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CONVERGENCE PROPERTIES OF NETS OF OPERATORS

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Abstract. We consider nets (T_j) of operators acting on complex functions, and we investigate the algebraic and the topological structure of the set $\{f : T_j(|f|^2) - |T_j f|^2 \to 0\}$. Our results extend and improve some known results from the literature, which are connected with Korovkin's theorem. Applications to Abel–Poisson-type operators and Bernstein-type operators are given.

1. Introduction and main result

Let X be a nonempty set, and let $B(X)$ be the algebra of all complex-valued bounded functions on X equipped with the supremum norm. Let $A(X)$ be a subalgebra of $B(X)$ closed under complex conjugation. Suppose that the constant function 1 is in $A(X)$.

A linear operator $T : A(X) \to B(X)$ is called *positive* if Tf is real-valued and *nonnegative* whenever $f \in A(X)$ is real-valued and nonnegative. The main result of this paper is the following.

Theorem 1.1. *Let $T_j : A(X) \to B(X)$ be a net of positive linear operators such that $T_j 1 = 1$ for all j. Let $E(X) := \{f \in A(X) : T_j(|f|^2) - |T_j f|^2 \to 0\}$. Then*

(a) *$E(X)$ is closed under complex conjugation, and $1 \in E(X)$,*
(b) *$E(X)$ is a closed subalgebra of $A(X)$.*

This result is related to, and motivated by, the results obtained in [2], [1], [4], and in [8]–[12] in connection with Korovkin's theorem. In particular, suppose

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that X is a compact Hausdorff space and that $A(X) = C(X)$ is the algebra of all continuous complex-valued functions defined on X. Suppose also that $E(X)$ separates the points of X; that is, for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $f \in E(X)$ such that $f(x_1) \neq f(x_2)$. Then from Theorem 1.1 and the Stone–Weierstrass theorem, we conclude that $E(X) = A(X) = C(X)$.

Details and applications will be presented in Section 3 (see Theorems 3.1, 3.2, and Corollary 3.3). In Examples 3.4 and 3.5 we construct nets of operators (of Abel–Poisson type and of Bernstein type, respectively) for which our general results can be applied.

2. Proof of Theorem 1.1

Let $T : A(X) \to B(X)$ be a positive linear operator such that $T1 = 1$, and let $f, g \in A(X)$. The following relations are well known (see, e.g., [12]):

$$\left|T(f\overline{g})\right|^2 \le T\big(|f|^2\big)T\big(|g|^2\big), \quad \text{(Cauchy–Schwarz)} \tag{2.1}$$

$$\operatorname{Re}(Tf) = T(\operatorname{Re} f); \qquad \operatorname{Im}(Tf) = T(\operatorname{Im} f), \tag{2.2}$$

$$|Tf|^2 \le T\big(|f|^2\big), \tag{2.3}$$

$$\overline{Tf} = T(\overline{f}), \tag{2.4}$$

$$\|Tf\| \le \|f\|. \tag{2.5}$$

We must also refer to the following proposition.

Proposition 2.1. *With the above notation,*

$$\left|T(\overline{f}g) - T(\overline{f})Tg\right|^2 \le \big(T\big(|f|^2\big) - |Tf|^2\big)\big(T\big(|g|^2\big) - |Tg|^2\big). \tag{2.6}$$

Proof. From (2.3) we get

$$\left|T(f + ag)\right|^2 \le T\big(|f + ag|^2\big), \quad a \in \mathbb{C}.$$

This implies, according to (2.4), that

$$(Tf + aTg)\big(T(\overline{f}) + \overline{a}T(\overline{g})\big) \le T\big((f + ag)(\overline{f} + \overline{a}\,\overline{g})\big),$$

and also that

$$\begin{aligned} &|Tf|^2 + aTgT(\overline{f}) + \overline{a}TfT(\overline{g}) + |a|^2|Tg|^2 \\ &\quad \le T\big(|f|^2\big) + aT(g\overline{f}) + \overline{a}T(f\overline{g}) + |a|^2T\big(|g|^2\big). \end{aligned} \tag{2.7}$$

Set $a = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, and

$$\begin{aligned} T\big(|g|^2\big) - |Tg|^2 &= \varphi_1, \\ T(\overline{f}g) - T(\overline{f})Tg &= \varphi_2 + i\psi_2, \\ T\big(|f|^2\big) - |Tf|^2 &= \varphi_3, \end{aligned}$$

where φ_1, φ_2, φ_3, ψ_2 are real functions; according to (2.3), $\varphi_1 \ge 0$, and $\varphi_3 \ge 0$. From (2.7) we deduce that

$$(\alpha^2 + \beta^2)\varphi_1 + (\alpha + i\beta)(\varphi_2 + i\psi_2) + (\alpha - i\beta)(\varphi_2 - i\psi_2) + \varphi_3 \ge 0,$$

for all $\alpha, \beta \in \mathbb{R}$.

This leads to

$$(\alpha^2+\beta^2)\varphi_1 + 2\alpha\varphi_2 - 2\beta\psi_2 + \varphi_3 \geq 0, \quad \alpha,\beta \in \mathbb{R}. \tag{2.8}$$

We will prove that

$$\varphi_2^2 + \psi_2^2 \leq \varphi_1\varphi_3. \tag{2.9}$$

Indeed, let $x \in X$; then $\varphi_1(x) \geq 0$. If $\varphi_1(x) = 0$, then (2.8) yields

$$2\alpha\varphi_2(x) - 2\beta\psi_2(x) + \varphi_3(x) = 0$$

for all $\alpha, \beta \in \mathbb{R}$; hence $\varphi_2(x) = \psi_2(x) = 0$, and (2.9) is proved.

If $\varphi_1(x) > 0$, we may take in (2.8) $\alpha = \frac{-\varphi_2(x)}{\varphi_1(x)}$ and $\beta = \frac{\psi_2(x)}{\varphi_1(x)}$; this yields (2.9). So (2.9) is proved, and it implies (2.6). □

Now we are in a position to prove Theorem 1.1. Item (a) is obvious. Let $f, g \in E(X)$. Clearly, $af \in E(X)$ for each $a \in \mathbb{C}$. Moreover,

$$\begin{aligned} T_j\big(|f+g|^2\big) - \big|T_j(f+g)\big|^2 &= T_j\big((f+g)(\overline{f}+\overline{g})\big) - (T_jf + T_jg)(\overline{T_jf} + \overline{T_jg}) \\ &= \big(T_j\big(|f|^2\big) - |T_jf|^2\big) + \big(T_j\big(|g|^2\big) - |T_jg|^2\big) + \big(T_j(f\overline{g}) - T_jfT_j(\overline{g})\big) \\ &\quad + \big(T_j(\overline{f}g) - T_j(\overline{f})T_jg\big). \end{aligned}$$

The first two terms tend to 0 since $f, g \in E(X)$. By using Proposition 2.1, we infer that the last two terms tend also to 0; hence $f+g \in E(X)$. This shows that $E(X)$ is a linear subspace of $A(X)$.

Now let $u := f^2\overline{f}$, $v := f\overline{f}$. Then by (2.6),

$$\begin{aligned} &\big|T_j\big(|f\overline{f}|^2\big) - \big|T_j(f\overline{f})\big|^2\big| \\ &\quad \leq \big|T_j(\overline{f}u) - T_j(\overline{f})T_j(u)\big| \\ &\qquad + \big|T_j(\overline{f})T_j(f\overline{v}) - T_j(\overline{f})T_j(f)T_j(\overline{v})\big| + \big|T_j(\overline{f})T_j(f)T_j(\overline{f}f) - T_j(f\overline{f})T_j(f\overline{f})\big| \\ &\quad \leq \big(T_j\big(|f|^2\big) - |T_jf|^2\big)^{\frac{1}{2}}\big(T_j\big(|u|^2\big) - |T_ju|^2\big)^{\frac{1}{2}} + \big|T_j(\overline{f})\big|\big(T_j\big(|f|^2\big) - |T_jf|^2\big)^{\frac{1}{2}} \\ &\qquad \times \big(T_j\big(|v|^2\big) - |T_jv|^2\big)^{\frac{1}{2}} + \big|T_j(f\overline{f})\big|\big|T_j\big(|f|^2\big) - |T_jf|^2\big|. \end{aligned}$$

By using (2.5) and the fact that

$$T_j\big(|f|^2\big) - |T_jf|^2 \to 0,$$

we conclude that

$$T_j\big(|f\overline{f}|^2\big) - \big|T_j(f\overline{f})\big|^2 \to 0;$$

that is, $f\overline{f} \in E(X)$ for each $f \in E(X)$. For $f, g \in E(X)$ we have

$$\overline{f} + ig \in E(X), \qquad \overline{f} + g \in E(X).$$

It follows that

$$\begin{aligned} f\overline{f} + g\overline{g} + i(fg - \overline{f}\overline{g}) &= (\overline{f} + ig)(\overline{\overline{f} + ig}) \in E(X), \\ f\overline{f} + g\overline{g} + fg + \overline{f}\overline{g} &= (\overline{f} + g)(\overline{\overline{f} + g}) \in E(X). \end{aligned}$$

Since $f\overline{f} + g\overline{g} \in E(X)$, we deduce that $fg - \overline{f}\overline{g} \in E(X)$ and that $fg + \overline{f}\overline{g} \in E(X)$. Thus $fg \in E(X)$, which shows that $E(X)$ is a subalgebra of $A(X)$.

Finally, let $f_n \in E(X)$, $f_n \to f \in A(X)$, and let $\epsilon > 0$. There exists n_1 such that

$$\big\| |f|^2 - |f_n|^2 \big\| \leq \frac{\epsilon}{3}$$

for all $n \geq n_1$. This entails that

$$\big\| T_j\big(|f|^2\big) - T_j\big(|f_n|^2\big) \big\| \leq \frac{\epsilon}{3} \quad \text{for } n \geq n_1 \text{ and all } j. \tag{2.10}$$

On the other hand,

$$\begin{aligned} \big| |T_j f_n|^2 - |T_j f|^2 \big| &= \big| |T_j f_n| - |T_j f| \big| \cdot \big| |T_j f_n| + |T_j f| \big| \\ &\leq |T_j f_n - T_j f| \big(|T_j f_n| + |T_j f| \big). \end{aligned}$$

Since $f_n \to f \in B(X)$, there exists $M > 0$ such that $\|f_n\| \leq M$, $n \geq 1$. Then $\| |T_j f_n| + |T_j f| \| \leq 2M$ for all n and all j. Moreover, there exists n_2 such that

$$\|T_j f_n - T_j f\| \leq \|f_n - f\| \leq \frac{\epsilon}{6M}$$

for all j and all $n \geq n_2$. Hence

$$\big\| |T_j f_n|^2 - |T_j f|^2 \big\| \leq \frac{\epsilon}{3} \tag{2.11}$$

for all j and all $n \geq n_2$.

Let $n_0 = \max\{n_1, n_2\}$. There exists j_0 such that

$$\big\| T_j\big(|f_{n_0}|^2\big) - |T_j f_{n_0}|^2 \big\| \leq \frac{\epsilon}{3}, \quad j \geq j_0. \tag{2.12}$$

From (2.10), (2.11), and (2.12) it follows that

$$\begin{aligned} \big\| T_j\big(|f|^2\big) - |T_j f|^2 \big\| &\leq \big\| T_j\big(|f|^2\big) - T_j\big(|f_{n_0}|^2\big) \big\| + \big\| T_j\big(|f_{n_0}|^2\big) - |T_j f_{n_0}|^2 \big\| \\ &\quad + \big\| |T_j f_{n_0}|^2 - |T_j f|^2 \big\| \leq \epsilon \end{aligned}$$

for all $j \geq j_0$. Thus $T_j(|f|^2) - |T_j f|^2 \to 0$; that is, $f \in E(X)$. This shows that $E(X)$ is closed and that the proof of Theorem 1.1 is finished. □

3. Remarks, applications, and examples

(I) Let $h \in A(X)$, and let $g \in E(X)$. By applying (2.6) to g and to $f = \overline{h}$, we get

$$\begin{aligned} \big| T_j(hg) - T_j(h)T_j(g) \big|^2 &\leq \big(T_j\big(|h|^2\big) - |T_j h|^2 \big)\big(T_j\big(|g|^2\big) - |T_j g|^2 \big) \\ &\leq 2\|h\|^2 \big(T_j\big(|g|^2\big) - |T_j g|^2 \big) \to 0. \end{aligned}$$

It follows that

$$T_j(hg) - T_j(h)T_j(g) \to 0$$

uniformly on X for all $h \in A(X)$, $g \in E(X)$. This is a kind of asymptotic multiplicativity of the net (T_j). The degree of nonmultiplicativity of linear operators is investigated in [7] and the references therein.

(II) Let $C[a, b]$ be the algebra of real-valued continuous functions defined on $[a, b]$, endowed with the supremum norm and usual ordering. As a consequence of the results of [8], the following result was presented in [10, Théorème 1].

Theorem 3.1. *Let* $A_n : C[a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ *be positive linear functionals* $A_n(1) \leq 1$, *and let* $x_0 \in [a,b]$. *Then*

$$S := \{f \in C[a,b] : A_n(f) \to f(x_0), A_n(f^2) \to f^2(x_0)\}$$

is a closed subalgebra of $C[a,b]$.

The next result was obtained in [11, Theorem 2].

Theorem 3.2. *Let* $A_n : C[a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$ *be positive linear functionals* $A_n(1) \leq 1$. *Then*

$$\sum := \{f \in C[a,b] : A_n(f^2) - A_n^2(f) \to 0\}$$

is a closed subalgebra of $C[a,b]$, *and* $S \subset \sum$.

Obviously, Theorem 1.1 is an extension of Theorem 3.2. At the same time, Theorem 1.1 extends the results of [9]. (This area of research is clearly related to Korovkin's theory, see, e.g. [3]; it also relates to the results mentioned above, see [2], [1], [4]–[6], [8], [12].)

(III) We conclude by presenting some applications and examples.

Let $C_{2\pi}([-\pi,\pi],\mathbb{C})$ be the algebra of complex-valued, continuous, and 2π-periodic functions defined on $[-\pi,\pi]$. Then $g : [-\pi,\pi] \to \mathbb{C}$, $g(t) = e^{it}$ is in $C_{2\pi}([-\pi,\pi],\mathbb{C})$.

Let

$$T_j : C_{2\pi}\big([-\pi,\pi],\mathbb{C}\big) \to C_{2\pi}\big([-\pi,\pi],\mathbb{C}\big)$$

be a net of positive linear operators such that $T_j 1 = 1$.

Corollary 3.3. *If* $|T_j g| \to 1$, *then*

$$T_j\big(|f|^2\big) - |T_j f|^2 \to 0 \quad \textit{for all } f \in C_{2\pi}\big([-\pi,\pi],\mathbb{C}\big).$$

Proof. According to Theorem 1.1, the set

$$V := \{f \in C_{2\pi}\big([-\pi,\pi],\mathbb{C}\big) : T_j\big(|f|^2\big) - |T_j f|^2 \to 0\}$$

is a subalgebra of $C_{2\pi}([-\pi,\pi],\mathbb{C})$, closed under complex conjugation, closed under uniform convergence, and $1 \in V$. We also have

$$T_j\big(|g|^2\big) - |T_j g|^2 = T_j 1 - |T_j g|^2 = 1 - |T_j g|^2 \to 0$$

so that $g \in V$. It follows that the trigonometric polynomials are in V. We infer that

$$V = C_{2\pi}\big([-\pi,\pi],\mathbb{C}\big),$$

and this concludes the proof. ☐

Example 3.4. We construct a net of operators satisfying the hypotheses of Corollary 3.3. Let

$$v_j : [-\pi,\pi] \to [-\pi,\pi], \quad j \in (0,1)$$

be continuous functions.

For $f \in C_{2\pi}([-\pi,\pi],\mathbb{C})$, let

$$T_j(f)(x) := \frac{1-j^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(v_j(x)-t)}{1-2j\cos t+j^2}\,dt, \quad x\in[-\pi,\pi], j\in(0,1).$$

Then T_j are positive linear operators, $T_j 1 = 1$,

$$T_j g(x) = j\big(\cos v_j(x) + i\sin v_j(x)\big), \quad j\in(0,1).$$

Thus $|T_j g| = j$, and $\lim_{j\to 1}|T_j g| = 1$.

According to Corollary 3.3,

$$\lim_{j\to 1}\big(T_j\big(|f|^2\big) - |T_j f|^2\big) = 0, \quad f\in C_{2\pi}\big([-\pi,\pi],\mathbb{C}\big).$$

In particular, if $\lim_{j\to 1} v_j(x) = x$ uniformly on $[-\pi,\pi]$, then $T_j\cos\to\cos$, $T_j\sin\to\sin$, and Korovkin's theorem guarantees that $T_j f\to f$, $f\in C_{2\pi}([-\pi,\pi],\mathbb{C})$ (see [3, Section 5.4.8]).

Example 3.5. Let $u_n : [0,1]\to[0,1]$, $n = 1,2,\ldots$, be continuous functions. For $f\in C([0,1],\mathbb{C})$ let

$$T_n f(x) = \sum_{k=0}^{n}\binom{n}{k}\big(u_n(x)\big)^k\big(1-u_n(x)\big)^{n-k} f\Big(\frac{k}{n}\Big), \quad x\in[0,1].$$

Then T_n are positive linear operators, $T_n 1 = 1$.

Let $g(x) := x$, $x\in[0,1]$. Then

$$T_n g(x) = u_n(x), \qquad T_n(g^2)(x) = u_n^2(x) + \frac{u_n(x)(1-u_n(x))}{n}.$$

Thus

$$T_n\big(|g|^2\big) - |T_n g|^2 = \frac{u_n(1-u_n)}{n}\to 0.$$

According to Theorem 1.1,

$$V := \big\{f\in C\big([0,1],\mathbb{C}\big) : T_n\big(|f|^2\big) - |T_n f|^2\to 0\big\}$$

is a subalgebra of $C([0,1],\mathbb{C})$, closed under complex conjugation, closed under uniform convergence, and containing the functions 1 and g.

We deduce that it contains all the algebraic polynomials, and so $V = C([0,1],\mathbb{C})$. This entails that

$$T_n\big(|f|^2\big) - |T_n f|^2\to 0, \quad f\in C\big([0,1],\mathbb{C}\big).$$

Of course T_n are Bernstein-type operators. Similar q-Bernstein-type operators can be constructed, but we omit the details.

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References

1. F. Altomare, "Frontieres abstraites et convergence de familles filtrées de formes linéaires sur les algébres de Banach commutatives" in *Initiation Seminar on Analysis*, Ann. I.S.U.P. **46**, Univ. Paris VI, Paris, 1981. Zbl 0514.46032. MR0670796. 1, 5
2. F. Altomare, "Quelques remarques sur les ensembles de Korovkin dans les espaces des fonctions continues complexes" in *Initiation Seminar on Analysis*, Ann. I.S.U.P. **46**, Univ. Paris VI, Paris, 1981. Zbl 0513.41011. MR0670791. 1, 5
3. F. Altomare, M. Campiti, *Korovkin-type Approximation Theory and its Applications*, de Gruyter, Berlin, 1994. Zbl 0924.41001. MR1292247. DOI 10.1515/9783110884586. 5, 6
4. W. R. Bloom and J. F. Sussich, *Positive linear operators and the approximation of continuous functions on locally compact abelian groups*, J. Aust. Math. Soc. (Ser. A) **30** (1980), 180–186. Zbl 0484.41045. MR0607929. 1, 5
5. S. O. Corduneanu, *Applications of the Bohr compactification*, Automat. Comput. Appl. Math. **11** (2002), no. 1, 53–58. MR2428246. 5
6. S. O. Corduneanu, *Korovkin-type theorems for almost periodic measures*, Colloq. Math. **93** (2002), no. 2, 277–284. Zbl 1015.43009. MR1930805. DOI 10.4064/cm93-2-7. 5
7. H. Gonska, I. Rasa, M.-D. Rusu, *Chebyshev-Grüss-type inequalities via discrete oscillations*, Bul. Acad. Stiinte Repub. Mold., Mat. **74** (2014), no. 1, 63–89. Zbl 1305.260en40. MR3242479. 4
8. W. M. Priestley, *A noncommutative Korovkin theorem*, J. Approx. Theory **16** (1976), no. 3, 251–260. Zbl 0316.41016. MR0397269. DOI 10.1016/0021-9045(76)90054-X. 1, 4, 5
9. I. Rasa, *Sur certaines suites de fonctionelles*, Rev. Anal. Numér. Théor. Approx. **4** (1975), no. 2, 171–178. Zbl 0361.41013. MR0615574. 1, 5
10. I. Rasa, *Sur certaines algèbres de fonctions continues et sur les suites de variables aléatoires uniformément bornées*, Rev. Anal. Numér. Théor. Approx. **6** (1977), no. 1, 81–84. Zbl 0391.60012. MR0613525. 1, 4
11. I. Rasa, *Nets of positive linear functionals on C(X)*, Rev. Anal. Numér. Théor. Approx. **19** (1990), no. 2, 173–175. Zbl 0731.46005. MR1218922. 1, 5
12. J. F. Sussich, *Korovkin's theorem for locally compact abelian groups*, M.Sc. thesis, Murdoch University, Perth, Australia, 1982. 1, 2, 5

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