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(p,σ)-ABSOLUTELY LIPSCHITZ OPERATORS

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Abstract. Due to recent advances in the theory of ideals of Lipschitz mappings, we introduce (p,σ)-absolutely Lipschitz mappings as an interpolating class between Lipschitz mappings and Lipschitz absolutely p-summing mappings. Among other results, we prove a factorization theorem that provides a reformulation to the one given by Farmer and Johnson for Lipschitz absolutely p-summing mappings.

1. Introduction and preliminaries

The fruitful development of the theory of absolute summability for linear operators (see, e.g., [8] for the general theory) produced several generalizations to the nonlinear context. This is the case of Lipschitz p-summing mappings (introduced by Farmer and Johnson in [9]), which quickly attracted the interest of many researchers trying to derive a parallel theory to the linear one (see, e.g., [5]–[7], [11]).

Midway between continuous linear operators and absolutely summing operators, a scale of linear operators (namely, (p,σ)-absolutely continuous operators $1 \leq p < \infty$, $0 \leq \sigma < 1$) was defined by Matter in [13] and [14] by applying an interpolative ideal procedure. The interpolated operator ideal $\Pi_{p,\sigma}$ of all (p,σ)-absolutely continuous operators was defined as an intermediate operator ideal between the ideal Π_p of the absolutely p-summing linear operators and the ideal of all continuous operators, and it shares similar properties with absolutely

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p-summing operators. Although it was first thought of as a tool for the study of super-reflexive Banach spaces, several works have focused on this class of operators: factorization properties and their representation as dual spaces of suitable tensor products can be found in [12] and [17]. The nonlinear version (multilinear, m-homogeneous polynomial) of this concept has been recently studied (see [1] and the references therein), and applications to the theory of Pettis or Bochner integrable functions can be found in [15].

Connecting both the linear and the Lipschitz theories, we study the class of (p,σ)-absolutely Lipschitz mappings. These have to be considered as an attempt to interpolate Lipschitz mappings with Lipschitz absolutely p-summing mappings. We pay attention to the domination/factorization theorem whose proof uses the abstract version of the Pietsch domination theorem given in [4] and [16]. When applying our factorization theorem to the particular class of absolutely p-summing Lipschitz mappings, we get an equivalent factorization to the one given by Farmer and Johnson in [9, Theorem 1].

Our article has the following organization. In Section 2 we extend to Lipschitz mappings the concept of the (p,σ)-absolutely continuous operator and we give the first results. Section 3 is devoted mainly to analyzing factorization theorems for (p,σ)-absolutely Lipschitz mappings and the duality for (p,σ)-absolutely Lipschitz operators is studied.

Let X denote a pointed metric space with a base point denoted by 0, and let E denote a Banach space. The Lipschitz mappings T from X to E that vanish at 0 form the Lipschitz space $\mathrm{Lip}_0(X,E)$, which is a Banach space under the Lipschitz norm Lip, where $\mathrm{Lip}(T)$ is the infimum of all constants $C \geq 0$ such that $\|T(x) - T(x')\| \leq Cd(x,x')$ for all $x, x' \in X$. For $E = \mathbb{R}$, we write $X^{\#} = \mathrm{Lip}_0(X) = \mathrm{Lip}_0(X,\mathbb{R})$; $B_{X^{\#}}$ is a compact Hausdorff space for the topology of pointwise convergence on X. A molecule on X is a real-valued function m on X with finite support that satisfies $\sum_{x\in X} m(x) = 0$. The real linear space of all molecules on X is denoted by $\mathcal{M}(X)$. A special role is played by the molecules of the form $m_{xx'} := \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A and $x, x' \in X$ as any $m \in \mathcal{M}(X)$ can be written as $m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}$ for some scalars λ_j and some $x_j, x'_j \in X$. We write

$$\|m\|_{\mathcal{M}(X)} = \inf\Big\{\sum_{j=1}^{n} |\lambda_j| d(x_j, x'_j), m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}\Big\},$$

where the infimum is taken over all representations of the molecule m. The space of Arens and Eells [3], denoted by Æ(X), is the completion of the normed space $(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)})$. The space X is isometrically embedded in Æ(X) via the mapping $\delta_X : X \to$Æ(X) given by $\delta_X(x) = m_{x0}$. In [5], vector-valued molecules were naturally considered. An E-valued molecule on X is a finitely supported function $m : X \longrightarrow E$ such that $\sum_{x\in X} m(x) = 0$. The vector space of all E-valued molecules on X is denoted by $\mathcal{M}(X,E)$. (For a general approach to the theory of Lipschitz mappings, we refer to [18].)

Recall that $T \in \mathrm{Lip}_0(X,E)$ is Lipschitz p-summing (in symbols $T \in \Pi_p^L(X,E)$) if there exists a constant $C \geq 0$ so that, for all $x_i, y_i \in X$ and all positive reals

a_i, $i = 1, \ldots, n$,

$$\sum_{i=1}^{n} a_i \big\| T(x_i) - T(y_i) \big\|^p \le C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^{n} a_i \big| f(x_i) - f(y_i) \big|^p.$$

The Lipschitz p-summing norm, $\pi_p^L(T)$, of T is the smallest constant $C \ge 0$ that fulfills the above inequality.

Lipschitz p-dominated operators between Banach spaces are treated in [7]. Let E and F be Banach spaces. A map $T \in \mathrm{Lip}_0(E, F)$ is *Lipschitz p-dominated* if there exist a Banach space Z and a linear operator $S \in \Pi_p(E, Z)$ such that $\|T(x) - T(x')\| \le \|S(x) - S(x')\|$ for all $x, x' \in E$.

Let $d_p^L(T)$ denote the infimum of all $\pi_p(S)$ when S varies over all linear p-summing operators defined on E that fulfill the above condition. This is a norm for the space $D_p^L(E, F)$ of all Lipschitz p-dominated mappings between E and F. Any mapping in $D_p^L(E, F)$ is Lipschitz and $\mathrm{Lip} \le d_p^L$. Note that

$$D_p^L(E, F) \subset \Pi_p^L(E, F). \tag{1.1}$$

2. (p, σ)-Absolutely Lipschitz mappings

Let $1 \le p < \infty$, and let $0 \le \sigma < 1$. Recall that a linear operator $T \in \mathcal{L}(E, F)$ between two Banach spaces E and F is called (p, σ)-*absolutely continuous* (see [13]) if there exist a Banach space G and a linear operator $S \in \Pi_p(E, G)$ such that

$$\big\| T(x) \big\| \le \|x\|^\sigma \big\| S(x) \big\|^{1-\sigma}, \quad x \in E. \tag{2.1}$$

Let $\pi_{p,\sigma}(T) = \inf \pi_p(S)^{1-\sigma}$, where the infimum is taken over all Banach spaces G and $S \in \Pi_p(E, G)$ such that (2.1) holds. By $\Pi_{p,\sigma}(E, F)$, we denote the Banach space of all (p, σ)-absolutely continuous operators between E and F.

Let us introduce the Lipschitz version of (p, σ)-absolutely continuous operators.

Definition 2.1. Let $1 \le p < \infty$, and let $0 \le \sigma < 1$. Let X be a pointed metric space, and let E be a Banach space. A base point preserving mapping $T \in \mathrm{Lip}_0(X, E)$ is called (p, σ)-*absolutely Lipschitz* if there exist a Banach space F and a Lipschitz operator $S \in \Pi_p^L(X, F)$ such that

$$\big\| T(x) - T(x') \big\| \le \big\| S(x) - S(x') \big\|^{1-\sigma} d(x, x')^\sigma$$

for all $x, x' \in X$. Let $\pi_{p,\sigma}^L(T)$ denote the infimum of all $\pi_p^L(S)^{1-\sigma}$ when S varies over all Lipschitz p-summing operators defined on X that fulfill the above condition.

The space of all (p, σ)-absolutely Lipschitz mappings between X and E is denoted by $\Pi_{p,\sigma}^L(X, E)$. An easy calculation shows that

$$\Pi_p^L \subset \Pi_{p,\sigma}^L \subset \mathrm{Lip}_0 \tag{2.2}$$

and $\mathrm{Lip} \le \pi_{p,\sigma}^L \le \pi_p^L$ for every $0 < \sigma < 1$. Section 3 contains a factorization theorem that provides a prototype of a (p, σ)-absolutely Lipschitz mapping in the sense that any other (p, σ)-absolutely Lipschitz mapping is the composition of this kind of prototype with Lipschitz mappings. That gives the whole spectrum of (p,σ)-absolutely Lipschitz mappings.

Remark 2.2. When $\sigma = 0$, $(p,0)$-absolutely Lipschitz mappings extend the notion of Lipschitz p-dominated operators that was in [7]. A Lipschitz p-dominated operator T is defined between Banach spaces, and satisfies a similar condition to the one given in Definition 2.1, but there the dominating mapping S is a linear absolutely p-summing operator. We don't know if the Lipschitz mapping S in our definition can be replaced with a linear one even if T is linear. In particular, we don't know if being (p,σ)-absolutely Lipschitz implies (p,σ)-absolute continuity whenever the mapping T is linear. The converse is of course clearly true. Our aim is to work in the setting of Lipschitz mappings.

Let $p > 1$, $1/p + 1/p' = 1$, and let $0 < \sigma < 1$. López Molina and Sánchez-Pérez in [12, Example 1.9] proved that the operator $u : \ell_{p'} \to \ell_{\frac{p}{1-\sigma}}$ defined by $u(e_i) = (\frac{1}{i})^{\frac{1}{p}} e_i$, where $(e_i)_{i=1}^{\infty}$ is the unit vector basis of $\ell_{p'}$, is (p,σ)-absolutely continuous and $u \notin \Pi_p(\ell_{p'}, \ell_{\frac{p}{1-\sigma}})$. Then u is trivially (p,σ)-absolutely Lipschitz, but, by Theorem 2 in [9], $u \notin \Pi_p^L(\ell_{p'}, \ell_{\frac{p}{1-\sigma}})$, and then the inclusion $\Pi_p^L \subset \Pi_{p,\sigma}^L$ is strict.

Proposition 2.3. *Let X be a pointed metric space, and let E be a Banach space. Then, for $1 \le p < \infty$ and $0 \le \sigma < 1$, the space $\Pi_{p,\sigma}^L(X,E)$ is a Banach space.*

Proof. We prove the triangle inequality. Consider $T_1, T_2 \in \Pi_{p,\sigma}^L(X,E)$, F_1, F_2 Banach spaces, and consider $S_i \in \Pi_p^L(X, F_i)$, $i = 1, 2$, such that

$$\big\|T_i(x) - T_i(x')\big\| \le \big\|S_i(x) - S_i(x')\big\|^{1-\sigma} d(x,x')^{\sigma} \quad \text{for all } x, x' \in X.$$

Let F be the ℓ_1-sum of F_1 and F_2, and let $I_i : F_i \to F$ be the canonical injections. The map

$$S := \pi_p^L(S_1)^{-\sigma} I_1 \circ S_1 + \pi_p^L(S_2)^{-\sigma} I_2 \circ S_2$$

belongs to $\Pi_p^L(X,F)$ and

$$\pi_p^L(S) \le \pi_p^L(S_1)^{1-\sigma} + \pi_p^L(S_2)^{1-\sigma}.$$

Using Hölder's inequality, we get

$$\begin{aligned}
&\big\|(T_1+T_2)(x) - (T_1+T_2)(x')\big\| \\
&\quad \le \big\|T_1(x) - T_1(x')\big\| + \big\|T_2(x) - T_2(x')\big\| \\
&\quad \le \sum_{i=1}^{2} \big\|\pi_p^L(S_i)^{-\sigma}\big(S_i(x) - S_i(x')\big)\big\|_{F_i}^{1-\sigma} \big(\pi_p^L(S_i)^{1-\sigma}\big)^{\sigma} d(x,x')^{\sigma} \\
&\quad \le \Big(\sum_{i=1}^{2} \big\|\pi_p^L(S_i)^{-\sigma}\big(S_i(x) - S_i(x')\big)\big\|_{F_i}\Big)^{1-\sigma} \Big(\sum_{i=1}^{2} \pi_p^L(S_i)^{1-\sigma}\Big)^{\sigma} d(x,x')^{\sigma} \\
&\quad = \big(\pi_p^L(S_1)^{1-\sigma} + \pi_p^L(S_2)^{1-\sigma}\big)^{\sigma} \big\|S(x) - S(x')\big\|_F^{1-\sigma} d(x,x')^{\sigma}
\end{aligned}$$

for all $x, x' \in X$. Thus $T_1 + T_2 \in \Pi^L_{p,\sigma}(X, E)$ and

$$\begin{aligned}\pi^L_{p,\sigma}(T_1 + T_2) &\leq \left(\pi^L_p(S_1)^{1-\sigma} + \pi^L_p(S_2)^{1-\sigma}\right)^\sigma \pi^L_p(S)^{1-\sigma} \\ &\leq \pi^L_p(S_1)^{1-\sigma} + \pi^L_p(S_2)^{1-\sigma}.\end{aligned}$$

Taking the infimum, we finally get that $\pi^L_{p,\sigma}(T_1 + T_2) \leq \pi^L_{p,\sigma}(T_1) + \pi^L_{p,\sigma}(T_2)$.

To prove the completeness of the space, take a sequence $(T_n)_n$ in $\Pi^L_{p,\sigma}(X, E)$ such that $\sum_{n=1}^\infty \pi^L_{p,\sigma}(T_n) < \infty$. Since $\mathrm{Lip} \leq \pi^L_{p,\sigma}$ and $(\mathrm{Lip}_0(X, E), \mathrm{Lip})$ is a Banach space, there exists $T := \sum_{n=1}^\infty T_n \in \mathrm{Lip}_0(X, E)$. We prove that $\sum_{n=1}^\infty T_n = T$ for $\pi^L_{p,\sigma}$. Let $\epsilon > 0$, and, for each $n \in \mathbb{N}$, let $S_n \in \Pi^L_p(X, F_n)$ be such that

$$\left\|T_n(x) - T_n(x')\right\| \leq \left\|S_n(x) - S_n(x')\right\|^{1-\sigma} d(x, x')^\sigma$$

for all $x, x' \in X$ and $\pi^L_p(S_n)^{1-\sigma} \leq \pi^L_{p,\sigma}(T_n) + \epsilon/2^n$. Then

$$\Big(\sum_{n=1}^\infty \pi^L_p(S_n)\Big)^{1-\sigma} \leq \sum_{n=1}^\infty \pi^L_p(S_n)^{1-\sigma} \leq \sum_{n=1}^\infty \pi^L_{p,\sigma}(T_n) + \epsilon < \infty.$$

Let $S = \sum_{n=1}^\infty \pi^L_p(S_n)^{-\sigma}(I_n \circ S_n) \in \Pi^L_p(X, F)$, where F is the ℓ_1-sum of all F_n and $I_n : F_n \to F$ is the natural inclusion. Hence

$$\begin{aligned}\left\|T(x) - T(x')\right\| &\leq \sum_{n=1}^\infty \left\|T_n(x) - T_n(x')\right\| \\ &\leq \sum_{n=1}^\infty \left\|S_n(x) - S_n(x')\right\|_{F_n}^{1-\sigma} d(x, x')^\sigma \\ &\leq \left\|S(x) - S(x')\right\|_F^{1-\sigma} \Big(\sum_{n=1}^\infty \pi^L_p(S_n)^{1-\sigma}\Big)^\sigma d(x, x')^\sigma.\end{aligned}$$

This implies $T \in \Pi^L_{p,\sigma}(X, E)$ and

$$\pi^L_{p,\sigma}(T) \leq \sum_{n=1}^\infty \pi^L_p(S_n)^{1-\sigma} \leq \sum_{n=1}^\infty \pi^L_{p,\sigma}(T_n) + \epsilon.$$

We have

$$\pi^L_{p,\sigma}\Big(T - \sum_{k=1}^n T_k\Big) = \pi^L_{p,\sigma}\Big(\sum_{k=n+1}^\infty T_k\Big) \leq \sum_{k=n+1}^\infty \pi^L_p(S_k)^{1-\sigma}.$$

Thus $\sum_{n=1}^\infty T_n = T$ for $\pi^L_{p,\sigma}$. □

Note that $\Pi^L_{p,0} = \Pi^L_p$ and $\pi^L_{p,0} = \pi^L_p$. Therefore, the class $\Pi^L_{p,\sigma}$ for $0 \leq \sigma < 1$ can be considered as an interpolating class between Π^L_p and Lip_0.

The next result is an extension of [7, Theorem 3.2] for (p, σ)-absolutely Lipschitz mappings. To prove the domination part, we will use an alternative technique: the unified abstract version of the Pietsch domination theorem given in [4, Theorem 2.2] (see also [16]).

Theorem 2.4. *Let $1 \leq p < \infty$, let $0 \leq \sigma < 1$, and let $T \in \mathrm{Lip}_0(X, E)$. The following statements are equivalent:*

(1) *$T \in \Pi^L_{p,\sigma}(X,E)$.*
(2) *There is a constant $C \geq 0$ and a regular Borel probability measure μ on $B_{X^\#}$ such that*

$$\left\|T(x)-T(x')\right\| \leq C\Big(\int_{B_{X^\#}} \left(\left|f(x)-f(x')\right|^{1-\sigma} d(x,x')^\sigma\right)^{\frac{p}{1-\sigma}} d\mu(f)\Big)^{\frac{1-\sigma}{p}}$$

for all $x, x' \in X$.
(3) *There is a constant $C \geq 0$ such that, for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and all $(a_i)_{i=1}^n \subset \mathbb{R}^+$, we have*

$$\Big(\sum_{i=1}^n a_i \left\|T(x_i)-T(x_i')\right\|^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}}$$
$$\leq C \sup_{f\in B_{X^\#}} \Big(\sum_{i=1}^n a_i \left(\left|f(x_i)-f(x_i')\right|^{1-\sigma} d(x_i,x_i')^\sigma\right)^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}}.$$

Furthermore, the infimum of the constants $C \geq 0$ in (2) and (3) is $\pi^L_{p,\sigma}(T)$.

Proof. (1)⇒(2) If $T \in \Pi^L_{p,\sigma}(X,E)$, then there exist a Banach space F and$S \in \Pi^L_p(X,F)$ such that

$$\left\|T(x)-T(x')\right\| \leq \left\|S(x)-S(x')\right\|^{1-\sigma} d(x,x')^\sigma$$

for all $x, x' \in X$. By [9, Theorem 1], since S is Lipschitz p-summing, then there exists a regular Borel probability measure μ on $B_{X^\#}$ such that

$$\begin{aligned}\left\|T(x)-T(x')\right\| &\leq \left\|S(x)-S(x')\right\|^{1-\sigma} d(x,x')^\sigma\\ &\leq \pi^L_p(S)^{1-\sigma}\Big(\int_{B_{X^\#}} \left|f(x)-f(x')\right|^p d\mu(f)\Big)^{\frac{1-\sigma}{p}} d(x,x')^\sigma\\ &= \pi^L_p(S)^{1-\sigma}\Big(\int_{B_{X^\#}} \left(\left|f(x)-f(x')\right|^{1-\sigma} d(x,x')^\sigma\right)^{\frac{p}{1-\sigma}} d\mu(f)\Big)^{\frac{1-\sigma}{p}}\end{aligned}$$

for all $x, x' \in X$.

(2)⇒(1) Let A be the natural isometric embedding from X into $C(B_{X^\#})$ composed with the formal identity from $C(B_{X^\#})$ into $L_\infty(\mu)$ given by $A(x)(f) = f(x)$, $x \in X, f \in B_{X^\#}$. Let $I_{\infty,p} : L_\infty(\mu) \longrightarrow L_p(\mu)$ be the canonical mapping $I_{\infty,p}(g) = g$. Note that $\pi^L_p(I_{\infty,p}) = 1$. Therefore, by (2),

$$\begin{aligned}&\left\|T(x)-T(x')\right\|\\ &\leq C\Big(\int_{B_{X^\#}} \left(\left|f(x)-f(x')\right|^{1-\sigma} d(x,x')^\sigma\right)^{\frac{p}{1-\sigma}} d\mu(f)\Big)^{\frac{1-\sigma}{p}}\\ &= \Big(\int_{B_{X^\#}} \left|C^{\frac{1}{1-\sigma}} I_{\infty,p}A(x)(f) - C^{\frac{1}{1-\sigma}} I_{\infty,p}A(x')(f)\right|^p d\mu(f)\Big)^{\frac{1-\sigma}{p}} d(x,x')^\sigma.\end{aligned}$$

Consequently, there is a Banach space $F = L_p(\mu)$ and a Lipschitz p-summing operator $S = C^{\frac{1}{1-\sigma}} I_{\infty,p} A$ such that

$$\big\|T(x) - T(x')\big\| \leq \big\|S(x) - S(x')\big\|^{1-\sigma} d(x,x')^{\sigma}$$

as required.

(2)⇒(3) If

$$\big\|T(x) - T(x')\big\| \leq C\Big(\int_{B_{X^\#}} \big(\big|f(x) - f(x')\big|^{1-\sigma} d(x,x')^{\sigma}\big)^{\frac{p}{1-\sigma}}\, d\mu(f)\Big)^{\frac{1-\sigma}{p}}$$

for all $x, x' \in X$, then, for $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}^+$ and $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in X$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} a_i \big\|T(x_i) - T(x'_i)\big\|^{\frac{p}{1-\sigma}} \\
&\quad \leq C^{\frac{p}{1-\sigma}} \sum_{i=1}^{n} \int_{B_{X^\#}} a_i \big(\big|f(x_i) - f(x'_i)\big|^{1-\sigma} d(x_i,x'_i)^{\sigma}\big)^{\frac{p}{1-\sigma}}\, d\mu(f) \\
&\quad = C^{\frac{p}{1-\sigma}} \int_{B_{X^\#}} \sum_{i=1}^{n} a_i \big(\big|f(x_i) - f(x'_i)\big|^{1-\sigma} d(x_i,x'_i)^{\sigma}\big)^{\frac{p}{1-\sigma}}\, d\mu(f) \\
&\quad \leq C^{\frac{p}{1-\sigma}} \sup_{f \in B_{X^\#}} \sum_{i=1}^{n} a_i \big(\big|f(x_i) - f(x'_i)\big|^{1-\sigma} d(x_i,x'_i)^{\sigma}\big)^{\frac{p}{1-\sigma}}.
\end{aligned}$$

(3)⇒(2) We will use the unified abstract version of the Pietsch domination theorem given in [4, Theorem 2.2].

Let $R : B_{X^\#} \times (X \times X \times \mathbb{R}) \times \mathbb{R} \to [0, \infty[$ be given by

$$R\big(f, (x, x', a), \lambda\big) = |a|^{\frac{1-\sigma}{p}} \big|f(x) - f(x')\big|^{1-\sigma} d(x,x')^{\sigma} |\lambda|,$$

and let $S : \mathrm{Lip}_0(X, E) \times (X \times X \times \mathbb{R}) \times \mathbb{R} \to [0, \infty[$ be given by

$$S\big(T, (x, x', a), \lambda\big) = |a|^{\frac{1-\sigma}{p}} \big\|T(x) - T(x')\big\| |\lambda|.$$

Then T is R-S-abstract $p/(1-\sigma)$-summing (see [4, Definition 2.1]):

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} S\big(T, (x_i, x'_i, a_i), \lambda_i\big)^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}} \\
&\quad = \Big(\sum_{i=1}^{n} |a_i| |\lambda_i|^{\frac{p}{1-\sigma}} \big\|T(x_i) - T(x'_i)\big\|^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}} \\
&\quad \leq C \sup_{f \in B_{X^\#}} \Big(\sum_{i=1}^{n} |a_i| |\lambda_i|^{\frac{p}{1-\sigma}} \big(\big|f(x_i) - f(x'_i)\big|^{1-\sigma} d(x_i,x'_i)^{\sigma}\big)^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}} \\
&\quad = C \sup_{f \in B_{X^\#}} \Big(\sum_{i=1}^{n} R\big(f, (x_i, x'_i, a_i), \lambda_i\big)^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}}.
\end{aligned}$$

Then, by [4, Theorem 2.2], there is a constant $C \geq 0$ and a regular Borel probability measure μ on $B_{X^\#}$ such that

$$S\big(T,(x,x',a),\lambda\big) \leq C\Big(\int_{B_{X^\#}} R\big(f,(x,x',a),\lambda\big)^{\frac{p}{1-\sigma}}\,d\mu(f)\Big)^{\frac{1-\sigma}{p}}$$

for all $(x,x',a) \in X \times X \times \mathbb{R}$ and $\lambda \in \mathbb{R}$. In particular, we have

$$\big\|T(x)-T(x')\big\| \leq C\Big(\int_{B_{X^\#}} \big(\big|f(x)-f(x')\big|^{1-\sigma} d(x,x')^\sigma\big)^{\frac{p}{1-\sigma}}\,d\mu(f)\Big)^{\frac{1-\sigma}{p}}$$

for all $x,x' \in X$. □

Remark 2.5. The notion of a (p,σ)-absolutely Lipschitz mapping can be defined for Lipschitz mappings between pointed metric spaces. Given pointed metric spaces X and Y, a map $T \in \mathrm{Lip}_0(X,Y)$ is called (p,σ)-*absolutely Lipschitz* if there exist a constant $k \geq 0$, a pointed metric space G, and a Lipschitz operator $S \in \Pi_p^L(X,G)$ such that

$$d\big(T(x),T(x')\big) \leq kd\big(S(x),S(x')\big)^{1-\sigma} d(x,x')^\sigma$$

for all $x,x' \in X$. In this case $\pi_{p,\sigma}^L(T)$ denotes the infimum of all $k\pi_p^L(S)^{1-\sigma}$. Theorem 2.4 can be easily adapted whenever $T \in \mathrm{Lip}_0(X,Y)$ and X and Y are pointed metric spaces.

Proposition 2.6 (Ideal property)**.** *Let X,Y,X_0,Y_0 be pointed metric spaces. If $v: X_0 \longrightarrow X$, $w: Y \longrightarrow Y_0$ are Lipschitz mappings and $T: X \longrightarrow Y$ is (p,σ)-absolutely Lipschitz, then wTv is (p,σ)-absolutely Lipschitz and*

$$\pi_{p,\sigma}^L(wTv) \leq \mathrm{Lip}(w)\,\mathrm{Lip}(v)\pi_{p,\sigma}^L(T).$$

Proof. Since T is (p,σ)-absolutely Lipschitz, then there exist a constant $k \geq 0$, a pointed metric space G, and a Lipschitz operator $S \in \Pi_p^L(X,G)$ such that

$$d\big(T(x),T(x')\big) \leq kd\big(S(x),S(x')\big)^{1-\sigma} d(x,x')^\sigma \quad \text{for all } x,x' \in X.$$

Let $x_0,x_0' \in X_0$. Then

$$\begin{aligned} d\big(wTv(x_0),wTv(x_0')\big) &\leq \mathrm{Lip}(w)d\big(Tv(x_0),Tv(x_0')\big) \\ &\leq k\,\mathrm{Lip}(w)d\big(S\circ v(x_0),S\circ v(x_0')\big)^{1-\sigma} d\big(v(x_0),v(x_0')\big)^\sigma \\ &\leq k\,\mathrm{Lip}(w)\,\mathrm{Lip}(v)^\sigma d\big(S\circ v(x_0),S\circ v(x_0')\big)^{1-\sigma} d(x_0,x_0')^\sigma. \end{aligned}$$

Since $S\circ v \in \Pi_p^L(X_0,G)$, it follows that $w\circ T\circ v \in \Pi_{p,\sigma}^L(X_0,Y_0)$ and

$$\begin{aligned} \pi_{p,\sigma}^L(wTv) &\leq k\,\mathrm{Lip}(w)\,\mathrm{Lip}(v)^\sigma \pi_p^L(S\circ v)^{1-\sigma} \\ &\leq k\,\mathrm{Lip}(w)\,\mathrm{Lip}(v)\pi_p^L(S)^{1-\sigma}. \end{aligned}$$

Taking the infimum, we get

$$\pi_{p,\sigma}^L(wTv) \leq \mathrm{Lip}(w)\,\mathrm{Lip}(v)\pi_{p,\sigma}^L(T).$$ □

Remark 2.7. Using (2.2), Proposition 2.3, and Proposition 2.6, it can be shown that all (p,σ)-absolutely Lipschitz mappings form a Lipschitz operator ideal (see [2]).

3. Factorization theorem

Let μ be a Borel probability measure on $B_{X^\#}$. Consider the canonical inclusion i:Æ$(X) \longrightarrow C(B_{X^\#})$ given by $i(\sum_{j=1}^n \lambda_j m_{x_j x_j'}) := \sum_{j=1}^n \lambda_j \langle m_{x_j x_j'}, \cdot\rangle$. On $i($Æ$(X))$, we define the semi-norm

$$\left\|i(m)\right\|_{p,\sigma} := \inf\Big\{\sum_{j=1}^n |\lambda_j| d(x_j,x_j')^\sigma \Big(\int_{B_{X^\#}} \left|f(x_j)-f(x_j')\right|^p d\mu(f)\Big)^{\frac{1-\sigma}{p}}\Big\},$$

where the infimum is taken over all representations of m of the form $m = \sum_{j=1}^n \lambda_j m_{x_j x_j'}$. Consider on $i \circ \delta_X(X)$ the pseudometric induced by $\|\cdot\|_{p,\sigma}$:

$$d_{p,\sigma}\big(i\circ\delta_X(x), i\circ\delta_X(x')\big) := \left\|i\circ\delta_X(x) - i\circ\delta_X(x')\right\|_{p,\sigma}$$

and the relation of equivalence $\mathcal{R}$ given by

$$i\circ\delta_X(x)\mathcal{R} i\circ\delta_X(x') \Leftrightarrow d_{p,\sigma}\big(i\circ\delta_X(x), i\circ\delta_X(x')\big) = 0.$$

We set $X^\mu_{p,\sigma} := \frac{i\circ\delta_X(X)}{\mathcal{R}}$, and let $q : i\circ\delta_X(X) \longrightarrow X^\mu_{p,\sigma}$ be the projection.

Note that, if we consider the canonical map $j_p : C(B_{X^\#}) \longrightarrow L_p(\mu)$, then $i\circ\delta_X(x)\mathcal{R} i\circ\delta_X(x')$ if and only if $j_p(i\circ\delta_X(x)) = j_p(i\circ\delta_X(x'))$. Hence $X^\mu_{p,\sigma}$ can be seen as a subset of $L_p(\mu)$ via the formal identity I.

Theorem 3.1. *Let $1 \leq p < \infty$, and let $0 \leq \sigma < 1$. Let X and Y be pointed metric spaces, and let $T \in \mathrm{Lip}_0(X,Y)$. The following statements are equivalent:*

1. $T \in \Pi^L_{p,\sigma}(X,Y)$,
2. *there exist a regular Borel probability measure μ on $B_{X^\#}$ and a Lipschitz operator $v : X^\mu_{p,\sigma} \to Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \big\downarrow{\scriptstyle \delta_X} & & \big\uparrow{\scriptstyle v} \\ \delta_X(X) & \xrightarrow{q\circ i} & X^\mu_{p,\sigma} \end{array}$$

Proof. (1)⇒(2) Assume first that $T \in \Pi^L_{p,\sigma}(X,Y)$. By Theorem 2.4 and Remark 2.5, there is a regular Borel probability measure μ on $B_{X^\#}$ such that

$$d\big(T(x),T(x')\big) \leq \pi^L_{p,\sigma}(T)\Big(\int_{B_{X^\#}} \big(\left|f(x)-f(x')\right|^{1-\sigma} d(x,x')^\sigma\big)^{\frac{p}{1-\sigma}} d\mu(f)\Big)^{\frac{1-\sigma}{p}}$$

for all $x,x' \in X$. Define $v(q\circ i\circ\delta_X(x)) := T(x)$, $x \in X$. If $x,x' \in X$ are so that $q(i\circ\delta_X(x)) = q(i\circ\delta_X(x'))$, then $0 = d_{p,\sigma}(i\circ\delta_X(x), i\circ\delta_X(x')) = \|\langle m_{xx'},\cdot\rangle\|_{p,\sigma}$. Therefore, given $\epsilon > 0$, there exists a representation of $m_{xx'}$, $m_{xx'} = \sum_{j=1}^n \lambda_j m_{x_j x_j'}$ such that

$$\sum_{j=1}^n |\lambda_j| d(x_j,x_j')^\sigma \Big(\int_{B_{X^\#}} \left|f(x_j)-f(x_j')\right|^p d\mu\Big)^{\frac{1-\sigma}{p}} < \epsilon.$$

Let $g \in B_{Y^\#}$. Then

$$\begin{aligned}
&\big|g\big(T(x)\big) - g\big(T(x')\big)\big| \\
&\quad = \big|\langle m_{xx'}, g \circ T\rangle\big| \\
&\quad \leq \sum_{j=1}^{n} |\lambda_j| \big|\langle m_{x_j x_j'}, g \circ T\rangle\big| \leq \sum_{j=1}^{n} |\lambda_j| d\big(T(x_j), T(x_j')\big) \\
&\quad \leq \pi_{p,\sigma}^{L}(T) \sum_{j=1}^{n} |\lambda_j| d(x_j, x_j')^\sigma \Big(\int_{B_{X^\#}} \big|f(x_j) - f(x_j')\big|^p \, d\mu\Big)^{\frac{1-\sigma}{p}} < \epsilon \pi_{p,\sigma}^{L}(T).
\end{aligned}$$

Letting $\epsilon \longrightarrow 0$, it follows that $g(T(x)) - g(T(x')) = 0$ for all $g \in B_{Y^\#}$. Hence $T(x) = T(x')$. This proves that v is well defined.

We now show that v is Lipschitz. Take $g \in B_{Y^\#}$, and let $m_{xx'} = \sum_{j=1}^{n} \lambda_j m_{x_j x_j'}$. Then, by Proposition 2.6,

$$\begin{aligned}
&\big|g \circ v\big(q \circ i \circ \delta_X(x)\big) - g \circ v\big(q \circ i \circ \delta_X(x')\big)\big| \\
&\quad = \big|g \circ T(x) - g \circ T(x')\big| = \big|\langle m_{xx'}, g \circ T\rangle\big| \\
&\quad \leq \pi_{p,\sigma}^{L}(T) \sum_{j=1}^{n} |\lambda_j| d(x_j, x_j')^\sigma \Big(\int_{B_{X^\#}} \big|f(x_j) - f(x_j')\big|^p \, d\mu\Big)^{\frac{1-\sigma}{p}}.
\end{aligned}$$

Taking the infimum over all representations of $m_{xx'}$, we get

$$\big|g \circ v\big(q \circ i \circ \delta_X(x)\big) - g \circ v\big(q \circ i \circ \delta_X(x')\big)\big| \leq \pi_{p,\sigma}^{L}(T) d_{p,\sigma}\big(i \circ \delta_X(x), i \circ \delta_X(x')\big)$$

for all $g \in B_{Y^\#}$. We conclude now that

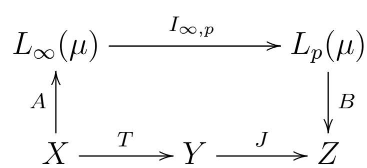
$$d\Big(v\big(q \circ i \circ \delta_X(x)\big), v\big(q \circ i \circ \delta_X(x')\big)\Big) \leq \pi_{p,\sigma}^{L}(T) d_{p,\sigma}\big(i \circ \delta_X(x), i \circ \delta_X(x')\big).$$

(2)⇒(1) Assume that T factors as in (2). By Proposition 2.6, it suffices to prove that $q \circ i : \delta_X(X) \longrightarrow X_{p,\sigma}^{\mu}$ is (p,σ)-absolutely Lipschitz, but this is clear as

$$d_{p,\sigma}\big(i \circ \delta_X(x), i \circ \delta_X(x')\big) = \big\|i(m_{xx'})\big\|_{p,\sigma} \leq \|m_{xx'}\|^\sigma \Big(\int_{B_{X^\#}} \big|f(x) - f(x')\big|^p \, d\mu\Big)^{\frac{1-\sigma}{p}}.$$

□

Farmer and Johnson [9, Theorem 1] proved that $\pi_p^L(T) \leq C$ if and only if for some (or any) isometric embedding J of Y into a 1-injective space Z there is a factorization



with μ a probability and $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) \leq C$.

Letting $\sigma = 0$ in Theorem 3.1, we obtain a factorization theorem for Lipschitz absolutely p-summing operators which is equivalent to the above. In that case, $X_{p,0}^{\mu} = j_p \circ i \circ \delta_X(X)$, where $j_p : C(B_{X^\#}) \to L_p(\mu)$ is the canonical mapping, and the induced metric $d_{p,0}$ generates the L_p-norm on $X^\mu_{p,0}$. Then Theorem 3.1 is a generalization of the Farmer and Johnson factorization.

Theorem 3.2. *Let $1 \leq p < \infty$. Let X and Y be pointed metric spaces. The following statements are equivalent for a mapping $T \in \mathrm{Lip}_0(X, Y)$ and a positive constant C:*

1. *$T \in \Pi^L_p(X, Y)$ and $\pi^L_p(T) \leq C$.*
2. *There exists a regular Borel probability measure μ on $B_{X^\#}$ such that*
$$d\big(T(x), T(x')\big) \leq C\Big(\int_{B_{X^\#}} \big|\langle \delta_X(x) - \delta_X(x'), f\rangle\big|^p \, d\mu(f)\Big)^{1/p}$$
for all $x, x' \in X$.
3. *There exist a regular Borel probability measure μ on $B_{X^\#}$ and a Lipschitz operator $v : X^\mu_{p,0} \to Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ \Big\downarrow{\scriptstyle i\circ\delta_X} & & \Big\uparrow{\scriptstyle v} \\ i \circ \delta_X(X) & \xrightarrow{\ j_p\ } & X^\mu_{p,0} \end{array}$$

Furthermore, the infimum of the constants $C \geq 0$ in (1) and (2) is $\pi^L_p(T)$.

Let us end showing the duality for (p, σ)-absolutely Lipschitz operators.

Let $1 \leq p, r < \infty$, and $0 \leq \sigma < 1$ such that $r' = \frac{p'}{1-\sigma}$, where p' is the conjugate of p; that is, $\frac{1}{p} + \frac{1}{p'} = 1$. For $x_1, \dots, x_n, x'_1, \dots, x'_n$ in X and scalars $\lambda_1, \dots, \lambda_n$, we define

$$\delta^{Lip}_{p,\sigma}\big((\lambda_j, x_j, x'_j)^n_{j=1}\big) := \sup_{f \in B_{X^\#}} \Big(\sum_{j=1}^n \big(|\lambda_j|\big|f(x_j) - f(x'_j)\big|^{1-\sigma} d(x_j, x'_j)^\sigma\big)^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}}.$$

If we denote

$$w^{Lip}_{\frac{p}{1-\sigma}}\big((\lambda_j, x_j, x'_j)^n_{j=1}\big) := \sup_{f \in B_{X^\#}} \Big(\sum_{j=1}^n \big(|\lambda_j|\big|f(x_j) - f(x'_j)\big|\big)^{\frac{p}{1-\sigma}}\Big)^{\frac{1-\sigma}{p}},$$

then we have

$$w^{Lip}_{\frac{p}{1-\sigma}}\big((\lambda_j, x_j, x'_j)^n_{j=1}\big) \leq \delta^{Lip}_{p,\sigma}\big((\lambda_j, x_j, x'_j)^n_{j=1}\big).$$

As a remark, the above inequality shows that $\Pi^L_{p/(1-\sigma)}(X, Y) \subset \Pi^L_{p,\sigma}(X, Y)$.

For a molecule $m \in \mathcal{M}(X, E)$, we define its (p, σ)-Chevet–Saphar norm by

$$cs_{p,\sigma}(m) = \inf\Big\{\big\|\big(\lambda_j\|v_j\|\big)^n_{j=1}\big\|_r \delta^{Lip}_{p',\sigma}\big((\lambda_j^{-1}, x_j, x'_j)^n_{j=1}\big) : m = \sum_{j=1}^n v_j m_{x_j x'_j}, \lambda_j > 0\Big\}.$$

We denote by $CS_{p,\sigma}(X, E)$ the space $\mathcal{M}(X, E)$ endowed with the norm $cs_{p,\sigma}$.

The following theorem can be proved as in [5, Theorem 4.3].

Theorem 3.3. *The spaces $CS_{p,\sigma}(X, E)^*$ and $\Pi^L_{p',\sigma}(X, E^*)$ are isometrically isomorphic via the canonical pairing $\langle m, T\rangle = \sum_{j=1}^n \langle v_j, T(x_j) - T(x'_j)\rangle$.*

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