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EQUIVALENT RESULTS TO BANACH'S CONTRACTION PRINCIPLE

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Abstract. We present two versions of the well-known Banach contraction principle: one in the context of extended metric spaces for which the distance mapping is allowed to be infinite, the other in the context of metric spaces endowed with a compatible binary relation. We also point out that these two results and the Banach contraction principle are actually equivalent.

1. Introduction

The well-known Banach fixed-point theorem [1] (also known as the contraction mapping principle) is probably one of the most famous fixed-point theorems. Due to its popularity, from both a theoretical point of view and an applications standpoint, the contraction mapping principle was subject to many generalizations, some of which arise from suitable generalizations of the notion of metric space.

In this paper, we present two such generalizations. Our first result (Theorem 3.1) is a version of the contraction mapping principle in the context of extended metric spaces (where the distance mapping is allowed to take the value ∞). We extend some earlier partial results obtained by Diaz [3], Jung [6], and Luxemburg [8] by establishing necessary and sufficient conditions for the

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existence of fixed points for contractions, specifying the basin of attraction of the fixed points and studying their uniqueness.

Our second result (Theorem 3.2) is related to a version of the contraction mapping principle in the context of metric spaces endowed with a graph due to Jachymski [5, Theorem 3.2]. In comparison, our result is stated in the language of binary relations and gives a full description of the basin of attraction of the fixed points for contractions. We refer to Remark 3.3 for a more detailed explanation.

The key point of this paper is that the two fixed-point results are actually equivalent to the original contraction mapping principle using a cyclic argument. To be specific, we will prove Theorem 3.1 using the contraction mapping principle, and then we will prove Theorem 3.2 via Theorem 3.1, while the contraction mapping principle easily follows as a particular case of Theorem 3.2. It is worth mentioning that equivalence results of this kind have been established for other important theorems in this field, the classical Brouwer fixed-point theorem being probably one of the most illustrative examples. (In this direction, we refer the reader to, for example, [4], [9], and the references therein.)

This article is organized as follows. In Section 2, we recall some basic facts about binary relations and extended metric spaces. In Section 3, we state our main results, while Section 4 contains all the details of the proofs.

2. Preliminaries

Let $\mathbb{Z}$ denote the set of integers, let $\mathbb{N}$ denote the set of positive integers, and let $\mathbb{N}_0$ denote the set of nonnegative integers.

2.1. **Binary relations.** We now recall some basic facts about binary relations, introduce some notation, and state some easy to check properties. We refer to [7, pp. 2–8] for some of the details.

Let X be a nonempty set.

(BR1) It is said that R is a *binary relation* over X if R is a nonempty subset of $X \times X$. In particular, any selfmap of X can be seen as a binary relation over X by identifying it with its graph.

(BR2) For any two binary relations S and R over X, their product (or composition) is the binary relation defined as

$$SR = \big\{(x, y) \in X \times X : \text{there exists } z \in X \text{ such that } (x, z) \in R \text{ and } (z, y) \in S\big\}.$$

This is an associative law with $\Delta_X := \{(x, x) : x \in X\}$ being the identity element; hence one can define the powers of any binary relation R over X by $R^{n+1} = R^n R$ for all $n \in \mathbb{N}_0$ (where $R^0 := \Delta_X$).

Next, let R be a binary relation over X

(BR3) For $x, y \in X$ and $n \in \mathbb{N}$, a vector $\zeta = (z_0, z_1, \dots, z_n) \in X^{n+1}$ is called an *R-chain (of order n, from x to y)* if $z_0 = x$, $z_n = y$, and $(z_i, z_{i+1}) \in R$ for all $i \in \{0, 1, \dots, n-1\}$. By this definition, R^n is the set of all pairs $(x, y) \in X \times X$ such that there exists an R-chain of order n from x to y. For $\zeta = (z_0, z_1, \dots, z_n) \in X^{n+1}$, let $\overline{\zeta} = (z_n, z_{n-1}, \dots, z_0)$. If ζ is an R-chain from x to y, then $\overline{\zeta}$ is an R-chain of the same order from y to x.

Let $x, y, z \in X$. If $\zeta = (z_0, z_1, \ldots, z_n)$ is an R-chain of order n from x to z and $\zeta' = (z'_0, z'_1, \ldots, z'_m)$ is an R-chain of order m from z to y, then $z_n = z = z'_0$ and the *composition of* ζ *with* ζ' defined by

$$\zeta \circ \zeta' := (z_0, z_1, \ldots, z_{n-1}, z, z'_1, z'_2, \ldots, z'_m) \in X^{n+m+1}$$

produces an R-chain of order $n + m$ from x to y.

(BR4) The inverse of R is the binary relation R^{-1} defined as

$$R^{-1} = \big\{(x, y) \in X \times X : (y, x) \in R\big\}.$$

Note that R^{-1} is not necessarily the inverse of R with respect to the composition law since RR^{-1} and $R^{-1}R$ are not in general identical with Δ_X.

(BR5) The binary relation R is said to be *reflexive* if $\Delta_X \subseteq R$, *symmetric* if $R^{-1} \subseteq R$ (this implies that $R^{-1} = R$), *transitive* if $R^2 \subseteq R$ (this implies that $R^n \subseteq R$ for all $n \in \mathbb{N}$), or an *equivalence* if it is at the same time reflexive, symmetric, and transitive. When R is an equivalence, by $[x]_R$ we denote the equivalence class of $x \in X$ with respect to R, that is, the set of all elements in X that are equivalent to x. Any two equivalent classes are either disjoint or equal; hence R partitions X into its equivalence classes. In this context, consider $R^{\#}$ the reflexive symmetric closure of R (i.e., the smallest reflexive and symmetric relation over X that includes R), consider R^+ the transitive closure of R (i.e., the smallest transitive relation over X that includes R), and consider $R^{\equiv}$ the equivalence closure of R (i.e., the smallest equivalence relation over X that includes R). It can be easily checked that

$$\begin{aligned} R^{\#} &= R \cup \Delta_X \cup R^{-1} \\ &= \big\{(x, y) \in X \times X : (x, y) \in R \text{ or } (y, x) \in R \text{ or } x = y\big\}, \\ R^+ &= \bigcup_{n \geq 1} R^n \\ &= \big\{(x, y) \in X \times X : \text{there exists a } R\text{-chain from } x \text{ to } y\big\}, \\ R^{\equiv} &= (R^{\#})^+ = \bigcup_{n \geq 1} (R^{\#})^n \\ &= \big\{(x, y) \in X \times X : \text{there exists a } R^{\#}\text{-chain from } x \text{ to } y\big\}. \end{aligned}$$

Next, let d be a metric over X.

(BR6) If $\zeta = (z_0, z_1, \ldots, z_n) \in X^{n+1}$ is an R-chain from x to y, then, by the triangle inequality,

$$d(x, y) \leq \sum_{i=0}^{n-1} d(z_i, z_{i+1}),$$

and we call $d(\zeta) := \sum_{i=0}^{n-1} d(z_i, z_{i+1})$ the *d-length of the R-chain* ζ.

(BR7) We say that R *is compatible with* d if, for every $x \in X$ and every sequence (x_n) convergent to x such that $(x_n, x_{n+1}) \in R$ for all $n \in \mathbb{N}$, there exists a subsequence (x'_n) of (x_n) such that $(x'_n, x) \in R$ for all $n \in \mathbb{N}$.

Next, let $F : X \to X$.

(BR8) A point $x \in X$ is called an *R-fixed point of F* if $(x, Fx) \in R$, and the set of the R-fixed points of F is denoted by $\mathrm{Fix}_R F$. In particular, with $R := \Delta_X$ we recover the usual notion of fixed point.

(BR9) We call F an *endomorphism of* R if $(Fx, Fy) \in R$ for all $(x, y) \in R$.
By replacing R with its reflexive symmetric closure $R^{\#}$, it can be easily checked that F is an endomorphism of $R^{\#}$ *if and only if* $(Fx, Fy) \in R^{\#}$ for all $(x, y) \in R$.
As a consequence, any endomorphism of R is an endomorphism of $R^{\#}$. In general, the converse is not true: consider, for example, a totally ordered set $(X, \leq)$ and let $R := \{(x, y) \in X \times X : x \leq y, x \neq y\}$. Then $R^{\#} = X \times X$ and any selfmap of X is an endomorphism of $R^{\#}$, while the endomorphisms of R are precisely the increasing maps (hence no constant map is an endomorphisms of R).

(BR10) Let R_F be the binary relation over X defined by

$$R_F = \big\{(x, y) \in X \times X : \text{there exist } p, q \in \mathbb{N}_0 \text{ such that } (F^p x, F^q y) \in R\big\}.$$

Obviously, $R \subseteq R_F$. It can be easily checked that if F is an endomorphism of R, and R is an equivalence, then R_F is also an equivalence, and the equivalence classes with respect to R_F are invariant under F.

2.2. **Extended metrics.** Allowing a metric to take the value ∞ leads to the concept of an *extended metric*. This notion has been introduced by Luxemburg [8] (under the name of *generalized metric*) and has been further investigated by Jung [6]. Here, we use the terminology from [2, p. 4].

(EM1) Let X be a nonempty set. A map $\rho : X \times X \to [0, \infty]$ is called an *extended metric* over X if it satisfies the following conditions:
($d1$) $\rho(x, y) = 0$ *if and only if* $x = y$;
($d2$) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
($d3$) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.
The pair (X, ρ) is called an *extended metric space*.

(EM2) The topology of an *extended metric space* is generated by the family of all open balls (with finite radius). Also, the notion of a Cauchy sequence and the completeness of an extended metric space are defined just like in the metric case.

Next, let (X, ρ) be an extended metric space.

(EM3) The relation Q defined by

$$Q = \big\{(x, y) \in X \times X : \rho(x, y) < \infty\big\}$$

is an equivalence over X that will be called the *canonical equivalence over* (X, ρ). Indeed, Q is reflexive by ($d1$), and symmetric by ($d2$), while the transitivity easily follows by ($d3$). The partitioning of X induced by the equivalence Q is called the *canonical decomposition of* X *induced by* ρ, while the members of the partition (the equivalence classes with respect to Q) are called the *metric components of* X *(with respect to* ρ*)*.

For $x \in X$, let X_x denote the metric component of X that contains x; that is, $X_x = \{y \in X : \rho(x, y) < \infty\}$.

These definitions are motivated by the fact that the restriction of ρ to pairs of elements from the same metric component is always finite; hence, it is a metric. In this way, any extended metric space is the disjoint reunion of a family of metric spaces when the distance to be infinite between elements of different metric spaces is considered; hence, there is an equivalence between extended metrics and disjoint union metrics (see [2, p. 90]). It also follows that an extended metric space is complete *if and only if* each of its metric components is complete with respect to the metric induced by the extended metric (see [2] and [6] for more details).

(EM4) Similar to the metric case, it can be shown that the extended metric space (X, ρ) is complete *if and only if* every sequence (x_n) in X that satisfies

$$\sum_{n=1}^{\infty} \rho(x_n, x_{n+1}) < \infty \tag{2.1}$$

is convergent. For the direct implication, if a sequence (x_n) in X satisfies (2.1), then all its terms belong to the same metric component and the sequence is a Cauchy sequence, and hence convergent. In proving the converse, let (x_n) be a Cauchy sequence in (X, ρ). Since there exists $n_0 \in \mathbb{N}$ such that $\rho(x_n, x_m) < 1$ for all $m, n \geq n_0$, it follows that all the terms x_n for $n \geq n_0$ belong to the same metric component. Next, by a simple argument, one can construct a subsequence (x'_n) of $(x_n)_{n \geq n_0}$ such that $\rho(x'_n, x'_{n+1}) \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$; hence the sequence (x'_n) satisfies the condition in (2.1) which leads to (x'_n) being convergent. Since $(x_n)_{n \geq n_0}$ is a Cauchy sequence (in a metric space) with a convergent subsequence, this leads by a classical argument to $(x_n)_{n \geq n_0}$ being convergent; hence (x_n) is convergent, which concludes the proof.

(EM5) Let $F : X \to X$ be an endomorphism of Q; that is, $\rho(Fx, Fy) < \infty$ for every $x, y \in X$ such that $\rho(x, y) < \infty$. Then the relation Q_F (see (BR10) in Section 2.1),

$$Q_F = \left\{(x, y) \in X \times X : \text{there exist } p, q \in \mathbb{N}_0 \text{ such that } \rho(F^p x, F^q y) < \infty\right\},$$

is an equivalence over X such that $Q \subseteq Q_F$. Every equivalence class of X with respect to Q_F is the reunion of a family of metric components of X, is invariant under F, and will be called an F*-component of* X *(with respect to* ρ*)*.

3. Main results

In this section, we state (without proof) the main results of the present article and relate them to previously known fixed-point theorems. The proofs will be presented in Section 4

Our first result is an extension of the contraction mapping principle in the context of extended metric spaces. By considering contractions in a complete extended metric space, our result gives a full description of the fixed points by establishing necessary and sufficient conditions for their existence by specifying their basin of attraction and by studying their uniqueness. Similar partial results were previously obtained in [3], [6], and [8].

Theorem 3.1. *Let (X, ρ) be a complete extended metric space $F : X \to X$ for which there exists $q \in (0,1)$ such that*

$$\rho(Fx, Fy) \leq q \cdot \rho(x,y) \quad \text{for all } x, y \in X, \tag{3.1}$$

and $Y \subseteq X$ a F-component of X with respect to ρ. Then F has a fixed point in Y if and only if *there exists $y \in Y$ such that $\rho(y, Fy) < \infty$. In the affirmative case,*

1. *F has a unique fixed point x_* in Y, and $x_* \in X_y$;*
2. *$X_{x_*} = X_y = \{x \in Y : \rho(x, Fx) < \infty\}$;*
3. *the basin of attraction of the fixed point x_* is Y (i.e., for $x \in X$, the sequence $(F^n x)$ is convergent to x_** if and only if *$x \in Y$).*

Our next result is a version of [5, Theorem 3.2], but it is stated in the language of binary relations. We refer to Remark 3.3 for more details on the comparison between the two results. Our result also contains [10, Corollary 2.12].

Theorem 3.2. *Let (X, d) be a complete metric space, let R be a binary relation over X such that $R^{\#}$ is compatible with d, and let $F : X \to X$ be an endomorphism of $R^{\#}$ for which there exists $q \in (0,1)$ such that*

$$d(Fx, Fy) \leq q \cdot d(x,y) \quad \text{for all } (x,y) \in R. \tag{3.2}$$

Let $Y \subseteq X$ be an equivalence class with respect to the equivalence $(R^{\equiv})_F$. Then F has a fixed point in Y if and only if *$Y \cap \mathrm{Fix}_{R^{\equiv}} F$ is nonempty (i.e., there exists $y \in Y$ such that $(y, Fy) \in R^{\equiv}$). In the affirmative case,*

1. *F has a unique fixed point x_* in Y, and $x_* \in [y]_{R^{\equiv}}$;*
2. *$[x_*]_{R^{\equiv}} = [y]_{R^{\equiv}} = Y \cap \mathrm{Fix}_{R^{\equiv}} F$;*
3. *the basin of attraction of the fixed point x_* is Y (i.e., for $x \in X$, the sequence $(F^n x)$ is convergent to x_** if and only if *$x \in Y$).*

Remark 3.3. There is a natural correspondence between the language of graph theory and that of binary relations; in effect, a (directed) graph is simply a binary relation over the set of its vertices. Assuming the notation and terminology in [5], we can easily compare the assumptions in Theorem 3.2 and [5, Theorem 3.2].

Let R be the binary relation associated to the graph G in [5] (i.e., $X = V(G)$ and $R = E(G)$), and let the selfmap $F : X \to X$ in Theorem 3.2 be f in [5].

The condition "$f : X \to X$ is a G-contraction" in [5, Theorem 3.2] is equivalent to f being an endomorphism of R for which there exists $q \in (0,1)$ such that

$$d(fx, fy) \leq q \cdot d(x,y) \quad \text{for all } (x,y) \in R. \tag{3.3}$$

In our result, we require that f be an endomorphism of $R^{\#}$ such that (3.3) holds, which in general is a less restrictive condition (see (BR9) in Section 2.1).

Condition (3.1) in [5, Theorem 3.2] translates to "R is compatible with d" compared to "$R^{\#}$ is compatible with d" in our Theorem 3.2. These two hypotheses are identical when G is an undirected graph (hence when R is symmetrical), while it appears that there is no obvious logical implication between them when G is directed (hence when R is not symmetrical). In this direction, further investigation is clearly needed, which leads to the open question of finding "good" examples for which only one of the two theorems can be applied.

With the assumption that G is undirected (which provides common ground for comparing the conclusions of the two theorems), the overall conclusions of Theorem 3.2 and [5, Theorem 3.2] are fairly similar. The main added value of Theorem 3.2 is that it provides a complete description of the basin of attraction of the fixed points.

We conclude with the announced equivalence result.

Theorem 3.4. *Theorems 3.1, 3.2, and the Banach contraction principle are equivalent.*

4. Proofs of the main results

Before proving the main results, we need the following auxiliary result, inspired by [11, Proposition 1].

Lemma 4.1. *Let (X, d) be a metric space, and let R be a reflexive and symmetric binary relation over X. Define the map $d_R : X \times X \to [0, \infty]$ by*

$$d_R(x, y) = \inf\big\{d(\zeta) : \zeta \text{ is a } R\text{-chain from } x \text{ to } y\big\}$$

with the usual convention that $\inf \emptyset = \infty$*. Then d_R is an extended metric over X having the following properties:*

(1) *$R^{\equiv}$ is the canonical equivalence over (X, d_R); that is, $d_R(x, y) < \infty$* if and only if *$(x, y) \in R^{\equiv}$;*
(2) *$d(x, y) \leq d_R(x, y)$ for all $x, y \in X$;*
(3) *$d(x, y) = d_R(x, y)$ for all $(x, y) \in R$.*

Moreover, if R is compatible with d and d is complete, then d_R is complete.

Proof. Clearly, d_R is correctly defined. Let $x, y \in X$.

By definition, $d_R(x, y) < \infty$ *if and only if* there exists an R-chain from x to y; that is, $(x, y) \in R^+ = R^{\equiv}$, which proves Conclusion 1.

Next, if $(x, y) \notin R^{\equiv}$, then $d(x, y) < \infty = d_R(x, y)$. If $(x, y) \in R^{\equiv}$, then let ζ be an arbitrary R-chain from x to y; hence, $d(x, y) \leq d(\zeta)$ by the triangle inequality (see (BR6) in Section 2.1). Since this inequality is true for every R-chain ζ from x to y, it follows by the definition of d_R that $d(x, y) \leq d_R(x, y)$, and hence we have Conclusion 2. In particular, if$(x, y) \in R$, then $\zeta := (x, y)$ is an R-chain from x to y; hence, $d_R(x, y) \leq d(\zeta) = d(x, y)$, which by Conclusion 2 leads to Conclusion 3.

We verify next that d_R is indeed an extended metric over X. It is obvious that $d_R(x, x) = 0$ for all $x \in X$ since (x, x) is an R-chain from x to x; also, if $d_R(x, y) = 0$, then $d(x, y) = 0$ by Conclusion 2, which leads to $x = y$. Next, d_R is symmetric based on the fact that the correspondence $\zeta \mapsto \overline{\zeta}$ is a bijection between the set of R-chains from x to y and the set of R-chains from y to x, and this correspondence preserves the d-length of the chains. Also, d_R satisfies the triangle inequality $d_R(x,y) \leq d_R(x,z) + d_R(z,y)$ for all $x, y, z \in X$. Indeed, let ζ be an R-chain from x to z, and let ζ' be an R-chain from z to y. Then $\zeta \circ \zeta'$ is an R-chain from x to y, and $d_R(x,y) \leq d(\zeta \circ \zeta') = d(\zeta) + d(\zeta')$. By taking the infimum over ζ and ζ', we obtain the desired inequality.

Next, assume that R is compatible with d, and that d is complete. In order to prove that d_R is also complete, it is enough to show (see (EM4) in Section 2.2) that every sequence (x_n) in X that satisfies

$$\sum_{n=1}^{\infty} d_R(x_n, x_{n+1}) < \infty \tag{4.1}$$

is d_R-convergent.

Let $n \in \mathbb{N}$. By (4.1), we have $d_R(x_n, x_{n+1}) < \infty$; hence $(x_n, x_{n+1}) \in R^{\equiv}$ by Conclusion 1, and by the definition of d_R there exists ζ_n an R-chain from x_n to x_{n+1} such that

$$d(\zeta_n) \leq d_R(x_n, x_{n+1}) + \frac{1}{2^n}.$$

Considering the (infinite) composition $\zeta_1 \circ \zeta_2 \circ \zeta_3 \circ \cdots$ of all the R-chains $\{\zeta_n : n \in \mathbb{N}\}$, we obtain a sequence (z_n) in X such that

$$(z_n, z_{n+1}) \in R \quad \text{for all } n \in \mathbb{N}, \tag{4.2}$$

(x_n) is a subsequence of (z_n), and

$$\sum_{n=1}^{\infty} d_R(z_n, z_{n+1}) = \sum_{n=1}^{\infty} d(z_n, z_{n+1}) = \sum_{n=1}^{\infty} d(\zeta_n) \leq \sum_{n=1}^{\infty} d_R(x_n, x_{n+1}) + \sum_{n=1}^{n} \frac{1}{2^n} < \infty;$$

hence, (z_n) is d_R-Cauchy and d-Cauchy. Since d is complete, it follows that (z_n) is d-convergent to some $x \in X$. Next, using (4.2) and the fact that d is compatible with R, there exists (y_n) a subsequence of (z_n) such that $(y_n, x) \in R$ for all $n \in \mathbb{N}$; hence,

$$d_R(y_n, x) = d(y_n, x) \to 0 \quad (\text{as } n \to \infty)$$

by Conclusion 3, proving that (y_n) is d_R-convergent.

In conclusion, (z_n) is a d_R-Cauchy sequence having a d_R-convergent subsequence; thus, (z_n) is d_R-convergent (see also (EM4) in Section 2.2). Since (x_n) is a subsequence of (z_n), we conclude that (x_n) is also d_R-convergent. □

For the following proof, (EM3) and (EM5) in Section 2.2 provide additional details.

Proof of Theorem 3.1 via the Banach contraction principle. Note by (3.1) that $\rho(Fx, Fy) < \infty$ for every $x, y \in X$ such that $\rho(x,y) < \infty$; thus, the existence of the F-components of X with respect to ρ is assured.

The direct implication is obvious, and so we prove the converse. Let $y \in Y$ such that $\rho(y, Fy) < \infty$; hence $Fy \in X_y \subseteq Y$. For arbitrary $x \in X_y$, it follows by (3.1) that $\rho(Fx, Fy) \leq q \cdot \rho(x,y) < \infty$, meaning that Fx and Fy belong to the same metric component of X, and hence to X_y. In conclusion, $F(X_y) \subseteq X_y$, and we can apply the Banach contraction principle in the complete metric space (X_y, ρ) to the restriction of F to X_y. It follows that F has a (unique) fixed point x_* in X_y, hence in Y. Also, the sequence $(F^n x)$ is convergent to x_* for every $x \in X_y$.

(1) We prove that x_* is the unique fixed point of F in Y. Let $y_* \in Y$ be a fixed point of F. Since Y is the F-component of X that contains x_* and y_*, there exist $p, q \in \mathbb{N}_0$ such that $\rho(F^p x_*, F^q y_*) < \infty$; hence $\rho(x_*, y_*) < \infty$ since $F^p x_* = x_*$ and $F^q y_* = y_*$ by the fixed-point property. As $x_* \in X_y$, we conclude that also $y_* \in X_y$; thus $y_* = x_*$ by the uniqueness of the fixed point of F in X_y.

(2) Obviously, $X_{x^*} = X_y$, since $x_* \in X_y$. Let $x \in X_y$. Since $F(X_y) \subseteq X_y$, it follows that $Fx \in X_y$; hence $\rho(x, Fx) < \infty$, which proves the direct inclusion
$$X_y \subseteq \{x \in Y : \rho(x, Fx) < \infty\}.$$
Conversely, let $x \in Y$ such that $\rho(x, Fx) < \infty$. By applying the results proved so far with y replaced by x and taking into account the uniqueness of the fixed point x_* in Y, it follows that $X_x = X_y$; hence, $x \in X_y$, which proves the inverse inclusion.

(3) By taking into account the fact that Y is the F-component of X which contains x_*, and that x_* is a fixed point, it follows that
$$Y = \{x \in X : \text{there exists } k \in \mathbb{N}_0 \text{ such that } \rho(x_*, F^k x) < \infty\}. \tag{4.3}$$
Let $Z \subseteq X$ be the basin of attraction of the fixed point x_*. Then so far we have proved that $X_{x^*} \subseteq Z$. From here, it follows that Z contains every $x \in X$ such that $F^k x \in X_{x^*}$ for some $k \in \mathbb{N}_0$; thus, $Y \subseteq Z$ by (4.3). Conversely, if $x \in Z$, then eventually $\rho(x_*, F^k x) < \infty$ for some $k \in \mathbb{N}_0$; hence $x \in Y$, which concludes the proof. □

Proof of Theorem 3.2 via Theorem 3.1. Considering the complete metric space (X, d) and the reflexive symmetric binary relation $R^\#$ over X, it follows by Lemma 4.1 that the map $\rho := d_{R^\#}$ is a complete extended metric over X.

We claim that F satisfies (3.1) in Theorem 3.1. Let $x, y \in X$. Obviously, the inequality in (3.1) is satisfied when $\rho(x, y) = \infty$, that is, when $(x, y) \notin R^\equiv$ (by Lemma 4.1(1)), and so we can assume that $(x, y) \in R^\equiv$. Next, let $\zeta = (z_0, z_1, \ldots, z_n)$ be an arbitrary $R^\#$-chain from x to y. Since F is an endomorphism of $R^\#$, it follows that $F\zeta := (Fz_0, Fz_1, \ldots, Fz_n)$ is an $R^\#$-chain from Fx to Fy. Also, by symmetry, the inequality in (3.2) takes place for all $(x, y) \in R^\#$; hence
$$\rho(Fx, Fy) \le d(F\zeta) \le q \cdot d(\zeta),$$
and, by taking the infimum after ζ, it follows that
$$\rho(Fx, Fy) \le q \cdot \rho(x, y),$$
which proves our claim.

Also, by Lemma 4.1(1), it follows (see (EM3) and (EM5) in Section 2.2) that $R^\equiv$ is the canonical equivalence over (X, ρ), and that the equivalence classes of X with respect to $R^\equiv$ are precisely the metric components of X with respect to ρ, while the equivalence classes of X with respect to $(R^\equiv)_F$ are the F-components of X with respect to ρ. Finally, $x \in X$ is a $R^{\equiv}$-fixed point of F *if and only if* $\rho(x, Fx) < \infty$. Now, the conclusions follow directly by Theorem 3.1. □

We conclude with the proof of the equivalence result.

Proof of Theorem 3.4. We proved Theorem 3.2 via Theorem 3.1, which in turn was proved via the Banach contraction principle. Finally, the Banach contraction principle can be obtained via Theorem 3.2 by letting R be $X \times X$. □

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