_{n_0})$.

If we let $K = \frac{\varepsilon}{2}P_{n_0}SP_{n_0}$, then K is compact with $\|K\| = \varepsilon/2 < \varepsilon$. By the preceding argument and by Lemma 2.6, we have

$$\begin{aligned}\overline{W(A)} \subsetneq W(T_{n_0}) &= W\Big(P_{n_0}\Big(A + \frac{\varepsilon}{2}S\Big)\Big|_{P_{n_0}\mathcal{H}}\Big) \\ &= W\big(P_{n_0}(A+K)\big|_{P_{n_0}\mathcal{H}}\big) \\ &\subseteq W(A+K).\end{aligned}$$

Note that

$$W_e(A+K) = W_e(A) \subseteq \overline{W(A)}.$$

We conclude that $W_e(A+K) \subseteq W(A+K)$. According to Lemma 2.5, we know that $W(A+K)$ is closed. On the other hand, by the construction of K it is obvious that $A+K$ is a unilateral weighted shift. The proof is complete. □

Remark 2.9. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is a *bilateral weighted shift* if there exists an orthonormal basis $\{e_n\}_{-\infty}^{+\infty}$, and if $\{a_n\}_{-\infty}^{+\infty} \subseteq \mathbb{C}$ such that $Ae_n = a_n e_{n+1}$ for all $n \in \mathbb{Z}$. Using similar arguments we can prove that the class of bilateral weighted shifts is also strongly numerically closed.

3. Triangular operators and block-diagonal operators

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *triangular* if T admits an upper triangular matrix; that is,

$$A = \begin{bmatrix} a_{11} & a_{11} & a_{11} & \cdots \\ & a_{11} & a_{11} & \cdots \\ & & a_{11} & \cdots \\ & 0 & & \ddots \end{bmatrix},$$

with respect to a suitable orthonormal basis.

The following theorem is the main result of this section.

Theorem 3.1. *The class of triangular operators is strongly numerically closed.*

Proof. Suppose that T is triangular with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Denote by P_n the orthogonal projection onto the finite-dimensional subspace spanned by $\{e_i : 1 \leq i \leq n\}$. Hence $\{P_n\}_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{PF}(\mathcal{H})$, and $P_n \to I$ in the strong operator topology.

For any fixed $\varepsilon > 0$, we can choose $a_1, a_2, \ldots, a_n \in \partial W_e(T)$ such that $\partial W_e(T) \subseteq \bigcup_{k=1}^{n} B(a_k, \frac{\varepsilon}{4})$, where $\partial W_e(T)$ denotes the boundary of $W_e(T)$. For each $1 \leq k \leq n$, there exist $\mu_{k,1}, \mu_{k,2}, \mu_{k,3}$ such that

$$|\mu_{k,j} - a_k| = \frac{\varepsilon}{2}, \quad j = 1, 2, 3, \qquad \text{and} \qquad B\Big(a_k, \frac{\varepsilon}{4}\Big) \subseteq \operatorname{conv}\{\mu_{k,j} : j = 1, 2, 3\}.$$

If we set $\lambda_{3k} = \lambda_{3k-1} = \lambda_{3k-2} = a_k$, and $\mu_{3(k-1)+j} = \mu_{k,j}$ for all $1 \leq k \leq n$, and $1 \leq j \leq 3$, then

$$W_e(T) = \operatorname{conv} \partial W_e(T) \subseteq \operatorname{conv}\Big\{\bigcup_{k=1}^{n} B\Big(a_k, \frac{\varepsilon}{4}\Big)\Big\} \subseteq \operatorname{conv}\{\mu_k : 1 \leq k \leq 3n\}.$$

Since $\lambda_1 \in W_e(T) \subseteq \overline{W(T)}$, then, by Lemma 2.7, there exists a sufficiently large integer m_1 and a unit vector $g_1 \in P_{m_1}\mathcal{H}$ such that

$$\big|\langle Tg_1, g_1\rangle - \lambda_1\big| = \big|\langle P_{m_1} T|_{P_{m_1}\mathcal{H}} g_1, g_1\rangle - \lambda_1\big| < \frac{\varepsilon}{8}.$$

Moreover note that $\lambda_2 \in W_e((I - P_{m_1})T|_{(I-P_{m_1})\mathcal{H}}) = W_e(T)$. Using Lemma 2.7 again, we see that there exists a sufficiently large m_2 with $m_2 > m_1$ and a unit vector $g_2 \in (P_{m_2} - P_{m_1})\mathcal{H}$ such that

$$\big|\langle Tg_2, g_2\rangle - \lambda_2\big| < \frac{\varepsilon}{8}.$$

Repeating the argument above, for each λ_k, $1 \leq k \leq 3n$, we can find a sufficiently large integer m_k with $m_k > m_{k-1}$ and a unit vector $g_k \in (P_{m_k} - P_{m_{k-1}})\mathcal{H}$ such that

$$\big|\langle Tg_k, g_k\rangle - \lambda_k\big| < \frac{\varepsilon}{8},$$

where $P_{m_0} = 0$. From the choices of $\{g_k\}_{k=1}^{3n}$, we know that $\{g_k\}_{k=1}^{3n}$ is a finite sequence of pairwise orthogonal unit vectors in $\mathcal{H}$.

If we set $F = \sum_{k=1}^{3n} (\mu_k - \omega_k) g_k \otimes g_k$, where $\omega_k = \langle Tg_k, g_k\rangle$ for each $1 \leq k \leq 3n$, then F is finite rank with $\|F\| < \varepsilon$. A simple computation then shows that

$\mu_k = \langle (T+F)g_k, g_k\rangle$ for each $1 \le k \le 3n$; that is, $\mu_k \in W(T+F)$ for each $1 \le k \le 3n$; hence

$$W_e(T+F) = W_e(T) \subseteq \operatorname{conv}\{\mu_k : 1 \le k \le 3n\} \subseteq W(T+F).$$

According to Lemma 2.5, we know that $W(T+F)$ is closed. On the other hand, if we let $M = \operatorname{span}\{e_k : 1 \le k \le m_{3n}\}$, then $T+F$ can be written as

$$T+F = \begin{bmatrix} A & * \\ 0 & T_1 \end{bmatrix} \begin{matrix} M \\ M^{\perp}, \end{matrix}$$

where T_1 is upper triangular under $\{e_k\}_{k=m_{3n}+1}^{\infty}$. Since A acts on the finite-dimensional space M, then A admits an upper triangular matrix under a suitable orthonormal basis $\{f_k\}_{k=1}^{m_{3n}}$ of M. If we let $f_k = e_k$ for $k > m_{3n}$, then it is clear that $\{f_n\}_{n=1}^{\infty}$ forms an orthonormal basis of $\mathcal{H}$ under which $T+F$ is triangular. This completes the proof. □

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *block-diagonal* if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ in $\mathcal{PF}(\mathcal{H})$ with $P_n \to I$ in the strong operator topology such that $P_nT = TP_n$ for all $n \ge 1$. With an argument similar to that in the proof of Theorem 3.1, we conclude the following theorem.

Theorem 3.2. *The class of block-diagonal operators is strongly numerically closed.*

Recall that an operator T is *quasidiagonal* if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ in $\mathcal{PF}(\mathcal{H})$ such that $P_n \to I$ in the strong operator topology, and $\|P_nT - TP_n\| \to 0$. By [10, Theorem 6.12], an operator $A \in \mathcal{B}(\mathcal{H})$ is quasidiagonal if and only if, for any $\varepsilon > 0$, T can be written as $A = A_\varepsilon + K_\varepsilon$, where A_ε is block-diagonal, and where K_ε is compact with $\|K_\varepsilon\| < \varepsilon$.

An operator T is *quasitriangular* if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ in $\mathcal{PF}(\mathcal{H})$ with $P_n \to I$ in the strong operator topology such that $\|(I - P_n)TP_n\| \to 0$. By [10, Theorem 6.4], an operator $A \in \mathcal{B}(\mathcal{H})$ is quasi-triangular if and only if, for any $\varepsilon > 0$, A can be written as $A = A_\varepsilon + K_\varepsilon$, where A_ε is triangular, K_ε is compact with $\|K_\varepsilon\| < \varepsilon$.

The following is an immediate corollary of Theorem 3.1 and Theorem 3.2.

Corollary 3.3. *The class of quasidiagonal operators and the class of quasitriangular operators are both strongly numerically closed.*

4. Normaloid operators and hyponormal operators

In [15], it is proved that the class of transaloid operators and the class of normal operators are both strongly numerically closed. In this section, we shall show that the class of normaloid operators and the class of hyponormal operators are also strongly numerically closed.

We begin with some fundamentals. Recall that an operator T is *normaloid* if $\|T\| = r(T)$. An operator T is *transaloid* if $T - \lambda$ is normaloid for all $\lambda \in \mathbb{C}$, and T is *hyponormal* if $T^*T - TT^* \ge 0$. For more results on these operators, see [5] and [15].

Let $T \in \mathcal{B}(\mathcal{H})$. Denote by $\ker T$ the kernel of T, and denote by $\operatorname{ran} T$ the range of T. If $\operatorname{ran} T$ is closed, and either $\dim \ker T$ or $\dim \ker T^*$ is finite, then T is called a *semi-Fredholm operator*. The *Wolf spectrum* $\sigma_{lre}(T)$ of T is defined by

$$\sigma_{lre}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}.$$

The set of normal eigenvalues of T is denoted by $\sigma_0(T)$; that is,

$$\sigma_0(T) = \big\{\lambda \in \mathbb{C} : \lambda \notin \sigma_{lre}(T),\ \text{and } \lambda \text{ is an isolated point of } \sigma(T)\big\}.$$

Let $\sigma_p(T)$ denote the set of all eigenvalues of T. It is well known that $\sigma_0(T) \subseteq \sigma_p(T)$. The reader is referred to [10, p. 5] and [4, p. 366] for more details about this terminology.

Lemma 4.1 ([4, p. 366])**.** *If we let $T \in \mathcal{B}(\mathcal{H})$, then $\partial\sigma(T) \subseteq \sigma_0(T) \cup \sigma_{lre}(T)$.*

Lemma 4.2. *If T is normaloid, then, given $\varepsilon > 0$, there exists a compact K with $\|K\| < \varepsilon$ and a unit vector $e \in \mathcal{H}$ such that*

$$T + K = \begin{bmatrix} \lambda & 0 \\ 0 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix},$$

where $\lambda \in \mathbb{C}$ and A acting on $(\mathbb{C}e)^\perp$ satisfies $\|A\| \le |\lambda| = \|T\|$.

Proof. As T is normaloid, there exists a $\lambda \in \sigma(T)$ such that $|\lambda| = \|T\|$. This implies that $\lambda \in \partial\sigma(T)$. According to Lemma 4.1, we have $\lambda \in \sigma_0(T)$ or $\lambda \in \sigma_{lre}(T)$. Note that $\sigma_0(T) \subseteq \sigma_p(T)$. If $\lambda \in \sigma_0(T)$, one can easily show that $\ker(T - \lambda)$ reduces T. Then

$$T = \begin{bmatrix} \lambda & 0 \\ 0 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix},$$

where e is a unit eigenvector for T corresponding to the eigenvalue λ, and where $A \in \mathcal{B}((\mathbb{C}e)^\perp)$ with $\|A\| \le |\lambda|$.

Now consider the case where $\lambda \in \sigma_{lre}(T)$. Note that, if $\sigma_{lre}(T) \subseteq \sigma_e(T) \subseteq W_e(T)$, then we have $\lambda \in W_e(T)$. Hence, for any fixed $\varepsilon > 0$, there exists a unit vector e such that $|\langle Te, e\rangle - \lambda| < \varepsilon/2$. Consequently T can be written as

$$T = \begin{bmatrix} \mu & F_1 \\ F_2 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix},$$

where $A : (\mathbb{C}e)^\perp \to (\mathbb{C}e)^\perp$ with $\|A\| \le \|T\| = |\lambda|$, $F_1 : (\mathbb{C}e)^\perp \to \mathbb{C}e$, and $F_2 : \mathbb{C}e \to (\mathbb{C}e)^\perp$ are rank 1 with $\|F_i\| < \varepsilon/2$ for $i = 1, 2$. Denote

$$K = \begin{bmatrix} \lambda - \mu & -F_1 \\ -F_2 & 0 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix}.$$

It is apparent that K is compact with $\|K\| < \varepsilon$ and that

$$T + K = \begin{bmatrix} \lambda & 0 \\ 0 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix}.$$

The proof is complete. □

Theorem 4.3. *The class of normaloid operators is strongly numerically closed.*

Proof. Suppose that T is normaloid. Then there exists a $\lambda \in \sigma(T)$ such that $|\lambda| = \|T\|$. It follows from Lemma 4.2 that, if $\varepsilon > 0$, then there exists a unit vector $e \in \mathcal{H}$, a compact operator K_1 with $\|K_1\| < \varepsilon/2$ and an operator A with $\|A\| \leq |\lambda|$ such that

$$T + K_1 = \begin{bmatrix} \lambda & 0 \\ 0 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix}.$$

Furthermore, for operator A, there exists a compact $\widetilde{K_2}$ with $\|\widetilde{K_2}\| < \varepsilon/2$ such that $W(A + \widetilde{K_2})$ is closed. If we let

$$\lambda = e^{i\theta}|\lambda|, \qquad K_2 = \begin{bmatrix} \varepsilon/2e^{i\theta} & 0 \\ 0 & \widetilde{K_2} \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix} \qquad \text{and} \qquad K = K_1 + K_2,$$

then K is compact with $\|K\| < \varepsilon$, and

$$T + K = T + K_1 + K_2 = \begin{bmatrix} \lambda + \varepsilon/2e^{i\theta} & 0 \\ 0 & A + \widetilde{K_2} \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^\perp \end{matrix}.$$

Note that $W(A + \widetilde{K_2})$ is closed; thus $W(T + K)$ is closed.

On the other hand, note that

$$\lambda + \frac{\varepsilon}{2}e^{i\theta} \in \sigma(T + K)$$

and that

$$\|A + \widetilde{K_2}\| \leq \|A\| + \|\widetilde{K_2}\| \leq |\lambda| + \frac{\varepsilon}{2} = \left|\lambda + \frac{\varepsilon}{2}e^{i\theta}\right|.$$

Then

$$\|T + K\| = \max\left\{\left|\lambda + \frac{\varepsilon}{2}e^{i\theta}\right|, \|A + \widetilde{K_2}\|\right\} = \left|\lambda + \frac{\varepsilon}{2}e^{i\theta}\right| \leq r(T + K) \leq \|T + K\|;$$

hence $T + K$ is normaloid. This completes the proof. □

For hyponormal operators we have the following result.

Theorem 4.4. *The class of hyponormal operators is strongly numerically closed.*

To give the proof of Theorem 4.4, we set forth some first principles.

Lemma 4.5 ([13])**.** *If T is hyponormal, then*

(i) *$T - \lambda$ is also hyponormal for any $\lambda \in \mathbb{C}$;*
(ii) *$\|T\| = r(T)$.*

The lemma above shows that hyponormal operators must be normaloid. The following lemma is well known. We write down its proof for the reader's convenience.

Lemma 4.6. *Let T be hyponormal. If $\lambda \in \sigma_p(T)$, then* $\ker(T - \lambda)$ *reduces T.*

Proof. If $\lambda \in \sigma_p(T)$, then $\ker(T - \lambda)$ is an invariant subspace of $T - \lambda$.

In addition, by Lemma 4.5 and by the definition of hyponormal operator, we have

$$\big\|(T - \lambda)x\big\| \geq \big\|(T - \lambda)^* x\big\|, \quad \forall x \in \mathcal{H};$$

hence $\ker(T-\lambda) \subseteq \ker(T-\lambda)^*$. This implies that $\ker(T-\lambda)$ is also an invariant subspace of $(T-\lambda)^*$; thus $\ker(T-\lambda)$ reduces $T-\lambda$, and, naturally, $\ker(T-\lambda)$ reduces T. $\square$

Lemma 4.7. *If T is hyponormal, then $\overline{W(T)} = \operatorname{conv}\sigma(T)$.*

Proof. It is well known that $\overline{W(T)} \supseteq \operatorname{conv}\sigma(T)$; therefore it suffices to show that $\overline{W(T)} \subseteq \operatorname{conv}\sigma(T)$. Otherwise, we have $\operatorname{conv}\sigma(T) \subsetneq \overline{W(T)}$. By their convexity there exists $\mu \in \mathbb{C}$ such that

$$\sup_{z\in W(T)} |z-\mu| > \sup_{z\in\sigma(T)} |z-\mu|.$$

Then $w(T-\mu) > r(T-\mu)$. It implies that $\|T-\mu\| > r(T-\mu)$. This contradicts Lemma 4.5; hence $\overline{W(T)} \subseteq \operatorname{conv}\sigma(T)$. The proof is complete. $\square$

Lemma 4.7 generalizes the fact that $\overline{W(N)} = \operatorname{conv}\sigma(N)$ for normal operator N. For normal operators, there is the following result.

Lemma 4.8 ([15, Proposition 5.1])**.** *Let $N \in \mathcal{B}(\mathcal{H})$ be normal. Then, given $\varepsilon > 0$, there exists a $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $N+K$ is normal, and*

$$\overline{W(N)} \subseteq \overline{W(N+K)} = W(N+K).$$

Recall that two operators $A, B \in \mathcal{B}(\mathcal{H})$ are said to be *approximately unitarily equivalent*, denoted by $A \simeq_a B$, if there exist unitary operators $\{U_n\}_{n=1}^{\infty}$ such that $\|U_nA - BU_n\| \to 0$.

Lemma 4.9 ([10, Proposition 4.21(iv)])**.** *If A and B are approximately unitarily equivalent, then, given $\varepsilon > 0$, there exists a unitary operator U and a compact operator K such that $\|K\| < \varepsilon$ and $A + K = U^*BU$.*

Lemma 4.10 ([10, Proposition 4.28])**.** *If we let T be hyponormal, and we let N be any normal operator with $\sigma(N) \subseteq \sigma_{lre}(T)$, then $T \simeq_a T \oplus N$.*

Proof of Theorem 4.4. Suppose that T is hyponormal. We may directly assume that $\sigma_0(T) = \emptyset$. In fact, if not, then we can assume that $\sigma_0(T) = \{\lambda_1, \lambda_2, \ldots\}$. Since $\sigma_0(T) \subseteq \sigma_p(T)$, by applying Lemma 4.6, T can be written as

$$T = \begin{bmatrix} N_0 & 0 \\ 0 & A \end{bmatrix} = \left[\begin{array}{ccc|c} \lambda_1 I_1 & & & \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ \hline & & & A \end{array}\right],$$

where N_0 is normal, and A is hyponormal with $\sigma_0(A) = \emptyset$. It follows from Lemma 4.8 that, given $\varepsilon > 0$, one can find a compact

$$\widetilde{K_0} = \begin{bmatrix} K_0 & 0 \\ 0 & 0 \end{bmatrix}$$

with $\|\widetilde{K_0}\| < \varepsilon$ such that $N_0 + K_0$ is normal, and that

$$\overline{W(N_0)} \subseteq \overline{W(N_0+K_0)} = W(N_0+K_0).$$

Note that

$$W(T+\widetilde{K_0}) = \operatorname{conv}\big\{W(N_0+K_0)\cup W(A)\big\}.$$

To finish the proof, it suffices to address A. We may directly assume that $\sigma_0(T)=\emptyset$.

According to Lemma 4.1, we know that $\partial\sigma(T)\subseteq\sigma_{lre}(T)$. If we let N be a diagonal operator with $\sigma(N)=\partial\sigma(T)$, then, by Lemma 4.10, we have $T\simeq_a T\oplus N$. It follows from Lemma 4.9 that, for any fixed $\varepsilon>0$, there exists a compact operator K_1 with $\|K_1\|<\frac{\varepsilon}{2}$ such that

$$T+K_1\cong\begin{bmatrix} T & 0\\ 0 & N\end{bmatrix}.$$

Without loss of generality, we may assume that

$$T+K_1=\begin{bmatrix} T & 0\\ 0 & N\end{bmatrix}.$$

Applying Lemma 4.8, we can find a compact operator

$$\widetilde{K_2}=\begin{bmatrix} 0 & 0\\ 0 & K_2\end{bmatrix}$$

with $\|\widetilde{K_2}\|=\|K_2\|<\varepsilon/2$ such that $N+K_2$ is normal, and

$$\overline{W(N)}\subseteq W(N+K_2)=\overline{W(N+K_2)}.$$

Set $K=K_1+\widetilde{K_2}$. It is completely apparent that K is compact with $\|K\|<\varepsilon$ and that

$$T+K=\begin{bmatrix} T & 0\\ 0 & N+K_2\end{bmatrix}$$

is hyponormal. Furthermore, by Lemma 4.7, we have

$$\overline{W(T)}=\operatorname{conv}\sigma(T)=\operatorname{conv}\partial\sigma(T)=\operatorname{conv}\sigma(N)=\overline{W(N)}\subseteq W(N+K_2);$$

hence

$$W(T+K)=\operatorname{conv}\big\{W(T)\cup W(N+K_2)\big\}=W(N+K_2),$$

and $W(T+K)$ is closed. Now the proof is complete. □

Recall that $T\in\mathcal{B}(\mathcal{H})$ is *subnormal* if there exists a Hilbert space $\mathcal{L}$ containing $\mathcal{H}$ and a normal operator $N\in\mathcal{B}(\mathcal{L})$ such that $N\mathcal{H}\subseteq\mathcal{H}$, and $T=N|_{\mathcal{H}}$. Recall also that T is *quasinormal* if T commutes with T^*T. It is well known that quasinormal operators and subnormal operators are always hyponormal. Using similar arguments, we can prove the following.

Proposition 4.11. *The class of subnormal operators and the class of quasinormal operators are both strongly numerically closed.*

In the rest of this section, we prove that the class of pure quasinormal operators is not strongly numerically closed. Recall that an operator T is *pure quasinormal* if T is quasinormal, and it has no nonzero reducing subspace M such that $T|_M$ is normal. In fact, for pure quasinormal operators, we have the following proposition.

Proposition 4.12. *If $T \in \mathcal{B}(\mathcal{H})$ is pure quasinormal, then $W(T)$ is always an open disk centered at the origin.*

Proof. By [5, p. 44], we may assume that

$$T = \begin{bmatrix} 0 & & & \\ A & 0 & & 0 \\ & A & 0 & \\ & & A & \ddots \\ & 0 & & \ddots \end{bmatrix},$$

where A is positive and injective acting on some Hilbert space $\mathcal{L}$. It is easy to show that T is unitarily equivalent to λT for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$; hence

$$W(T) = W(\lambda T) = \lambda W(T)$$

for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. This implies that $W(T)$ has circular symmetry. By the convexity of $W(T)$, we know that $W(T)$ is a disk centered at the origin.

To finish the proof, it suffices to show that $W(T)$ is open. If not, and we suppose that $W(T)$ is closed disk centered at the origin, then $w(T) \in W(T)$. Moreover, it is easy to prove that

$$w(T) = r(T) = \|T\|.$$

By [8, p. 316], we have $w(T) \in \sigma_p(T)$; however a simple calculation shows that $\sigma_p(T) = \emptyset$. This is a contradiction; hence $W(T)$ is an open disk centered at the origin. This completes the proof. □

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