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THE $\lambda^{+r}(\mu)$-STATISTICAL CONVERGENCE

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Abstract. Let $\lambda = (\lambda_n)_{n\geq 1}$be a nondecreasing sequence of positive numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ for all n, and let $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \ldots$. Then for any given nonzero sequence μ, we define by $\Delta^+(\mu)$ the operator that generalizes the operator of the first difference and is defined by $\Delta^+(\mu)x_k = \mu_k(x_k - x_{k+1})$. In this article, for any given integer $r \geq 1$, we deal with the $\lambda^{+r}(\mu)$-statistical convergence that generalizes in a certain sense the well-known λ_E^r-statistical convergence. The main results consist in determining sets of sequences χ and χ' of the form s_ξ^0 satisfying $\chi \subset [V,\lambda]_0(\Delta^{+r}(\mu)) \subset \chi'$ and sets κ and κ' of the form s_ξ satisfying $\kappa \leq [V,\lambda]_\infty(\lambda^{+r}(\mu)) \leq \kappa'$. This study is justified since the infinite matrix associated with the operator $\Delta^{+r}(\mu)$ cannot be explicitly calculated for all r.

1. Introduction and preliminaries

1.1. **Statistical convergence.** The notion of *statistical convergence*, first introduced by Fast [8], was later studied by Fridy ([9], [10]), Fridy and Orhan [11], and Connor [4]. The sequence $X = (x_n)_{n\geq 1}$ is said to be *statistically convergent to the number L* if

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \leq n : |x_k - L| \geq \varepsilon\right\}\right| = 0 \quad \text{for all } \varepsilon > 0,$$

where the vertical bars indicate the number of elements in the enclosed set. Note that *lacunary statistical convergence*, which was introduced by Fridy and Orhan

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([11], [12]) and further studied by Alotaibi et al. [1], can be considered as a special case of statistical convergence. Recall that [6] defined the λ_E^{+r}-*statistical convergence* where $\lambda = (\lambda_n)_n$ is a nondecreasing sequence of positive numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ for all n (see also [2], [3], [7], [5], [19]). For a given Banach space E, the sequence $(x_n)_{n\geq 1}$ is said to be λ_E^{+r}-*statistically convergent* if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : \|\Delta^{+r} x_k - L\| \geq \varepsilon\right\}\right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \ldots$; $\Delta^+ x_k = x_k - x_{k+1}$; and $\Delta^{+r} x_k = \sum_{j=0}^{r} (-1)^j \binom{r}{j} x_{k+j}$.

In the present article, we will take $E = \mathbb{R}$ or $\mathbb{C}$, $L = 0$ and replace Δ^{+r} by $\Delta^{+r}(\mu) = (D_\mu \Delta^+)^r$ and deal with the $\lambda^{+r}(\mu)$-*statistical convergence to zero*. (In all that follows, the subscripts are greater or equal to 1.) We will also consider the infinite matrix of *first difference* $\Delta^+ = (a_{nm})_{n,m\geq 1}$ defined by $a_{nn} = 1$, $a_{n,n+1} = -1$ and $a_{nm} = 0$ otherwise. Let D_μ be the diagonal matrix defined by $[D_\mu]_{nn} = \mu_n$ for all n and consider the set U of all sequences such that $u_n \neq 0$ for all n. Then we let $\Delta^+(\mu) = D_\mu \Delta^+$ for $\mu \in U$.

From the *generalized de la Vallée-Poussin mean* defined by

$$t_n(X) = \frac{1}{\lambda_n} \sum_{k\in I_n} x_k \quad \text{for } X = (x_n)_n,$$

we are led to define the following sets for $r \geq 1$ integer

$$[V,\lambda]_0\big(\Delta^{+r}(\mu)\big) = \Big\{X \in s : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} \big|\Delta^{+r}(\mu) x_k\big| = 0\Big\},$$

$$[V,\lambda]_\infty\big(\Delta^{+r}(\mu)\big) = \Big\{X \in s : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \big|\Delta^{+r}(\mu) x_k\big| < \infty\Big\}.$$

In the case when $\lambda_n = n$, we will write the previous sets $[V]_0(\Delta^{+r}(\mu))$ and $[V]_\infty(\Delta^{+r}(\mu))$. For the convenience of the reader, note that

$$[V,\lambda]_0\big(\Delta^{+1}(\mu)\big) = \Big\{X \in s : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} \big|\mu_k (x_k - x_{k+1})\big| = 0\Big\},$$

and that in the previous definitions of $[V,\lambda]_0(\Delta^{+r}(\mu))$ and $[V,\lambda]_\infty(\Delta^{+r}(\mu))$, for $r = 2$, we have

$$\Delta^{+2}(\mu) x_k = \mu_k^2 x_k - \mu_k(\mu_k + \mu_{k+1}) x_{k+1} + \mu_k \mu_{k+1} x_{k+2} \quad \text{for all } k.$$

Note that there is no general expression for $\Delta^{+r}(\mu) x_k$ where r is any given integer. Now we can state the definition of $\lambda^{+r}(\mu)$-statistical convergence to zero.

Definition 1.1. A sequence $X = (x_n)_{n\geq 1}$ is said to be $\lambda^{+r}(\mu)$-statistically convergent to zero if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : \left|\Delta^{+r}(\mu) x_k\right| \geq \varepsilon\right\}\right| = 0.$$

In this case, we write $x_k \to 0S_\lambda(\Delta^{+r}(\mu))$. If $\lambda_n = n$ for all n, then we write $x_k \to 0S(\Delta^{+r}(\mu))$.

Now we explicitly give both a relation between $X \in [V,\lambda]_0(\Delta^{+r}(\mu))$ and the condition $x_k \to 0S_\lambda(\Delta^{+r}(\mu))$ and a relation between the condition $x_k \to 0S(\Delta^{+r}(\mu))$ and the condition $x_k \to 0S_\lambda(\Delta^{+r}(\mu))$. In this way we have the next results.

Theorem 1.2. *Let $\alpha, \mu \in U^+$, and let $r \geq 1$ be an integer. Then*

(a) *$X \in [V,\lambda]_0(\Delta^{+r}(\mu))$ implies that $x_k \to 0S_\lambda(\Delta^{+r}(\mu))$ and the inclusion is proper;*

(b) *if $X \in l_\infty(\Delta^{+r}(\mu))$ and $x_k \to 0S_\lambda(\Delta^{+r}(\mu))$, then $X \in [V,\lambda]_0(\Delta^{+r}(\mu))$, where $l_\infty(\Delta^{+r}(\mu)) := \{X \in s : \sup_k \mid \Delta^{+r}(\mu)x_k \mid < \infty\}$.*

Proof. (a) This follows easily from the following:

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\Delta^{+r}(\mu)x_k\right| \geq \frac{\varepsilon}{\lambda_n}\left|\left\{k \in I_n : \left|\Delta^{+r}(\mu)x_k\right| \geq \varepsilon\right\}\right|.$$

The following example shows that the inclusion is proper. Let $X = (x_n)_{n\geq 1}$ be defined such that

$$\Delta^{+r}(\mu)x_k = \begin{cases} k & \text{for } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $X \notin l_\infty$, and for $0 < \varepsilon \leq 1$,

$$\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\Delta^{+r}(\mu)x_k\right| \geq \varepsilon\right\}\right| = \frac{[\sqrt{\lambda_n}]}{\lambda_n} \to 0 \quad (n \to \infty);$$

that is, $x_k \to 0S_\lambda(\Delta^{+r}(\mu))$. But

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\Delta^{+r}(\mu)x_k\right| \nrightarrow 0;$$

that is, $x \notin [V,\lambda]_0(\Delta^{+r}(\mu))$.

(b) Let $X \in l_\infty$. Then $|x_k| \leq M$ for all k, where $M > 0$. For $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\Delta^{+r}(\mu)x_k\right| &= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |\Delta^{+r}(\mu)|\geq\epsilon}}\left|\Delta^{+r}(\mu)x_k\right| + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |\Delta^{+r}(\mu)|<\epsilon}}\left|\Delta^{+r}(\mu)x_k\right| \\ &\leq \frac{M}{\lambda_n}\left|\left\{k \in I_n : \left|\Delta^{+r}(\mu)x_k\right| \geq \varepsilon\right\}\right| + \varepsilon.\end{aligned}$$

Hence $x_k \to 0S_\lambda(\Delta^{+r}(\mu))$ implies that $X \in [V,\lambda]_0(\Delta^{+r}(\mu))$. □

Remark 1.3. It is easy to see that $S_\lambda(\Delta^{+r}(\mu)) \subseteq S(\Delta^{+r}(\mu))$ for all λ, since $\lambda_n/n \leq 1$.

Now we establish the reverse inclusion.

Theorem 1.4. *We have that $S^0(\Delta^{+r}(\mu)) \subseteq S^0_\lambda(\Delta^{+r}(\mu))$ if and only if*

$$\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0, \tag{1.1}$$

where by $X \in S^0(\Delta^{+r}(\mu))$ (or $X \in S^0_\lambda(\Delta^{+r}(\mu))$), we mean that $x_k \to 0S(\Delta^{+r}(\mu))$ (or that $x_k \to 0S_\lambda(\Delta^{+r}(\mu))$).

Proof. For $\varepsilon > 0$, we have

$$\{k \in I_n : |\Delta^{+r}(\mu)x_k| \geq \varepsilon\} \subset \{k \leq n : |\Delta^{+r}(\mu)x_k| \geq \varepsilon\}.$$

Therefore,

$$\frac{1}{n}\left|\{k \leq n : |\Delta^{+r}(\mu)x_k| \geq \varepsilon\}\right| \geq \frac{1}{n}\left|\{k \in I_n : |\Delta^{+r}(\mu)x_k| \geq \varepsilon\}\right| \geq \frac{\lambda_n}{n}.\frac{1}{\lambda_n}\left|\{k \in I_n : |\Delta^{+r}(\mu)x_k| \geq \varepsilon\}\right|.$$

Letting $n \to \infty$ and using (1.1), we get the inclusion.

Conversely, suppose that

$$\lim_{n\to\infty} \inf \frac{\lambda_n}{n} = 0.$$

Choose a subsequence $(n(j))_{j\geq 1}$ such that $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$. Define a sequence $X = (x_k)_{k\geq 1}$ such that

$$\Delta^{+r}(\mu)x_k = \begin{cases} 1 & \text{for } k \in I_{n(j)}, j = 1, 2, 3, \ldots; \\ 0 & \text{otherwise.} \end{cases}$$

Then $X \in [V]_0(\Delta^{+r}(\mu))$, and hence by Theorem 1.2(a), we have $X \in 0S(\Delta^{+r}(\mu))$. But $X \notin [V, \lambda]_0(\Delta^{+r}(\mu))$, and therefore by Theorem 1.2(b), we have $X \notin 0S_\lambda(\Delta^{+r}(\mu))$. To get a subset for each of the sets $[V]_0(\Delta^{+r}(\mu))$ and $[V]_\infty(\Delta^{+r}(\mu))$, we need the following. □

1.2. **Some properties of the set $\widehat{C}_1$.** We write c_0, c, and l_∞ for the sets of *null*, *convergent* and *bounded sequences*, respectively. If we then put $U^+ = \{(u_n)_n \in s : u_n > 0 \text{ for all } n\}$ and use Wilansky's notation (see [20]), we define, for any sequence $\alpha = (\alpha_n)_n \in U^+$ and for any set of sequences E, the set

$$(1/\alpha)^{-1} * E = \{(x_n)_n \in s : (x_n/\alpha_n)_n \in E\}.$$

To simplify, we will write $D_\alpha E = (1/\alpha)^{-1} * E$, where D_α is the diagonal matrix defined in Section 1.1, and we put $s_\alpha = D_\alpha l_\infty$, $\overset{0}{s}_\alpha = D_\alpha c_0$, and $s^{(c)}_\alpha = D_\alpha c$. Recall that a *Banach space $E \subset s$ is a BK space if each projection $X \mapsto P_n(X) = x_n$ is continuous*. Each of the spaces $D_\alpha E$, where $E \in \{l_\infty, c_0, c\}$, *is a BK space normed by* $\|X\|_{s_\alpha} = \sup_n(|x_n|/\alpha_n)$, and $\overset{\circ}{s}_\alpha$ has AK (see [14]).

Now let $\alpha = (\alpha_n)_n$, $\beta = (\beta_n)_n \in U^+$. By $S_{\alpha,\beta}$, we denote the set of all infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that

$$\|A\|_{S_{\alpha,\beta}} = \sup_n\Big(\frac{1}{\beta_n}\sum_{m=1}^{\infty}|a_{nm}|\alpha_m\Big) < \infty.$$

The set $S_{\alpha,\beta}$ with norm $\|A\|_{S_{\alpha,\beta}}$ is a *Banach space* with the norm.

Let E and F be any subsets of s. When A maps E into F, we will write $A \in (E, F)$ (see [13]). So for every $X \in E$, $AX \in F$ ($AX \in F$ will mean that, for each $n \geq 1$, the series defined by $A_n(X) = \sum_{m=1}^{\infty} a_{nm} x_m$ is convergent and $(A_n(X))_n \in F$). It was shown in [17] that $A \in (s_\alpha, s_\beta)$ if and only if $A \in S_{\alpha,\beta}$. So we can write $(s_\alpha, s_\beta) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$, we obtain the *Banach algebra with identity* $S_{\alpha,\beta} = S_\alpha$ (see [17]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$. We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$.

If $\alpha = (r^n)_{n \geq 1}$, Γ_α, S_α, s_α, s_α^0 and $s_\alpha^{(c)}$ are denoted by Γ_r, S_r, s_r, s_r^0, and $s_r^{(c)}$, respectively. When $r = 1$, we obtain $s_1 = l_\infty$, $s_1^0 = c_0$, and $s_1^{(c)} = c$, and putting $e = (1, 1, \ldots)$ we have $S_1 = S_e$. It is well known (see [13]) that $(s_1, s_1) = (c_0, s_1) = (c, s_1)$. We also have $A \in (c_0, c_0)$ if and only if $A \in S_1$ and $\lim_{n\to\infty} a_{nm} = 0$ for all $m \geq 1$.

For any subset E of s, we put

$$AE = \{Y \in s : Y = AX \text{ for some } X \in E\}.$$

If F is a subset of s, we denote

$$F(A) = F_A = \{X \in s : Y = AX \in F\}.$$

Define now the set $\widehat{C}_1$ of sequences $\alpha \in U^+$ satisfying the condition $\sup_n (\sum_{k=1}^{n} \alpha_k)/\alpha_n < \infty$. Let Δ be the well-known operator defined by $\Delta x_n = x_n - x_{n-1}$ for all n, with $x_0 = 0$. Note that $\Delta^T = \Delta^+$. Therefore, we get the next result (see [14]).

Lemma 1.5. *Let $\alpha \in U^+$. The following conditions are equivalent:*

(i) $\alpha \in \widehat{C}_1$,
(ii) *the operator Δ is bijective from s_α to itself,*
(iii) *the operator Δ is bijective from s_α^0 to itself.*

It can easily be deduced that if $\alpha \in \widehat{C}_1$, then for any given integer $r \geq 1$ the operator Δ^r *is bijective from s_α to itself, and* $s_\alpha(\Delta^r) = s_\alpha$. It is the same for the operator Δ considered as an operator from s_α^0 to itself. Recall the following result.

Lemma 1.6 ([14, Corollary 3.4, p. 1795])**.** *Let $r \geq 1$ be an integer. The following properties are equivalent:*

(i) $\alpha \in \widehat{C}_1$,
(ii) $s_\alpha(\Delta) = s_\alpha$,
(iii) $s_\alpha(\Delta^r) = s_\alpha$,
(iv) $C(\alpha)(\Sigma^{r-1}\alpha) \in l_\infty$.

In the remainder of this article, we will also use the next result (see [16, Proposition 2.1, p. 1656]), where Γ is the set of all $\alpha \in U^+$ such that $\overline{\lim}_{n\to\infty}(\alpha_{n-1}/\alpha_n) < 1$.

Lemma 1.7. *Let $\alpha \in U^+$. Then*

(i) *if $\alpha \in \widehat{C}_1$, then there are $K > 0$ and $\gamma > 1$ such that $\alpha_n \geq K\gamma^n$ for all n;*

(ii) *the condition* $\alpha \in \Gamma$ *implies that* $\alpha \in \widehat{C}_1$ *and there is a real* $b > 0$ *such that*

$$\left[C(\alpha)\alpha\right]_n \leq \frac{1}{1-\chi} + b\chi^n \quad \text{for } n \geq q+1 \text{ and } \chi = \gamma_q(\alpha) \in]0,1[.$$

To state some results on the sets $[V]_0(\Delta^+(\mu))$ and $[V]_\infty(\Delta^+(\mu))$, we will give some properties on the sets $s_\alpha(\Delta^{+r}(\mu))$ and $s_\alpha^0(\Delta^{+r}(\mu))$.

2. Some properties of the sets $s_\alpha(\Delta^{+r}(\mu))$ and $s_\alpha^0(\Delta^{+r}(\mu))$ for $r \geq 1$ integer

First, recall the following lemma (see [18, Theorem 3.5]).

Lemma 2.1. *Let* α, $\beta \in U^+$. *Then* $s_\alpha^0 = s_\beta^0$ *if and only if there are* K_1, $K_2 > 0$ *such that*

$$K_1 \leq \frac{\alpha_n}{\beta_n} \leq K_2 \quad \text{for all } n.$$

We also need the next result.

Lemma 2.2. *Let* $r \geq 1$ *be an integer, and let* $\alpha \in U^+$. *The operator represented by the infinite matrix* Δ^{+r} *is surjective from* $s_\alpha^0(\Delta^{+r})$ *to* s_α^0.

Proof. Consider the matrix $\Delta^{+r}(e_1, e_2, \dots, e_r)$ obtained from Δ^{+r} by addition of the rows e_r, $e_{r-1}, \dots,$ e_1, respectively. Similarly for any $B = (b_n)_n \in s$, we will write $B(u_1, u_2, \dots, u_r)$ for the vector obtained from B by addition of the scalars u_r, $u_{r-1}, \dots,$ u_1; that is, $B(u_1, u_2, \dots, u_r) = (u_1, u_2, \dots, u_r, b_1, \dots, b_n, \dots)^T$. Since the infinite matrix $\Delta^{+r}(e_1, e_2, \dots, e_r)$ is a triangle, the equation $\Delta^{+r}X = B$, where $B \in s_\alpha^0$ has infinitely solutions in $s_\alpha^0(\Delta^{+r})$ given by

$$X = \left[\Delta^{+r}(e_1, e_2, \dots, e_r)\right]^{-1} B(u_1, u_2, \dots, u_r)$$

for all scalars u_r, $u_{r-1}, \dots,$ u_1. This completes the proof. □

We will now use the convention $x_n = 1$ for all $n \leq 0$, and put $(x_{n-r})_n$ for the sequence $(1, \dots, 1, x_1, \dots, x_{n-r}, \dots)$, where x_1 is in the $(r+1)$-position. The following theorem extends some results given in [15] and [16].

Theorem 2.3. *Let* $\alpha \in U^+$ *and* $r \geq 1$ *be an integer. Then*

(i) $(\alpha_{n-1}/\alpha_n)_n \in l_\infty$ *if and only if* $s^0_{(\alpha_{n-1})_n} \subset s_\alpha^0(\Delta^+)$;

(ii) *the following statements are equivalent:*
  (a) $\alpha \in \widehat{C}_1$,
  (b) $s_\alpha^0(\Delta^+) = s^0_{(\alpha_{n-1})_n}$,
  (c) $s_\alpha^0(\Delta^+) \subset s^0_{(\alpha_{n-1})_n}$;

(iii) (a) $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ *if and only if, for any given integer* $r \geq 1$,

$$s_\alpha^0 \subset s_\alpha^0(\Delta^+) \subset \cdots \subset s_\alpha^0(\Delta^{+r});$$

  (b) *the conditions* $\alpha \in \widehat{C}_1$ *and* $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ *are equivalent to* $s_\alpha^0(\Delta^+) = s_\alpha^0$;

(iv) *the following statements are equivalent:*
  (a) $s_\alpha^0(\Delta^+) = s_\alpha^0$,

(b) $s_\alpha^0(\Delta^{+k}) = s_\alpha^0(\Delta^{+(k+1)})$ *for all* $k \geq 0$ *integer,*
(c) $s_\alpha^0(\Delta^{+k}) = s_\alpha^0$ *for all* $k \geq 0$ *integer.*

Proof. (i) We have $s^0_{(\alpha_{n-1})_n} \subset s^0_\alpha(\Delta^+)$ if and only if $I \in (s^0_{(\alpha_{n-1})_n}, s^0_\alpha(\Delta^+))$. Thus $\Delta^+ X \in s^0_\alpha$ for all $X \in s^0_{(\alpha_{n-1})_n}$ and $\Delta^+ \in (s^0_{(\alpha_{n-1})_n}, s^0_\alpha)$. This means $D_{1/\alpha}\Delta^+ D_{(\alpha_{n-1})_n} \in (c_0, c_0)$, where $D_{1/\alpha}\Delta^+ D_{(\alpha_{n-1})_n} = (a_{nm})_{n,m\geq 1}$ with $a_{nn} = \alpha_{n-1}/\alpha_n$, $a_{n,n+1} = 1$ and $a_{nm} = 0$, otherwise. Using the characterization of (c_0, c_0) given in the Introduction, we have $D_{1/\alpha}\Delta^+ D_{(\alpha_{n-1})_n} \in S_1$ and $(\alpha_{n-1}/\alpha_n)_n \in l_\infty$. We conclude that $s^0_{(\alpha_{n-1})_n} \subset s^0_\alpha(\Delta^+)$ if and only if $(\alpha_{n-1}/\alpha_n)_n \in l_\infty$.

(ii). First show that (c) implies (a). We have $I \in (s^0_\alpha(\Delta^+), s^0_{(\alpha_{n-1})_n})$ if and only if

$$B = \Delta^+ X \in s^0_\alpha \quad \text{implies that } X \in s^0_{(\alpha_{n-1})_n} \text{ for all } X \in s. \tag{2.1}$$

Using elementary calculations (see [15]), and Lemma 2.2 for every $B \in s^0_\alpha$, there is $X \in s^0_\alpha(\Delta^+)$ such that $B = \Delta^+ X$ with $x_n = u - \sum_{k=1}^{n-1} b_k$ for all $u \in \mathbb{C}$. Then (2.1) implies that, for every $B = (b_n)_n \in s^0_\alpha$,

$$\frac{u - \sum_{k=1}^{n-1} b_k}{\alpha_{n-1}} \to 0 \qquad (n \to \infty) \quad \text{for any given } u \in \mathbb{C}.$$

So if we take $u = 0$, then $(\sum_{k=1}^{n-1} b_k)/\alpha_{n-1} \to 0$ $(n \to \infty)$; that is, for every $B \in s^0_\alpha$ we have $\Sigma^- B \in s^0_{(\alpha_{n-1})_n}$, where

$$\Sigma^- = \begin{pmatrix} 0 & & & \\ 1 & 0 & & 0 \\ 1 & 1 & 0 & \\ . & . & . & . \end{pmatrix}.$$

We conclude that $\Sigma^- \in (s^0_\alpha, s^0_{(\alpha_{n-1})_n})$; that is, $\Sigma^- \in S_{\alpha,(\alpha_{n-1})_n}$, which means that

$$\frac{\alpha_1 + \cdots + \alpha_{n-1}}{\alpha_{n-1}} = O(1) \quad (n \to \infty) \qquad \text{and} \qquad \alpha \in \widehat{C}_1.$$

Thus we have shown that (c) implies (a).

Let us show now that (a) implies (b). Assume that $\alpha \in \widehat{C}_1$ and let $X \in s^0_\alpha(\Delta^+)$. We have $B = \Delta^+ X \in s^0_\alpha$ and for every n, we have $x_n = u - \sum_{k=1}^{n-1} b_k$ for $u \in \mathbb{C}$. As we have seen above, the condition $\alpha \in \widehat{C}_1$ implies that $\Sigma^- \in S_{\alpha,(\alpha_{n-1})_n}$ and $\Sigma^- B \in s^0_{(\alpha_{n-1})_n}$ for all $B \in s^0_\alpha$. Then

$$\frac{x_n}{\alpha_{n-1}} = \frac{u - \sum_{k=1}^{n-1} b_k}{\alpha_{n-1}} = o(1) \quad (n \to \infty);$$

that is, $X \in s^0_{(\alpha_{n-1})_n}$, and we conclude that $s^0_\alpha(\Delta^+) \subset s^0_{(\alpha_{n-1})_n}$. Now $\alpha \in \widehat{C}_1$ implies that $(\alpha_{n-1}/\alpha_n)_{n\geq 1} \in l_\infty$, and by (i), $s^0_{(\alpha_{n-1})_n} \subset s^0_\alpha(\Delta^+)$. So $s^0_\alpha(\Delta^+) = s^0_{(\alpha_{n-1})_n}$ and we have shown that (a) implies (b). We conclude that (b) implies (c) and that (c) implies (a) and that (a) implies (b). This completes the proof of (ii).

(iii)(a) Let $X \in s^0_\alpha$. Then condition $(\alpha_{n+1}/\alpha_n)_{n\geq 1} \in l_\infty$ implies that

$$\frac{x_n - x_{n+1}}{\alpha_n} = \frac{x_n}{\alpha_n} - \frac{x_{n+1}}{\alpha_{n+1}}\frac{\alpha_{n+1}}{\alpha_n} \to 0 \quad (n \to \infty)$$

and $s_\alpha^0 \subset s_\alpha^0(\Delta^+)$. Now assume that, for any given integer $i \geq 1$, $X \in s_\alpha^0(\Delta^{+i})$; then $Y = \Delta^{+i}X \in s_\alpha^0$, and with $s_\alpha^0 \subset s_\alpha^0(\Delta^+)$, then $Y \in s_\alpha^0(\Delta^+)$ and $X \in s_\alpha^0(\Delta^{+(i+1)})$. So we have $s_\alpha^0(\Delta^{+i}) \subset s_\alpha^0(\Delta^{+(i+1)})$, which shows the necessity.

Conversely, the inclusion $s_\alpha^0 \subset s_\alpha^0(\Delta^+)$ implies that $\Delta^+ X \in s_\alpha^0$ for all $X \in s_\alpha^0$. Thus $\Delta^+ \in (s_\alpha^0, s_\alpha^0)$ and $(\alpha_{n+1}/\alpha_n)_n \in l_\infty$.

(iii)(b) If $\alpha \in \widehat{C}_1$ and $(\alpha_n/\alpha_{n-1})_n \in l_\infty$, then there are K_1, $K_2 > 0$ such that

$$K_1 \leq \frac{\alpha_n}{\alpha_{n-1}} \leq K_2 \quad \text{for all } n.$$

Then by Lemma 2.1, we have $s^0_{(\alpha_{n-1})_n} = s_\alpha^0$. By (ii), we conclude that the condition $\alpha \in \widehat{C}_1$ implies that $s_\alpha^0(\Delta^+) = s^0_{(\alpha_{n-1})_n} = s_\alpha^0$.

Conversely, assume that $s_\alpha^0(\Delta^+) = s_\alpha^0$. Then $I \in (s_\alpha^0, s_\alpha^0(\Delta^+))$ and $\Delta^+ \in (s_\alpha^0, s_\alpha^0)$; that is, $(\alpha_n/\alpha_{n-1})_n \in l_\infty$. Now as we have just seen in the proof of (c) implying (a) in (ii), for every $B \in s_\alpha^0$ there is $X \in s$ such that $B = \Delta^+ X$ with $x_n = u - \sum_{k=1}^{n-1} b_k$ for all $u \in \mathbb{C}$. Then for every $B \in s_\alpha^0$ with $B = \Delta^+ X$, we have that $X \in s_\alpha^0$; that is,

$$\frac{u - \sum_{k=1}^{n-1} b_k}{\alpha_n} \to 0 \quad (n \to \infty) \quad \text{for any given } u \in \mathbb{C}.$$

Then if we take $u = 0$, we have $\Sigma^- \in (s_\alpha^0, s^0_{(\alpha_{n-1})_n})$; that is, $\Sigma^- \in S_{\alpha,(\alpha_{n-1})_n}$. So

$$\frac{\alpha_1 + \cdots + \alpha_{n-1}}{\alpha_{n-1}} = O(1) \quad (n \to \infty) \qquad \text{and} \qquad \alpha \in \widehat{C}_1.$$

The proof of (iv) is elementary and left to the reader. □

We immediately deduce the following corollary.

Corollary 2.4. *Let $r \geq 1$ be an integer, and assume that $(\alpha_n/\alpha_{n-1})_n \in l_\infty$. Then $s_\alpha^0(\Delta^{+r}) \subset s_\alpha^0$ implies that $s_\alpha^0(\Delta^{+r}) = s_\alpha^0$.*

Proof. By Theorem 2.3(iii)(a), the condition $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ implies that $s_\alpha^0 \subset s_\alpha^0(\Delta^{+r})$. So the condition $s_\alpha^0(\Delta^{+r}) \subset s_\alpha^0$ implies that $s_\alpha^0(\Delta^{+r}) = s_\alpha^0$. □

Remark 2.5. Note that in Theorem 2.3, the conditions $\alpha \in \widehat{C}_1$ and $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ are equivalent to $s_\alpha^0(\Delta^+) = s^0_{(\alpha_{n-1})_n} = s_\alpha^0$.

Corollary 2.6. *Let $\alpha \in U^+$.*

(i) *Let $r \geq 1$ be an integer. The condition $\alpha \in \widehat{C}_1$ implies $s_\alpha^0(\Delta^{+r}) = s^0_{(\alpha_{n-r})_n}$.*
(ii) *The conditions $\alpha \in \widehat{C}_1$ and $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ are equivalent to $s_\alpha^0(\Delta^{+k}) = s_\alpha^0$ for all $k \geq 0$.*

Proof. For statement (i), the condition $\alpha \in \widehat{C}_1$ implies that $s_\alpha^0(\Delta^+) = s^0_{(\alpha_{n-1})_n}$ by Theorem 2.3(ii). Now let $j \geq 1$ be an integer, and assume that $s_\alpha^0(\Delta^{+j}) = s^0_{(\alpha_{n-j})_n}$. Then $X \in s_\alpha^0(\Delta^{+(j+1)})$ if and only if $\Delta^{+(j+1)}X \in s_\alpha^0$, which in turn is $\Delta^+ X \in s_\alpha^0(\Delta^{+j}) = s^0_{(\alpha_{n-j})_n}$. So $s_\alpha^0(\Delta^{+(j+1)}) = s^0_{(\alpha_{n-j})_n}(\Delta^+)$. Since $\alpha \in \widehat{C}_1$, then $(\alpha_{n-j})_n \in \widehat{C}_1$ and $s^0_{(\alpha_{n-j})_n}(\Delta^+) = s_\alpha^0(\Delta^{+(j+1)}) = s^0_{(\alpha_{n-(j+1)})_n}$. This shows (i).

For statement (ii), the conditions $\alpha \in \widehat{C}_1$ and $\alpha_n/\alpha_{n-1} = O(1)$ $(n \to \infty)$ imply that $s^0_\alpha(\Delta^+) = s^0_{(\alpha_{n-1})_n} = s^0_\alpha$ by Theorem 2.3(iii)(b). This implies that $s^0_\alpha(\Delta^{+k}) = s^0_{(\alpha_{n-k})_n} = s^0_\alpha$ by what we have just shown. Conversely, assume that $s^0_\alpha(\Delta^{+k}) = s^0_\alpha$ for all $k \geq 0$; then obviously $s^0_\alpha(\Delta^+) = s^0_\alpha$, and by Theorem 2.3 this implies that $\alpha \in \widehat{C}_1$ and $\alpha_n/\alpha_{n-1} = O(1)$ $(n \to \infty)$. ☐

Corollary 2.7. *Let $\alpha \in U^+$. The following conditions are equivalent:*

(i) $s^0_\alpha(\Delta^+) \neq s^0_\alpha$;
(ii) *there is $k \geq 0$ integer such that $s^0_\alpha(\Delta^{+k}) \neq s^0_\alpha(\Delta^{+(k+1)})$;*
(iii) *either $\alpha \notin \widehat{C}_1$ or there is a sequence of integers $(n_i)_i$ strictly increasing to infinity such that $\alpha_{n_i}/\alpha_{n_i-1} \to \infty$ $(i \to \infty)$.*

Proof. Here (i) is equivalent to (ii) by Theorem 2.3(iv), and (i) is equivalent to (iii) by Theorem 2.3(iii). ☐

Corollary 2.8. *Let $\alpha \in U^+ \backslash \widehat{C}_1$. Then $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ if and only if*

$$s^0_{(\alpha_{n-1})_n} \subsetneq s^0_\alpha(\Delta^+).$$

Proof. This result comes from Theorem 2.3 parts (i) and (iii). Indeed, by Theorem 2.3(ii), $\alpha \in U^+ \backslash \widehat{C}_1$ if and only if $s^0_{(\alpha_{n-1})_n} \neq s^0_\alpha(\Delta^+)$. ☐

Using Theorem 2.3 and the equivalence (ii) and (iii) in Corollary 2.8, we get the following.

Proposition 2.9. *Let $\alpha \in U^+ \backslash \widehat{C}_1$. Then $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ if and only if, for any given integer $k \geq 1$,*

$$s^0_\alpha \subsetneq s^0_\alpha(\Delta^+) \subsetneq \cdots \subsetneq s^0_\alpha(\Delta^{+k}).$$

We immediately deduce the following.

Theorem 2.10. *Let $r \geq 1$ be an integer, and let $\alpha \in U^+$. Then we have the following.*

(i) *For any given $\beta \in U^+$, the condition*

$$s^0_\alpha(\Delta^{+r}) \subset s^0_\beta \subset s^0_\alpha(\Delta^{+(r+1)}) \tag{2.2}$$

implies that $s^0_\alpha = s^0_\beta = s^0_\alpha(\Delta^{+r})$.

(ii) *Condition (2.2) implies that $(\alpha_n/\alpha_{n-1})_n \in l_\infty$, $\alpha \in \widehat{C}_1$ and $s^0_\alpha = s^0_{(\alpha_{n+j})_n}$ for all $j \geq 1$.*

Proof. (i) First, the inclusion $s^0_\alpha(\Delta^{+r}) \subset s^0_\alpha(\Delta^{+(r+1)})$ implies that

$$\Delta^{+r}X = Y \in s^0_\alpha \quad \text{implies } \Delta^+ Y \in s^0_\alpha \text{ for all } Y \in s.$$

By Lemma 2.2, the operator represented by the infinite matrix Δ^{+r} is surjective from $s^0_\alpha(\Delta^{+r})$ to s^0_α. So $\Delta^+ \in S_\alpha$ and $(\alpha_{n+1}/\alpha_n)_n \in l_\infty$, and by Theorem 2.3(iii), we have $s^0_\alpha \subset s^0_\alpha(\Delta^{+r})$.

Now the condition $s^0_\beta \subset s^0_\alpha(\Delta^{+(r+1)})$ implies that $I \in (s^0_\beta, s^0_\alpha(\Delta^{+(r+1)}))$ and that $\Delta^{+(r+1)} \in S_{\beta,\alpha}$; and since all the diagonal entries of $\Delta^{+(r+1)}$ are equal to 1, we successively get $\beta/\alpha \in l_\infty$ and $s^0_\beta \subset s^0_\alpha$. Then

$$s^0_\alpha \subset s^0_\alpha(\Delta^{+r}) \subset s^0_\beta \subset s^0_\alpha$$

and $s^0_\alpha = s^0_\beta = s^0_\alpha(\Delta^{+r})$.

(ii) By (i) and Theorem 2.3(iii), condition (2.2) implies that $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ and $\alpha \in \widehat{C}_1$. Then the condition $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ implies that $s^0_\alpha \supset s^0_{(\alpha_{n+1})_n} \supset \cdots \supset s^0_{(\alpha_{n+j})_n}$ and $\alpha \in \widehat{C}_1$ implies that $(\alpha_{n-1}/\alpha_n)_n \in l_\infty$ and $s^0_\alpha \subset s^0_{(\alpha_{n+1})_n} \subset \cdots \subset s^0_{(\alpha_{n+j})_n}$ for all $j \geq 1$. This gives the conclusion. □

Concerning the sets $s^0_\alpha(\Delta^{+r}(\mu))$ and $s_\alpha(\Delta^{+r}(\mu))$, we deduce from the preceding the next result.

Lemma 2.11. *Let α, $\mu \in U^+$, and let $r \geq 1$ be an integer. If*

$$\alpha/\mu, \big(\alpha_{n-1}/(\mu_{n-1}\mu_n)\big)_n, \ldots, \big(\alpha_{n-r+1}/(\mu_{n-r+1}\cdots\mu_n)\big)_n \in \widehat{C_1} \tag{2.3}$$

holds, then we have

$$s^0_\alpha\big(\Delta^{+r}(\mu)\big) = s^0_{(\frac{\alpha_{n-r}}{\mu_{n-r}\cdots\mu_{n-1}})_n} \qquad \textit{and} \qquad s_\alpha\big(\Delta^{+r}(\mu)\big) = s_{(\frac{\alpha_{n-r}}{\mu_{n-r+1}\cdots\mu_{n-1}})_n}.$$

Proof. It is enough to show that under (2.3), $s^0_\alpha(\Delta^{+r}(\mu)) = s^0_{(\alpha_{n-r}/(\mu_{n-r}\cdots\mu_{n-1}))_n}$, the other proof being similar. Since $s^0_\alpha(\Delta^+(\mu)) = s^0_{\alpha/\mu}(\Delta^+)$ by Corollary 2.6(i), we have $s^0_\alpha(\Delta^+(\mu)) = s^0_{(\alpha_{n-1}/\mu_{n-1})_n}$ for $\alpha/\mu \in \widehat{C_1}$ and the lemma holds for $r = 1$. Now let j be an integer with $1 \leq j \leq r-1$ and assume that

$$s^0_\alpha\big(\Delta^{+j}(\mu)\big) = s_{(\frac{\alpha_{n-j}}{\mu_{n-j}\cdots\mu_{n-1}})_n}$$

for $\alpha/\mu, (\alpha_{n-1}/(\mu_{n-1}\mu_n))_n, \ldots, (\alpha_{n-j+1}/(\mu_{n-j+1}\cdots\mu_n))_n \in \widehat{C_1}$. Then

$$\begin{aligned} s^0_\alpha\big(\Delta^{+(j+1)}(\mu)\big) &= \big\{X \in s : \Delta^{+j}(\mu)\big(\Delta^+(\mu)X\big) \in s_\alpha\big\} \\ &= \big\{X \in s : \Delta^+(\mu)X \in s_\alpha\big(\Delta^{+j}(\mu)\big)\big\} \\ &= s^0_{(\alpha_{n-j}/(\mu_{n-j}\cdots\mu_{n-1}))_n}\big(\Delta^+(\mu)\big). \end{aligned}$$

Now the condition

$$\left(\frac{\alpha_{n-j}}{\mu_{n-j}\cdots\mu_{n-1}}\frac{1}{\mu_n}\right)_n \in \widehat{C_1} \tag{2.4}$$

implies that

$$s^0_{(\frac{\alpha_{n-j}}{\mu_{n-j}\cdots\mu_{n-1}})_n}\big(\Delta^+(\mu)\big) = s^0_{(\frac{\alpha_{n-1-j}}{\mu_{n-1-j}\cdots\mu_{n-2}}\frac{1}{\mu_{n-1}})_n} = s^0_{(\frac{\alpha_{n-(j+1)}}{\mu_{n-(j+1)}\cdots\mu_{n-1}})_n}.$$

Since condition (2.4) is equivalent to

$$\Big(\frac{\alpha_{n-(j+1)+1}}{\mu_{n-(j+1)+1}\cdots\mu_n}\Big)_n \in \widehat{C_1},$$

we have shown how $\alpha/\mu, (\alpha_{n-1}/(\mu_{n-1}\mu_n))_n, \dots, (\alpha_{n-(j+1)+1}/(\mu_{n-(j+1)+1}\cdots \mu_n))_n \in \widehat{C_1}$ implies that

$$s^0_\alpha\big(\Delta^{+(j+1)}(\mu)\big) = s^0_{(\frac{\alpha_{n-(j+1)}}{\mu_{n-(j+1)}\cdots\mu_{n-1}})_n}.$$

This completes the proof. $\square$

Remark 2.12. We immediately see that $\alpha \in \widehat{C_1}$ successively implies that

$$s_\alpha(\Delta^r) = s_\alpha, \qquad \overset{\circ}{s}_\alpha(\Delta^r) = \overset{\circ}{s}_\alpha, \qquad s_\alpha(\Delta^{+r}) = s_{(\alpha_{n-r})_n} \qquad \text{and}$$
$$\overset{\circ}{s}_\alpha(\Delta^{+r}) = \overset{\circ}{s}_{(\alpha_{n-r})_n}.$$

3. Properties of the sets $[V]_0(\Delta^{+r}(\mu))$ and $[V]_\infty(\Delta^{+r}(\mu))$

In this section, we give some conditions to have $x_k \to 0S(\Delta^{+r}(\mu))$. Among other things we also make explicit interesting subsets of $[V]_0(\Delta^{+r}(\mu))$ and $[V]_\infty(\Delta^{+r}(\mu))$. In the following we will use the condition

$$\sup_n\Big(\frac{1}{n}\sum_{k=1}^{n}\alpha_k\Big) < \infty \tag{3.1}$$

for given sequence $\alpha \in U^+$. We will put $\xi = (n)_n$, and we will denote by $C(\xi) = (c_{nm})_{n,m\geq 1}$, or C_1 for short, the *Cesàro operator* defined by $c_{nm} = 1/n$ for $m \leq n$ and $c_{nm} = 0$ otherwise. It can easily be seen that $C_1 \in (c_0, c_0)$ and $c_0(\Delta^{+r}(\mu)) \subset [V]_0(\Delta^{+r}(\mu))$. Since $Y = \Delta^{+r}(\mu)X \in c_0$ implies that $C_1(|Y|) \in c_0$, it is interesting to explicitly define a set $E \subset s$ such that

$$c_0\big(\Delta^{+r}(\mu)\big) \subsetneq E \subset [V]_0\big(\Delta^{+r}(\mu)\big). \tag{3.2}$$

We will see that we can take $E = s^0_\alpha(\Delta^{+r}(\mu))$ for $\alpha \notin l_\infty$ satisfying (3.1). In this way we are led to state the following result.

Proposition 3.1. *Let $\alpha \in U^+$. Assume there is a map $\varphi : \mathbb{N}^* \mapsto \mathbb{N}^*$ strictly increasing satisfying the following conditions:*

(i) $n^2/\varphi(n) = O(1)$ *$(n \to \infty)$,*
(ii) $\alpha_{\varphi(n)}/n = O(1)$ *$(n \to \infty)$,*
(iii) $(\sum_{k\in I, k\leq n}\alpha_k)/n = O(1)$ *$(n \to \infty)$ where $I = [\varphi(\mathbb{N}^*)]^c$.*

Then $\alpha \in U^+\backslash l_\infty$ satisfies (3.1).

Proof. Let $n \geq 1$ be an integer and let i_n be the greatest integer less than n, for which $\varphi(i_n) \leq n$. So we have $\varphi(i_n) \leq n < \varphi(i_n + 1)$. Then there is $K > 0$ such that

$$\begin{aligned}\frac{\sum_{k=1}^{n}\alpha_k}{n} &= \frac{\alpha_{\varphi(1)} + \cdots + \alpha_{\varphi(i_n)} + \sum_{k\in I, k\leq n}\alpha_k}{n}\\ &\leq K\frac{1+2+\cdots+i_n}{n} + \frac{1}{n}\sum_{k\in I, k\leq n}\alpha_k\end{aligned}$$

$$\leq K\frac{\frac{i_n(i_n+1)}{2}}{\varphi(i_n)}+\frac{1}{n}\sum_{k\in I,k\leq n}\alpha_k=O(1)\quad(n\to\infty).$$

So we have shown that $\alpha\notin l_\infty$ and $(\sum_{k=1}^n\alpha_k)/n=O(1)$ $(n\to\infty)$.

For example, let α be a sequence satisfying $\alpha_{2^i}=i$ for all $i\geq 1$ and $\alpha_k\leq K$ for given $K>0$ and for all $k\in I$, where $I=\{k\in\mathbb{N}^*: k\neq 2^i \text{ for all } i\geq 1\}$. Then (3.1) holds. □

Now we can state the main result.

Theorem 3.2. *Let α, $\mu\in U^+$, and let $r\geq 1$ be an integer. Assume that (2.3) and (3.1) hold. Then we have*

$$x_k\to 0S\big(\Delta^{+r}(\mu)\big)$$
for all sequences $(x_n)_n\in s$ *such that* $\lim_{n\to\infty}\frac{\mu_{n-1}\cdots\mu_{n-r}}{\alpha_{n-r}}x_n=0.$

Proof. We have $C(\xi)|\Delta^{+r}(\mu)X|\in c_0$ if and only if

$$\big|\Delta^{+r}(\mu)X\big|\in\Delta s^0_{(n)_n}.\tag{3.3}$$

Now (3.3) is satisfied when

$$\big|\Delta^{+r}(\mu)X\big|\in s^0_\alpha\tag{3.4}$$

with

$$s^0_\alpha\subset\Delta s^0_{(n)_n}.\tag{3.5}$$

Thus by Lemma 2.11, condition (2.3) implies that (3.4) holds if and only if

$$X\in s^0_\alpha\big(\Delta^{+r}(\mu)\big)=s^0_{(\alpha_{n-r}/(\mu_{n-r}\cdots\mu_{n-1}))_n}$$

and (3.1) means $D_{1/\xi}\Sigma D_\alpha\in(c_0,c_0)$, which is equivalent to (3.5). Now for $\varepsilon>0$, we get

$$\frac{1}{n}\sum_{k=1}^n\big|\Delta^{+r}(\mu)x_k\big|\geq\frac{1}{n}\sum_{k\in I_\varepsilon(n)}\big|\Delta^{+r}(\mu)x_k\big|\geq\frac{\varepsilon}{n}\big|\big\{k\leq n:\big|\Delta^{+r}(\mu)x_k\big|\geq\varepsilon\big\}\big|.$$

We easily conclude, since as we have just seen

$$\lim_{n\to\infty}\frac{\mu_{n-1}\cdots\mu_{n-r}}{\alpha_{n-r}}x_n=0\quad\text{implies that }\frac{1}{n}\sum_{k=1}^n\big|\Delta^{+r}(\mu)x_k\big|\to 0\quad(n\to\infty).$$ □

Remark 3.3. From Theorem 3.2 under (2.3) and (3.1), we have

$$s^0_{\left(\frac{\alpha_{n-r}}{\mu_{n-1}\cdots\mu_{n-r}}\right)}\subset[V]_0\big(\Delta^{+r}(\mu)\big).$$

Using a similar argument, we also have $s_{\left(\frac{\alpha_{n-r}}{\mu_{n-1}\cdots\mu_{n-r}}\right)}\subset[V]_\infty(\Delta^{+r}(\mu))$.

Remark 3.4. Note that for $\mu = (2^{-n})_n$, we immediately see that

$$c_0\big(\Delta^{+}(\mu)\big) = s_2^0 \subset [V]_0\big(\Delta^{+}(\mu)\big),$$

and that

$$c_0\big(\Delta^{+2}(\mu)\big) = s^0_{(1/2^{-(2n-3)})_n} = s_4^0 \subset [V]_0\big(\Delta^{+2}(\mu)\big).$$

In fact for $\alpha = e$, we get $\alpha/\mu = (2^n)_n$, $(\alpha_{n-1}/\mu_{n-1}\mu_n)_n = (2^{2n-1})_n \in \widehat{C_1}$.

Proposition 3.5. *Let β, $\mu \in U^+$, and let $r \geq 1$ be an integer. Assume that $(n/\beta_n)_n \in l_\infty$ and $\beta/\mu, (\beta_{n-1}/(\mu_{n-1}\mu_n))_n, \dots, (\beta_{n-r+1}/(\mu_{n-r+1}\cdots\mu_n))_n \in \widehat{C_1}$. Then we have*

$$\begin{aligned} [V]_0\big(\Delta^{+r}(\mu)\big) &\subset s^0_{(\beta_{n-r}/(\mu_{n-r}\cdots\mu_{n-1}))_n} \qquad \text{and} \\ [V]_\infty\big(\Delta^{+r}(\mu)\big) &\subset s_{(\beta_{n-r}/(\mu_{n-r}\cdots\mu_{n-1}))_n}. \end{aligned} \tag{3.6}$$

Proof. As we have seen above, $X \in [V]_0(\Delta^{+r}(\mu))$ if and only if

$$\big|\Delta^{+r}(\mu)X\big| \in \Delta s^0_{(n)_n}.$$

Now since $(n/\beta_n) \in l_\infty$, we have $D_{1/\beta}\Delta D_{(n)_n} \in (c_0, c_0)$ and $\Delta s^0_{(n)_n} \subset s^0_\beta$. Therefore, $|\Delta^{+r}(\mu)X| \in s^0_\beta$ and $X \in s^0_\beta(\Delta^{+r}(\mu))$. Finally by Lemma 2.11, we have

$$s^0_\beta\big(\Delta^{+r}(\mu)\big) = s^0_{(\beta_{n-r}/(\mu_{n-r}\cdots\mu_{n-1}))_n}$$

and the first inclusion given in (3.6) holds. The other inclusion can be shown similarly. □

Corollary 3.6. *Let β, $\mu \in U^+$, and assume that $(n/\beta_n)_{n\geq 1} \in l_\infty$ and $\beta/\mu \in \widehat{C_1}$. Then we have*

$$[V]_0\big(\Delta^{+}(\mu)\big) \subset s^0_{(\beta_{n-1}/\mu_{n-1})_n} \qquad \text{and} \qquad [V]_\infty\big(\Delta^{+}(\mu)\big) \subset s_{(\beta_{n-1}/\mu_{n-1})_n}.$$

We also have the next result.

Proposition 3.7. *Let α, $\mu \in U^+$ satisfy the conditions $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ and $(\sum_{k=1}^n \alpha_k\mu_k)/n = O(1)$ $(n \to \infty)$. Then $x_k \to 0S(\Delta^+(\mu))$ for all $X \in s^0_\alpha$.*

Proof. Let $X \in s^0_\alpha$. First, we have $D_{(1/n)_n}\Sigma D_{\mu\alpha} = (\sigma_{nm})_{n,m\geq 1}$ with $\sigma_{nm} = \mu_m\alpha_m/n$ for $m \leq n$ and $\sigma_{nm} = 0$ for $m > n$; and since $(\sum_{k=1}^n \alpha_k\mu_k)/n = O(1)$ $(n \to \infty)$, we deduce that $D_{(1/n)_n}\Sigma D_{\mu\alpha} \in S_1$ and $\sigma_{nm} \to 0$ $(n \to \infty)$ for all m. So by the characterization of (c_0, c_0), we conclude that $D_{(1/n)_n}\Sigma D_{\mu\alpha} \in (c_0, c_0)$. Then $(\Sigma D_\mu)s^0_\alpha \subset s^0_{(n)_n}$ and $s^0_\alpha \subset D_{1/\mu}\Delta s^0_{(n)_n}$. Now, by Corollary 2.6 the condition $(\alpha_n/\alpha_{n-1})_n \in l_\infty$ implies that $s^0_\alpha \subset s^0_\alpha(\Delta^+)$; we then have $|\Delta^+X| \in s^0_\alpha$ for all $X \in s^0_\alpha$. Then $|\Delta^+X| \in D_{1/\mu}\Delta s^0_{(n)_n}$ (i.e., $C((n)_n)|\Delta^+(\mu)X| \in c_0$ for all $X \in s^0_\alpha$). This concludes the proof. □

These results lead us to compare Theorem 3.2 with Proposition 3.7. In this way we can finish the next remark.

Remark 3.8. Note that Theorem 3.2 does not imply Proposition 3.7, and conversely Proposition 3.7 does not imply Theorem 3.2. Indeed, let α be a sequence defined by $\alpha_{2^i} = i$ for all $i \geq 1$ and $\alpha_k = 1$ for all $k \in I$, where $I = \{k \in \mathbb{N}^* : k \neq 2^i \text{ for all } i \geq 1\}$. For $\mu = (2^{-n})_n$, we deduce that Theorem 3.2 holds for $r = 1$ and that Proposition 3.7 cannot be satisfied. Now take $\alpha = 1/\mu = (2^n)_n$; then $x_k \to 0 S(\Delta^+(\mu))$ by Proposition 3.7 and α does not satisfy (2.3) and Theorem 3.2 with $r = 1$.

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