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ON AN APPROXIMATION OF 2-DIMENSIONAL WALSH–FOURIER SERIES IN MARTINGALE HARDY SPACES

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Abstract. In this paper, we investigate convergence and divergence of partial sums with respect to the 2-dimensional Walsh system on the martingale Hardy spaces. In particular, we find some conditions for the modulus of continuity which provide convergence of partial sums of Walsh–Fourier series. We also show that these conditions are in a sense necessary and sufficient.

1. Introduction

It is well known (for details, see, e.g., [1], [6]) that the Walsh–Paley system is not a Schauder basis in $L_1(G)$. Moreover, there exists (for details, see [10]) a function in the dyadic martingale Hardy space $H_p(G)$ $(0 < p \leq 1)$ for which the corresponding partial sums are not bounded in $L_p(G)$. However, Simon [11, Theorem 1] (see also [2], [4], [12]) proved that if $0 < p \leq 1$, then there is an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[p]} n}\sum_{k=1}^{n}\frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p(G)}^p, \quad (n = 2, 3, \ldots), \tag{1.1}$$

for all $f \in H_p(G)$, where $[p]$ denotes the integer part of p.

When $0 < p < 1$, Tephnadze [13, Theorem 1] proved that the sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ in (1.1) cannot be improved. In [15, Theorem 1] he proved that

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when $0 < p \leq 1$, the weighted maximal operator

$$\widetilde{S}_p^* f := \sup_{n\in\mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1}\log^{[p]}(n+1)}$$

is bounded from the Hardy space $H_p(G)$ to the space $L_p(G)$. Moreover, for any nondecreasing function $\varphi : \mathbb{N}_+ \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-1}\log^{[p]}(n+1)}{\varphi(n)} = +\infty,$$

there exists a martingale $f \in H_p(G), 0 < p \leq 1$, such that

$$\sup_{n\in\mathbb{N}} \Big\| \frac{S_n f}{\varphi(n)} \Big\|_p = \infty.$$

Applying the results of [15, Theorem 1], it was also proved that the following theorems are true (see Tephnadze [15, Theorems 2–4]).

Theorem T1. *Let $0 < p \leq 1$, $f \in H_p(G)$, and $2^k < n \leq 2^{k+1}$. Then there is an absolute constant c_p, depending only on p, such that*

$$\|S_n f - f\|_{H_p(G)} \leq c_p n^{1/p-1} \log^{[p]} n \omega_{H_p(G)}\Big(\frac{1}{2^k}, f\Big).$$

Theorem T2.

(a) *Let $0 < p \leq 1$, $f \in H_p(G)$, and let*

$$\omega_{H_p(G)}\Big(\frac{1}{2^n}, f\Big) = o\Big(\frac{1}{2^{n(1/p-1)} n^{[p]}}\Big), \quad \text{as } n \to \infty.$$

Then

$$\|S_k f - f\|_p \to 0, \quad \text{when } k \to \infty.$$

(b) *For every $p \in [0, 1]$ there exists a martingale $f \in H_p(G)$ for which*

$$\omega_{H_p(G)}\Big(\frac{1}{2^n}, f\Big) = O\Big(\frac{1}{2^{n(1/p-1)} n^{[p]}}\Big), \quad \text{as } n \to \infty$$

and

$$\|S_k f - f\|_p \nrightarrow 0, \quad \text{as } k \to \infty.$$

For the definition of the modulus of continuity w_{H_p} and other undefined notation in this Introduction, see Section 2.

For the 2-dimensional case, it is well known (see [6]) that the Walsh–Paley system is not a Schauder basis in $L_1(G^2)$. Moreover, there exists (for details, see [10]) a function in the dyadic martingale Hardy space $H_p(G^2)$ $(0 < p \leq 1)$ for which the corresponding partial sums are not bounded in $L_p(G^2)$. However, Weisz [18, Theorem 1] proved that if $\alpha \geq 0$ and $0 < p \leq 1$, then there exists an absolute constant c_p, depending only on p, such that

$$\sup_{n,m\geq 2} \Big(\frac{1}{\log n \log m}\Big)^{[p]} \sum_{2^{-\alpha}\leq k/l\leq 2^\alpha, (k,l)\leq(n,m)} \frac{\|S_{k,l} f\|_p^p}{(kl)^{2-p}} \leq c_p \|f\|_{H_p(G^2)}^p,$$

for all $f \in H_p(G^2)$, where $[p]$ denotes the integer part of p. Goginava and Gogoladze [5, Theorem 1] proved that for any $f \in H_1(G^2)$, there exists an absolute constant c such that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_1}{n\log^2 n} \le c\|f\|_{H_1(G^2)}, \tag{1.2}$$

for all $f \in H_1(G^2)$. Memić, Simon, and Tephnadze [8, Theorem 3.1] (see also [14]) considered the generalized estimate (1.2) and proved that for any $0 < p \le 1$ and $f \in H_p(G^2)$, there exists an absolute constant c_p, depending only on p, such that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_p^p}{n^{3-2p}\log^{2[p]} n} \le c_p\|f\|_{H_p(G^2)}^p, \tag{1.3}$$

for all $f \in H_p(G^2)$. The authors in [8] and [14] also proved that the sequence $\{1/(n^{3-2p}\log^{2[p]} n)\}_{n=1}^{\infty}$ in (1.3) cannot be improved.

In this article, we investigate the 2-dimensional analogies of Theorems T1 and T2 for $0 < p < 1$, and we find some conditions for the modulus of continuity that provide convergence of the partial sums $S_{k,l}$ with respect to the Walsh–Fourier system but in the case when indexes are restricted by the condition $2^{-\alpha} \le k/l \le 2^{\alpha}$. We also show that these conditions are in a sense necessary and sufficient.

The article is organized as follows. We present some definitions and notation in Section 2. Section 3 is reserved for some necessary lemmas, some of which are new and of independent interest. The main results are presented and proved in Section 4.

2. Definitions and notation

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2; that is, $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition; the measure (denoted by μ) and the topology are the product measure and product topology, respectively. The compact abelian group G is called the *Walsh group*. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &:= G,\\ I_n(x) &:= I_n(x_0, \dots, x_{n-1})\\ &:= \big\{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\big\}, \quad (x \in G, n \in \mathbb{N}). \end{aligned}$$

These sets are called the *dyadic intervals*. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of $G, I_n := I_n(0)$ $(n \in \mathbb{N})$. Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G$ the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$. Let $\overline{I_n} := G \backslash I_n$.

It is evident that

$$\overline{I_N}=\bigcup_{s=0}^{N-1} I_s\backslash I_{s+1}. \tag{2.1}$$

If $n\in\mathbb{N}$, then $n=\sum_{i=0}^{\infty}n_i2^i$, where $n_i\in\{0,1\}$ $(i\in\mathbb{N})$; that is, n is expressed in the base 2 number system. Denote $|n|:=\max\{j\in\mathbb{N}:n_j\neq 0\}$, that is, $2^{|n|}\leq n<2^{|n|+1}$. It is easy to show that for every odd number n, it holds that $n_0=1$, and we can write $n=1+\sum_{i=1}^{|n|}n_j2^i$, where $n_j\in\{0,1\}$, $j\in\mathbb{N}_+$.

For $k\in\mathbb{N}$ and $x\in G$, let us denote by

$$r_k(x):=(-1)^{x_k},\quad (x\in G,k\in\mathbb{N})$$

the kth Rademacher function. The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x):=\prod_{k=0}^{\infty}\left(r_k(x)\right)^{n_k}=r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1}n_kx_k}\quad (x\in G,n\in\mathbb{N}_+).$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1}w_k(x).$$

Recall that (for details, see, e.g., [6])

$$D_{2^n}(x)=\begin{cases}2^n & x\in I_n,\\ 0 & x\in\overline{I}_n.\end{cases} \tag{2.2}$$

Let $n\in\mathbb{N}$ and $n=\sum_{i=0}^{\infty}n_i2^i$. Then

$$D_n(x)=w_n(x)\sum_{j=0}^{\infty}n_jw_{2^j}(x)D_{2^j}(x). \tag{2.3}$$

Set $G^2:=G\times G$. The norm (or quasinorm) of the space $L_p(G^2)$ is defined by

$$\|f\|_p:=\left(\int_{G^2}|f|^p\,d\mu\right)^{1/p}\quad (0<p<\infty).$$

The space weak-$L_p(G^2)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_p}:=\sup_{\lambda>0}\lambda\mu(f>\lambda)^{1/p}<+\infty.$$

The rectangular partial sums of the 2-dimensional Walsh–Fourier series of the function $f\in L_2(G^2)$ are defined as

$$S_{M,N}f(x,y):=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\widehat{f}(i,j)w_i(x)w_j(y),$$

where the number

$$\widehat{f}(i,j)=\int_{G^2}f(x,y)w_i(x)w_j(y)\,d\mu(x,y)$$

is said to be the $(i,j)th$ *Walsh–Fourier coefficient* of the function f. It is well known (for details, see, e.g., [10]) that

$$S_{M,N}f(x,y)=\int_{G^2} f(x,y)D_M(x-t)D_N(y-s)\,d\mu(x,y).$$

We also consider the maximal operator $\widetilde{S}^{*,p}$ defined by

$$\widetilde{S}^{*,p}f=\sup_{m,n\geq 1}\frac{|S_{m,n}f|}{(m+n)^{2/p-2}}. \tag{2.4}$$

The σ-algebra generated by the dyadic 2-dimensional $I_n(x)\times I_n(y)$ square of measure $2^{-n}\times 2^{-n}$ will be denoted by F_n $(n\in\mathbb{N})$. Denote by $f=(f_n,n\in\mathbb{N})$ the 1-parameter martingale with respect to F_n $(n\in\mathbb{N})$ (for details, see, e.g., Weisz [16], [19], [20]; see also [7]). The maximal function f^* of a martingale f is defined by

$$f^*:=\sup_{n\in\mathbb{N}}|f_n|.$$

The dyadic maximal function f^* of $f\in L_1(G^2)$ is given by

$$f^*(x,y):=\sup_{n\in\mathbb{N}}\frac{1}{\mu(I_n(x)\times I_n(y))}\Big|\int_{I_n(x)\times I_n(y)} f(s,t)\,d\mu(s,t)\Big|,\quad (x,y)\in G^2.$$

If $f\in L_1(G^2)$, then it is easy to show that the sequence $(S_{2^n,2^n}f:n\in\mathbb{N})$ is a martingale and that its maximal function coincides with the dyadic maximal function of $f\in L_1(G^2)$. The dyadic Hardy martingale space $H_p(G^2)$ $(0<p<\infty)$ consists of all functions for which

$$\|f\|_{H_p(G^2)}:=\|f^*\|_p<\infty.$$

If $f=(f_n,n\in\mathbb{N})$ is a martingale, then the Walsh–Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i,j):=\lim_{k\to\infty}\int_G f_k(x,y)w_i(x)w_j(y)\,d\mu(x,y).$$

The Walsh–Fourier coefficients of $f\in L_1(G^2)$ are the same as those of the martingale $(S_{2^n,2^n}f:n\in\mathbb{N})$ obtained from f. We define the concept of the 2-dimensional modulus of continuity in $H_p(G^2)$ $(p>0)$ as follows:

$$\omega_{H_p(G^2)}\Big(\frac{1}{2^n},f\Big):=\|f-S_{2^n,2^n}f\|_{H_p(G^2)}.$$

The 1-dimensional modulus of continuity $w_{H_p(G)}$ is defined similarly (see, e.g., [3]). A bounded measurable function a is a p-atom if there exists a dyadic 2-dimensional square $I\times I$ such that

$$\int_{I\times I} a\,d\mu=0,\qquad \|a\|_\infty\leq\mu(I\times I)^{-1/p},\qquad \operatorname{supp}(a)\subset I\times I.$$

3. Lemmas

Weisz in [16, Theorem 2.2] and [19, Theorem 1.14] proved that dyadic Hardy martingale spaces $H_p(G^2)$ for $0 < p \leq 1$ have atomic characterizations (see also [7]).

Lemma 3.1. *A martingale $f = (f_n : n \in \mathbb{N})$ is in $H_p(G^2), 0 < p \leq 1$, if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n,2^n} a_k = f_n \tag{3.1}$$

and

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \backsim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (3.1).

Weisz [17, Theorem 1] (see also [16], [19]) also proved the following fact.

Lemma 3.2. *Suppose that an operator T is σ-sublinear and that, for some $0 < p \leq 1$,*

$$\int_{\overline{I \times I}} |Ta|^p \, d\mu \leq c_p < \infty,$$

for every p-atom a, where $I \times I$ denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_p \leq c_p \|f\|_{H_p(G^2)}.$$

In [15, Lemma 2] the following was proved.

Lemma 3.3. *Let $x \in I_s \backslash I_{s+1}$, $s = 0, \dots, N-1$. Then*

$$\int_{I_N} \big| D_n(x+t) \big| \, d\mu(t) \leq \frac{c2^s}{2^N},$$

where c is an absolute constant.

We also need the following estimates of the 2-dimensional Dirichlet kernels of independent interest.

Lemma 3.4. *Let $m, n \in \mathbb{N}$, and let $(x,y) \in I_N \times (I_s \backslash I_{s+1})$, $s = 0, \dots, N-1$. Then, for every $\varepsilon > 0$, we have*

$$\int_{I_N \times I_N} \big| D_m(x+t) D_n(y+s) \big| \, d\mu(t) \, d\mu(s) \leq \frac{cm^\varepsilon 2^s}{2^{N(1+\varepsilon)}},$$

where c is an absolute constant.

Proof. By combining (2.2) and (2.3), we can conclude that

$$|D_m| \leq m$$

and that

$$|D_m| \leq 2^s, \quad \text{for } I_s \backslash I_{s+1}.$$

Hence,

$$\begin{aligned}
&\int_{I_N} \left|D_m(x+t)\right| d\mu(t) \\
&\quad \leq m^{\varepsilon} \sum_{s=N}^{\infty} \int_{I_s \backslash I_{s+1}} \left|D_m(x+t)\right|^{1-\varepsilon} d\mu(t) \\
&\quad \leq cm^{\varepsilon} \sum_{s=N}^{\infty} \int_{I_s \backslash I_{s+1}} 2^{s(1-\varepsilon)} \, d\mu(t) \\
&\quad \leq cm^{\varepsilon} \sum_{s=N}^{\infty} 2^{-\varepsilon s} \leq \frac{cm^{\varepsilon}}{2^{\varepsilon N}}.
\end{aligned}$$

Therefore, by using Lemma 3.3, we obtain

$$\begin{aligned}
&\int_{I_N \times I_N} \left|D_m(x+t) D_n(y+s)\right| d\mu(t) \, d\mu(s) \\
&\leq \int_{I_N} \left|D_m(x+t)\right| d\mu(t) \int_{I_N} \left|D_n(y+s)\right| d\mu(s) \\
&\leq \frac{cm^{\varepsilon} 2^s}{2^{N(1+\varepsilon)}}.
\end{aligned}$$

Thus the proof is complete. □

Lemma 3.5. *Let $m, n \in \mathbb{N}$, and let $(x, y) \in (I_s \backslash I_{s+1}) \times I_N$, $s = 0, \ldots, N-1$. Then, for every $\varepsilon > 0$, we have*

$$\int_{I_N \times I_N} \left|D_m(x+t) D_n(y+s)\right| d\mu(t) \, d\mu(s) \leq \frac{cn^{\varepsilon} 2^s}{2^{N(1+\varepsilon)}},$$

where c is an absolute constant.

Proof. The proof is quite analogous to that of Lemma 3.4. Hence, we leave out the details. □

Lemma 3.6. *Let $m, n \in \mathbb{N}$, and let $(x, y) \in (I_{s_1} \backslash I_{s_1+1}) \times (I_{s_2} \backslash I_{s_2+1})$, $s_1, s_2 = 0, \ldots, N-1$. Then*

$$\int_{I_N} \left|D_n(x+t) D_m(y+s)\right| d\mu(t) \, d\mu(s) \leq \frac{c 2^{s_1+s_2}}{2^{2N}},$$

where c is an absolute constant.

Proof. By applying Lemma 3.3, we obtain

$$\int_{I_N\times I_N}\left|D_m(x+t)D_n(y+s)\right|d\mu(t)\,d\mu(s)$$
$$\leq\int_{I_N}\left|D_m(x+t)\right|d\mu(t)\int_{I_N}\left|D_n(y+s)\right|d\mu(s)\leq\frac{c2^{s_1+s_2}}{2^{2N}}.$$

Thus the proof is complete. □

4. The main results

Our main results read as follows.

Theorem 4.1.

(a) *Let $0<p<1$ and $f\in H_p(G^2)$. Then the maximal operator $\widetilde{S}^{*,p}$ defined by (2.4) is bounded from the martingale Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.*

(b) *(Sharpness). Let $\varphi:\mathbb{N}\to[1,\infty)$ be a nondecreasing function satisfying the condition*

$$\sup_{m,n\in\mathbb{N}}\frac{(m+n)^{2/p-2}}{\varphi(m,n)}=+\infty.$$

Then

$$\sup_{m,n\in\mathbb{N}}\left\|\frac{S_{m,n}f}{\varphi(m,n)}\right\|_{\text{weak-}L_p(G^2)}=\infty.$$

Theorem 4.2. *Let $0<p<1$, $2^{-\alpha}<m/n\leq 2^{\alpha}$, and $2^k<m,n\leq 2^{k+1+[\alpha]}$. Then there exists an absolute constant c_p such that*

$$\|S_{m,n}f-f\|_p\leq c_p2^{k(2/p-2)}\omega_{H_p(G^2)}\Big(\frac{1}{2^k},f\Big).$$

Theorem 4.3.

(a) *Let $0<p<1$, $2^{-\alpha}\leq m/n\leq 2^{\alpha}$, and*

$$\omega_{H_p(G^2)}\Big(\frac{1}{2^k},f\Big)=o\Big(\frac{1}{2^{k(2/p-2)}}\Big),\quad \text{as } k\to\infty.$$

Then

$$\|S_{m,n}f-f\|_{H_p(G^2)}\to 0,\quad \text{when } n\to\infty.$$

(b) *(Sharpness). Let $0<p<1$ and $2^{-\alpha}<m/n\leq 2^{\alpha}$. Then there exists a martingale $f\in H_p(G^2)$ such that*

$$\omega_{H_p(G^2)}\Big(\frac{1}{2^k},f\Big)=O\Big(\frac{1}{2^{k(2/p-2)}}\Big),\quad \text{as } k\to\infty$$

and

$$\|S_{m,n}f-f\|_{\text{weak-}L_p(G^2)}\nrightarrow 0,\quad \text{as } n\to\infty.$$

Proof of Theorem 4.1. Since $\widetilde{S}_p^*$ is bounded from L_∞ to L_∞, by using Lemma 3.2 we conclude that the proof of part (a) will be complete if we show that

$$\int_{\overline{I\times I}} \left|\widetilde{S}_p^* a(x,y)\right|^p d\mu(x)\, d\mu(y) \le c < \infty, \quad \text{when } 0<p<1, \tag{4.1}$$

for every p-atom a, where $I\times I$ denotes the support of the atom.

Let a be an arbitrary p-atom with support $I\times I$ and $\mu(I\times I)=2^{-2N}$. We may assume that $I\times I = I_N\times I_N$, where $I_N := I_N(0)$. It is easy to see that $S_{m,n}a=0$ when $m\le 2^N$ and $n\le 2^N$. Therefore, we can suppose that $m>2^N$ or that $n>2^N$. Since $\|a\|_\infty \le 2^{2N/p}$, we find that

$$\begin{aligned}
\left|S_{m,n}(a)\right| &\le \int_{I_N\times I_N} \left|a(t_1,t_2)\right|\left|D_{m,n}(x+t_1,y+t_2)\right| d\mu(t_1)\, d\mu(t_2) \\
&\le \|a\|_\infty \int_{I_N\times I_N} \left|D_{m,n}(x+t_1,y+t_2)\right| d\mu(t_1)\, d\mu(t_2) \\
&\le 2^{2N/p} \int_{I_N\times I_N} \left|D_{m,n}(x+t_1,y+t_2)\right| d\mu(t_1)\, d\mu(t_2).
\end{aligned} \tag{4.2}$$

Let $0<p<1$ and $(x,y)\in I_N\times(I_{s_2}\backslash I_{s_2+1})$. We choose ε, so that $2/p-2-\varepsilon>0$ and then from Lemma 3.4 it follows that

$$\begin{aligned}
\frac{|S_{m,n}a(x,y)|}{(m+n+1)^{2/p-2}} &\le \frac{2^{2N/p}2^{s_2}m^\varepsilon}{(m+n+1)^{2/p-2}2^{N(\varepsilon+1)}} \\
&\le \frac{2^{2N/p}2^{s_2}(m+n)^\varepsilon}{(m+n+1)^{2/p-2}2^{N(\varepsilon+1)}} \\
&\le \frac{2^{N(2/p-2-\varepsilon)}2^{s_2}2^N}{(m+n+1)^{2/p-2-\varepsilon}} \le 2^{s_2}2^N.
\end{aligned} \tag{4.3}$$

According to (2.1) and (4.3), we have

$$\begin{aligned}
&\int_{I_N\times\overline{I_N}} \left|\widetilde{S}_p^* a(x,y)\right|^p d\mu(x)\, d\mu(y) \\
&\quad = \sum_{s_2=0}^{N-1} \int_{I_N\times(I_{s_2}\backslash I_{s_2+1})} \left|\widetilde{S}_p^* a(x,y)\right|^p d\mu(x)\, d\mu(y) \\
&\quad \le \sum_{s_2=0} \frac{2^{ps_2}}{2^{s_2}} < c_p < \infty.
\end{aligned} \tag{4.4}$$

If we apply (2.1), (4.2), and Lemma 3.5 analogously to (4.4), we get

$$\begin{aligned}
&\int_{\overline{I_N}\times I_N} \left|\widetilde{S}_p^* a(x,y)\right|^p d\mu(x)\, d\mu(y) \\
&\quad = \sum_{s_1=0}^{N-1} \int_{(I_{s_1}\backslash I_{s_1+1})\times I_N} \left|\widetilde{S}_p^* a(x,y)\right|^p d\mu(x)\, d\mu(y) \\
&\quad \le \sum_{s_1=0} \frac{2^{ps_1}}{2^{s_1}} < c_p < \infty.
\end{aligned} \tag{4.5}$$

Let $0 < p < 1$, and let $(x,y) \in (I_{s_1}\backslash I_{s_1+1}) \times (I_{s_2}\backslash I_{s_2+1})$. By using Lemma 3.6 now, we get

$$\frac{|S_{m,n}a(x,y)|}{(m+n+1)^{1/p-1}} \leq \frac{2^{2N(1/p-1)}2^{s_1+s_2}}{(m+n+1)^{1/p-1}} \leq 2^{s_1+s_2}. \tag{4.6}$$

In view of (2.1) and (4.6), we can conclude that

$$\begin{aligned}\int_{\overline{I_N}\times\overline{I_N}} &\left|\widetilde{S}_p^* a(x,y)\right|^p d\mu(x)\, d\mu(y) \\ &= \sum_{s_1=0}^{N-1}\sum_{s_2=0}^{N-1}\int_{(I_{s_1}\backslash I_{s_1+1})\times(I_{s_2}\backslash I_{s_2+1})}\left|\widetilde{S}_p^* a(x,y)\right|^p d\mu(x)\, d\mu(y) \\ &\leq \sum_{s_1=0}^{N-1}\sum_{s_2=0}^{N-1}\frac{2^{(s_1+s_2)p}}{2^{s_1+s_2}} < c_p < \infty. \end{aligned}\tag{4.7}$$

Since

$$\overline{I_N \times I_N} = (I_N \times \overline{I_N}) \cup (\overline{I_N} \times I_N) \cup (\overline{I_N} \times \overline{I_N}),$$

by combining (4.4), (4.5), and (4.7), we get that (4.1) holds for every p-atom, and the proof of part (a) is complete.

Now, we prove the second part of the theorem. Let $\varphi : \mathbb{N} \to [1, \infty)$ be a nondecreasing function, and let $\{\alpha_k : k \in \mathbb{N}\}$ be a sequence of natural numbers satisfying the condition

$$\lim_{k\to\infty}\frac{(2^{\alpha_k}+1)^{2/p-2}}{\varphi(2^{\alpha_k}+1,1)} = +\infty.$$

Set, for $k \in \mathbb{N}_+$,

$$f_k(x,y) = \big(D_{2^{\alpha_k+1}}(x) - D_{2^{\alpha_k}}(x)\big)D_{2^{\alpha_k}}(y).$$

It is evident that

$$\widehat{f}_k(i,j) = \begin{cases} 1 & \text{if } (i,j) \in \{2^{\alpha_k},\ldots,2^{\alpha_k+1}-1\} \times \{1,\ldots,2^{\alpha_k}-1\}, \\ 0 & \text{otherwise.}\end{cases}$$

Therefore,

$$\begin{aligned} &S_{i,j}(f_k;x,y) \\ &= \begin{cases} (D_i(x) - D_{2^{\alpha_k}}(x))D_j(y) \\ \qquad \text{if } (i,j) \in \{2^{\alpha_k},\ldots,2^{\alpha_k+1}-1\} \times \{1,\ldots,2^{\alpha_k}-1\}, \\ f_k(x,y) \\ \qquad \text{if } i \geq 2^{\alpha_k+1}, \text{ and } j \geq 2^{\alpha_k}, \\ 0 \quad \text{otherwise.}\end{cases}\end{aligned}\tag{4.8}$$

From (4.8) it follows that

$$\begin{aligned}\|f_k\|_{H_p} &= \Big\|\sup_{n\in\mathbb{N}} S_{2^n,2^n} f_k\Big\|_p \\ &= \Big\|\big(D_{2^{\alpha_k+1}}(x) - D_{2^{\alpha_k}}(x)\big)D_{2^{\alpha_k}}(y)\Big\|_p \\ &\leq \Big\|D_{2^{\alpha_k}}(x)D_{2^{\alpha_k}}(y)\Big\|_p \leq 2^{2\alpha_k(1-1/p)}.\end{aligned}\tag{4.9}$$

Let $(x, y) \in G^2$. Moreover, (4.8) also implies that

$$\frac{|S_{2^{\alpha_k}+1,1}(f_k; x, y)|}{\varphi(2^{\alpha_k}+1, 1)} = \frac{|(D_{2^{\alpha_k}+1}(x) - D_{2^{\alpha_k}}(x))D_1(y)|}{\varphi(2^{\alpha_k}+1, 1)}$$
$$= \frac{|w_{2^{\alpha_k}}(x)w_0(y)|}{\varphi(2^{\alpha_k}+1, 1)} = \frac{1}{\varphi(2^{\alpha_k}+1, 1)}.$$

Hence, by also using (4.9), we find that

$$\frac{\frac{1}{\varphi(2^{\alpha_k}+1,1)}(\mu\{(x, y) \in G^2 : \frac{S_{2^{\alpha_k}+1,1}(f_k;x,y)}{\varphi(2^{\alpha_k}+1,1)} \geq \frac{1}{\varphi(2^{\alpha_k}+1,1)}\})^{1/p}}{\|f_k\|_{H_p}}$$
$$\geq \frac{1}{\varphi(2^{\alpha_k}+1, 1)2^{2\alpha_k(1-1/p)}} \geq \frac{(2^{\alpha_k}+1)^{2/p-2}}{\varphi(2^{\alpha_k}+1, 1)} \to \infty, \quad \text{as } k \to \infty.$$

The proof is complete. □

Proof of Theorem 4.2. Let $0 < p < 1$, let $2^{-\alpha} \leq m/n \leq 2^{\alpha}$, and let $2^k < m, n \leq 2^{k+1+[\alpha]}$. According to Theorem 4.1, we can conclude that

$$\|S_{m,n}f\|_p \leq c_p^1(m+n)^{2/p-2}\|f\|_{H_p(G^2)} \leq c_p^2 2^{k(2/p-2)}\|f\|_{H_p(G^2)}.$$

Hence,

$$\begin{aligned}
&\|S_{m,n}f - f\|_p^p \\
&\quad\leq \|S_{m,n}f - S_{2^k,2^k}f\|_p^p + \|S_{2^k,2^k}f - f\|_p^p \\
&\quad= \big\|S_{m,n}(S_{2^k,2^k}f - f)\big\|_p^p + \|S_{2^k,2^k}f - f\|_p^p \\
&\quad\leq c_p^2(2^{k(2-2p)} + 1)\omega^p_{H_p(G^2)}\Big(\frac{1}{2^k}, f\Big)
\end{aligned}$$

and

$$\|S_{m,n}f - f\|_p \leq c_p 2^{k(2/p-2)}\omega_{H_p(G^2)}\Big(\frac{1}{2^k}, f\Big). \tag{4.10}$$

The proof is complete. □

Proof of Theorem 4.3. Let $0 < p < 1$, $f \in H_p(G^2)$, $2^{-\alpha} \leq m/n \leq 2^{\alpha}$, $2^k < m, n \leq 2^{k+1+[\alpha]}$, and

$$\omega_{H_p(G^2)}\Big(\frac{1}{2^k}, f\Big) = o\Big(\frac{1}{2^{k(2/p-2)}}\Big) \quad \text{as } k \to \infty.$$

By using (4.10), we immediately get that

$$\|S_{m,n}f - f\|_p \to \infty \quad \text{when } n \to \infty,$$

and the proof of part (a) is complete.

To prove part (b) of the theorem, we use a similar construction of martingale, which was used in [9]. Let

$$f_n = \sum_{\{k;\alpha_k+1<n\}} \lambda_k a_k,$$

where

$$\lambda_k = 2^{-\alpha_k(2/p-2)}$$

and

$$a_k(x,y)=2^{\alpha_k(2/p-2)}\big(D_{2^{\alpha_k+1}}(x)-D_{2^{\alpha_k}}(x)\big)\big(D_{2^{\alpha_k+1}}(y)-D_{2^{\alpha_k}}(y)\big).$$

Since

$$S_{2^n,2^n}a_k=\begin{cases}a_k & \alpha_k+1<n,\\ 0 & \alpha_k+1\geq n,\end{cases}$$

$$\operatorname{supp}(a_k)=I^2_{\alpha_k},\qquad \int_{I^2_{\alpha_k}} a_k\,d\mu=0,\qquad \|a_k\|_\infty\leq 2^{2\alpha_k/p}=(\operatorname{supp} a_k)^{-1/p}$$

from Lemma 3.1 and given the fact that

$$\sum_{k=0}^{\infty}|\mu_k|^p<\infty,$$

we conclude that $f\in H_p(G^2)$.

Moreover,

$$\begin{aligned}
&f-S_{2^n,2^n}f\\
&\quad=(f_1-S_{2^n,2^n}f_1,\dots,f_n-S_{2^n,2^n}f_n,\dots,f_{n+k}-S_{2^n,2^n}f_{n+k})\\
&\quad=(0,\dots,0,f_{n+1}-f_n,\dots,f_{n+k}-f_n,\dots)\\
&\quad=\Big(0,\dots,0,\sum_{i=n}^{n+k}\frac{a_i(x,y)}{2^{i(2/p-2)}},\dots\Big),\qquad k\in\mathbb{N}_+,
\end{aligned}\tag{4.11}$$

is a martingale and (4.11) is its atomic decomposition. By using Lemma 3.1, we find that

$$\omega_{H_p}\Big(\frac{1}{2^n},f\Big):=\|f-S_{2^n,2^n}f\|_{H_p}\leq\sum_{i=n}^{\infty}\frac{1}{2^{i(2/p-2)}}\leq\frac{c}{2^{n(2/p-2)}}.$$

Moreover, it is easy to show that

$$\widehat{f}(i,j)=\begin{cases}1 & \text{if } (i,j)\in\{2^{\alpha_k},\dots,2^{\alpha_k+1}-1\}\times\{2^{\alpha_k},\dots,2^{\alpha_k+1}-1\},\\ & k\in\mathbb{N},\\ 0 & \text{if } (i,j)\notin\bigcup_{k=1}^{\infty}\{2^{\alpha_k},\dots,2^{\alpha_k+1}-1\}\times\{2^{\alpha_k},\dots,2^{\alpha_k+1}-1\}.\end{cases}\tag{4.12}$$

In view of (4.12), we can conclude that

$$S_{2^{\alpha_k+1},2^{\alpha_k+1}}f(x,y)=S_{2^{\alpha_k},2^{\alpha_k}}f(x,y)+w_{2^{\alpha_k}}(x)w_{2^{\alpha_k}}(y)=:I+II.\tag{4.13}$$

It is obvious that

$$|II|=\big|w_{2^{2\alpha_k}}(x)w_{2^{\alpha_k}}(y)\big|=1.$$

Hence,

$$\begin{aligned}
&\|II\|^p_{\text{weak-}L_p(G^2)}\\
&\quad\geq\frac{1}{2^p}\Big(\mu\Big\{(x,y)\in G\times G:|II|\geq\frac{1}{2}\Big\}\Big)\\
&\quad\geq\frac{1}{2^p}\mu(G\times G)\geq\frac{1}{2^p}.
\end{aligned}\tag{4.14}$$

Since (for details, see, e.g., Weisz [16], [19])

$$\|f - S_{2^n,2^n} f\|_{\text{weak-}L_p(G^2)} \to 0, \quad \text{as } n \to \infty,$$

according to (4.13) and (4.14), we get

$$\begin{aligned}\limsup_{k\to\infty} &\|f - S_{2^{\alpha_k+1},2^{\alpha_k+1}} f\|^p_{\text{weak-}L_p(G^2)} \\ &\geq \limsup_{k\to\infty} \|II\|^p_{\text{weak-}L_p(G^2)} \\ &\quad - \limsup_{k\to\infty} \|f - S_{2^{\alpha_k},2^{\alpha_k}} f\|^p_{\text{weak-}L_p(G^2)} \geq c > 0.\end{aligned}$$

The proof is complete. □

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