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NOTE ON LINEARITY OF REARRANGEMENT-INVARIANT SPACES

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ABSTRACT. For some rearrangement-invariant functional $\|\cdot\|_X$ having the lattice property, we give a characterization of linearity of the set $\{f : \|f\|_X < \infty\}$. Afterwards we apply this general abstract theorem in the case of Orlicz–Lorentz spaces.

1. Introduction

The concept of *rearrangement-invariant spaces* plays an important role in analysis and its applications. Since the concept was first presented in the 1930s, it has served a number of important applications, such as partial differential equations and Sobolev space theory. It has been intensively studied since the 1950s, starting with the famous pioneering paper [7] in which the so-called *classical Lorentz spaces* were introduced. It was also shown in that very same article, however, that the functional which governs these spaces is not necessarily a norm in general. In certain cases these "spaces" do not even need to be linear sets. The main reason for this fact is that the operator which associates a measurable function with its nonincreasing rearrangement is not subadditive. Many authors have studied functional properties of these spaces. In 1990, Sawyer in [9] characterized the normability of classical Lorentz spaces. Many papers, with characterizations of linearity (see [4]) quasinormability (see [3]) and normability ([2]) followed (see also [5]).

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General properties of Banach lattices were also studied in [6], and the notion of space symetrization was considered. In the following pages we will adopt a similar approach when studying the problem of linearity of a rearrangement-invariant lattice.

Our principal goal in this paper is to establish necessary and sufficient conditions for a rearrangement-invariant lattice to be linear set. It turns out that under these circumstances such conditions depend on finiteness of the dilation operator.

We also point out some applications of this result (including an alternative approach to the characterization of linearity of Orlicz–Lorentz) which enjoy the above-mentioned properties. In the case of linearity, the result is known (see [4]) but it assumes the Δ_2-condition of the function φ. In the present article, we will present a stronger version of this theorem without that restriction.

2. Preliminaries and main theorem

Let $(\mathcal{R}, \mu)$ be a nonatomic, σ-finite measure space. Denote the set of all real-valued μ-measurable functions on $\mathcal{R}$ by $\mathcal{M}(\mathcal{R})$. In the special case when $\mathcal{R} = (0, \infty)$, we write $\mathcal{M}(0, \infty)$. For $f \in \mathcal{M}(\mathcal{R})$, we will define the *distribution function* by

$$f_*(t) := \mu\big\{|f| > t\big\}$$

and the *nonincreasing rearrangement* of f by

$$f^*(t) := \inf\big\{s \in [0, \infty) : f_*(s) \leq t\big\}, \quad t \in [0, \infty).$$

For $a > 0$, we denote the *dilation operator* E_a by

$$E_a g(t) = g(a^{-1}t) \quad \text{for } g \in \mathcal{M}(0, \infty) \text{ and } t \in (0, \infty).$$

It is known that the operation $f \mapsto f^*$ is not subadditive, and instead we have the following inequality

$$(f + g)^*(s) \leq E_2 f^*(s) + E_2 g^*(s) \quad \text{for every } f, g \in \mathcal{M}(\mathcal{R}) \text{ and } s \in (0, \infty). \tag{2.1}$$

We will also use the term *weight* for a positive locally integrable function defined on $(0, \infty)$. For a weight w, we will define function W by the following formula:

$$W(t) := \int_0^t w(s)\, \mathrm{d}s, \quad t \in [0, \infty).$$

When a functional $\|\cdot\|_X : \mathcal{M}(\mathcal{R}) \to [0, \infty]$ is given, we denote

$$X := \big\{f \in \mathcal{M}(\mathcal{R}) : \|f\|_X < \infty\big\}.$$

Definition 2.1. We call X a *rearrangement-invariant (RI) lattice* if the following conditions are satisfied:

(P1) If $f^* = g^*$, then $\|f\|_X = \|g\|_X$.
(P2) If $|f| \leq |g|$ μ-a.e., then $\|f\|_X \leq \|g\|_X$.
(P3) $\|af\|_X = |a|\|f\|_X$.

We call $\|\cdot\|_X$ a *quasinorm* if (P1)–(P3) hold and, moreover, if the inequality

$$\|f+g\|_X \leq C\big(\|f\|_X + \|g\|_X\big)$$

holds for some $C \in (0,\infty)$ independent on f and g. If X is an RI lattice and if there exists a functional

$$\|\cdot\|_{\bar{X}} : \mathcal{M}(0,\infty) \to [0,\infty]$$

satisfying

$$\|f\|_X = \|f^*\|_{\bar{X}},$$

then we say that $\|\cdot\|_{\bar{X}}$ is the *representation functional* of X.

We will use the following immediate consequence of *Hardy's lemma* (see [1, Proposition 3.6, p. 56], see also [9]): for given weights w, v we have

$$\sup_{f\in\mathcal{M}(\mathcal{R})} \frac{\int_0^\infty f^*(s)w(s)\,\mathrm{d}s}{\int_0^\infty f^*(s)v(s)\,\mathrm{d}s} = \sup_{t>0}\frac{W(t)}{V(t)}.$$

Although the following lemma is a simple observation and a kind of folklore, let us make it convenient to the reader by listing it with a proof, which is a little bit technical.

Lemma 2.2. *Let $(\mathcal{R},\mu)$ be a nonatomic σ-finite measure space. Let $h \in \mathcal{M}(0,\mu(\mathcal{R}))$ be a nonnegative, nonincreasing, and right-continuous function. Then there exists a function $f \in \mathcal{M}(\mathcal{R})$ such that $f^* = h$.*

Proof. Let us first suppose that h is a simple function. Let

$$h = \sum_{i=1}^{l} a_i I_i,$$

where I_i are disjointed intervals. Since $(\mathcal{R},\mu)$ is nonatomic, there exist $A_i \subset \mathcal{R}$ disjointed with $\mu(A_i) = |I_i|$ (see [1, Lemma 2.5, p. 46]). If we set

$$f := \sum_{i=1}^{l} a_i \chi_{A_i},$$

then we have $f^* = h$, as desired.

For general nonincreasing nonnegative function h, we will find simple functions h_n such that $0 \leq h_n \uparrow h$ and f_n such that $f_{n+1} \geq f_n$ and $f_n^* = h_n$. Then if we define

$$f := \lim_n f_n,$$

we have $f^* = h$ (by [1, Proposition 1.7, p. 41]).

Now, let us conctruct such a sequence in the following way. For $k, l \in \mathbb{N}$, set

$$H_l^k := \left\{\frac{l}{2^k} < h \leq \frac{l+1}{2^k}\right\}$$

and find $F_l^k \subset \mathcal{R}$ such that $\mu(F_l^k) = |H_l^k|$, $F_j^k \cap F_i^k = \emptyset$ for $i \neq j$ and $F_{2l}^{k+1} \cup F_{2l+1}^{k+1} = F_l^k$. Define

$$f_n := \sum_{i=1}^{n2^n-1} \frac{i}{2^n} \chi_{F_i^n} + n\chi_{F_n},$$

where $F_n = \bigcup_{j \geq n} F_j^1$, and set

$$h_n := f_n^*.$$

One readily checks that such a sequence has the required properties. □

Our main result is statement (i) in the following theorem.

Theorem 2.3. *Let X be an RI lattice for which there exists a representation functional $\|\cdot\|_{\bar{X}}$.*

(i) *Assume that the space*

$$\bar{X} := \big\{ f \in \mathcal{M}(0,\infty) : \|f\|_{\bar{X}} < \infty \big\}$$

is a linear set. Then the space X is a linear set if and only if the following implication holds:

$$\textit{If } \|f^*\|_{\bar{X}} < \infty, \ \textit{then } \|E_2 f^*\|_{\bar{X}} < \infty. \tag{2.2}$$

(ii) *Assume that $\|\cdot\|_{\bar{X}}$ is a quasinorm. Then $\|\cdot\|_X$ is a quasinorm if and only if there exists a positive constant C such that*

$$\|E_2 f^*\|_{\bar{X}} \leq C\|f^*\|_{\bar{X}}. \tag{2.3}$$

(iii) *Assume that $\|\cdot\|_{\bar{X}}$ is a norm. Then $\|\cdot\|_X$ is a norm if and only if*

$$\|E_2 f^*\|_{\bar{X}} \leq 2\|f^*\|_{\bar{X}}. \tag{2.4}$$

Proof. (i) Assume first that (2.2) holds. Let there be f, g such that $\|f\|_X < \infty$ and $\|g\|_X < \infty$. Then, by (2.1) and (2.2), we get

$$\|f+g\|_X = \big\|(f+g)^*\big\|_{\bar{X}} \leq \|E_2 f^* + E_2 g^*\|_{\bar{X}} < \infty.$$

Conversely, let f be such that $\|f^*\|_{\bar{X}} < \infty$ but $\|E_2 f^*\|_{\bar{X}} = \infty$. Let us first suppose that $\mu(\mathcal{R}) = \infty$. Then there exists two sets of infinite measure $E, M \subset \mathcal{R}$ such that $E \cap M = \emptyset$. Now (E,μ) and (M,μ) are two measure spaces; therefore, according to Lemma 2.2, there exist functions $\tilde{h} \in \mathcal{M}(E)$ and $\tilde{g} \in \mathcal{M}(M)$, with

$$\tilde{g}^* = \tilde{h}^* = f^*.$$

Let us extend them by zero at the rest of $\mathcal{R}$ to functions h and g. We have

$$g^* = h^* = f^*$$

and

$$(g+h)_*(t) = 2f_*(t).$$

Hence

$$(g+h)^*(s) = E_2 f^*(s) \quad \text{for } s \in (0,\infty).$$

Therefore, $g, h \in X$, but $g + h \notin X$. Consequently, X is not a linear set.

In the case when $r := \mu(\mathcal{R}) < \infty$, we find a μ-measurable set $E \subset \mathcal{R}$ such that

$$\Big\{|f| > f^*\Big(\frac{r}{2}\Big)\Big\} \subset E \subset \Big\{|f| \geq f^*\Big(\frac{r}{2}\Big)\Big\}, \tag{2.5}$$

and $\mu(E) = r/2$. Then there exists h, g with disjointed supports satisfying

$$h^* = g^* = (f\chi_E)^*,$$

and we have

$$\|h+g\|_X = \big\|(h+g)^*\big\|_{\bar{X}} \geq \|E_2 f^*\|_{\bar{X}}.$$

Therefore, $g, h \in X$ but $g + h \notin X$, and hence, again, X is not a linear set.

For proof of the statement (ii), see [6, Lemma 1.4]. The proof of (iii) is analogous to that of (ii). □

Remark 2.4. Let $\delta > 1$. Then the conditions (2.2) and (2.3) can be respectively replaced by

$$\|f^*\|_{\bar{X}} < \infty \quad \text{then } \|E_\delta f^*\|_{\bar{X}} < \infty$$

and

$$\|E_\delta f^*\|_{\bar{X}} \leq C\|f^*\|_{\bar{X}}.$$

3. Applications

In this section, we will illustrate the results obtained on the particular example of Lorentz–Orlicz spaces. We start with a general definition of a general structure that covers such spaces. These spaces first appeared in [8, Definition 7.2].

Definition 3.1. Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$. Let w be a weight. Then we define the functional

$$\|f\|_{\Lambda_{\varphi,w}} := \inf\Big\{\lambda : \int_0^\infty \varphi\Big(\frac{f^*(s)}{\lambda}\Big)w(s)\,\mathrm{d}s \leq 1\Big\}$$

and the set

$$\Lambda_{\varphi,w} := \big\{f \in \mathcal{M}(\mathcal{R}) : \|f\|_{\Lambda_{\varphi,w}} < \infty\big\}.$$

We note that, clearly, $\Lambda_{\varphi,w}$ is an RI lattice. Furthermore, the representation functional is therefore defined as

$$\|f\|_{L^\varphi_w} := \inf\Big\{\lambda : \int_0^\infty \varphi\Big(\frac{|f(s)|}{\lambda}\Big)w(s)\,\mathrm{d}s \leq 1\Big\}, \quad f \in \mathcal{M}(0,\infty).$$

Note that

$$L^\varphi_w = \Big\{f \in \mathcal{M}(0,\infty) : \exists\lambda \in (0,\infty) : \int_0^\infty \varphi\Big(\frac{|f(s)|}{\lambda}\Big)w(s)\,\mathrm{d}s < \infty\Big\}.$$

The following lemma is a classical result (for a more general form, see [8]).

Lemma 3.2. *Let φ and w be as in Definition 3.1. Then*

(i) $\|\cdot\|_{L^\varphi_w}$ *has lattice property,*
(ii) L^φ_w *is a linear set.*

The following theorem is known in a weaker form (with the additional assumption of $\varphi \lesssim E_2\varphi$) (see [4, Theorem 4.1]). We will point an alternative proof based on Theorem 2.3 removing the assumption.

Theorem 3.3. *Let φ and w have the same properties as in Definition 3.1. Then the following conditions are equivalent:*

(i) $\Lambda_{\varphi,w}$ *is linear,*

(ii)

$$\sup_{t>0}\frac{W(2t)}{W(t)} < \infty.$$

Proof. According to Lemma 3.2, the representation space meets the assumptions of Theorem 2.3. Let us first prove that (ii) implies (i). We have

$$\begin{aligned}
&\int_0^\infty \varphi\Big(\frac{f^*(\frac{s}{2})}{\lambda}\Big)w(s)\,\mathrm{d}s \\
&\quad= 2\int_0^\infty \varphi\Big(\frac{f^*(t)}{\lambda}\Big)w(2t)\,\mathrm{d}t \\
&\quad\le 2\sup_f\Big(\frac{\int_0^\infty \varphi(\frac{f^*(t)}{\lambda})w(2t)\,\mathrm{d}t}{2\int_0^\infty \varphi(\frac{f^*(t)}{\lambda})w(t)\,\mathrm{d}t}\Big)\int_0^\infty \varphi\Big(\frac{f^*(t)}{\lambda}\Big)w(t)\,\mathrm{d}t \qquad (3.1)\\
&\quad= \sup_{t>0}\frac{W(2t)}{W(t)}\int_0^\infty \varphi\Big(\frac{f^*(t)}{\lambda}\Big)w(t)\,\mathrm{d}t \\
&\quad\le C\int_0^\infty \varphi\Big(\frac{f^*(t)}{\lambda}\Big)w(t)\,\mathrm{d}t,
\end{aligned}$$

where the last inequality follows immediately from Hardy's lemma. This proves (i) via Theorem 2.3(i).

Now, let us assume that condition (ii) is violated. Then we may pick a sequence $\{t_n\}$ such that

$$\frac{W(2t_n)}{W(t_n)} > 4^n \quad \text{for all } n\in\mathbb{N}.$$

Since the function $\frac{W(2t)}{W(t)}$ is continuous, and therefore locally bounded on $(0,\infty)$, we may assume that either $t_n \uparrow \infty$ or $t_n \downarrow 0$. Let us first suppose that $t_n \uparrow \infty$. Since in this case the weight cannot be integrable, we may assume (by a picking suitable subsequence if necessary) that

$$W(t_n) \ge 2W(t_{n-1}), \qquad W(2t_n) > 2W(2t_{n-1}), \qquad \text{and} \qquad \int_{t_{k-1}}^{t_k} w(s)\,\mathrm{d}s\uparrow.$$

Then

$$\frac{W(2t_n)-W(2t_{n-1})}{W(t_n)-W(t_{n-1})} \ge c4^n \quad \text{for some } c>0 \text{ and all } n\in\mathbb{N}.$$

For our technical convenience, we set $t_0 := 0$. Now, let us define a sequence of functions $\{f_n\}\subset\mathcal{M}(\mathcal{R})$ with pairwise disjoint supports and such that

$$f_n^* = \chi_{(0,t_n-t_{n-1})}\varphi^{-1}\Big(\Big(2^n\int_{t_{n-1}}^{t_n} w\Big)^{-1}\Big) \quad \text{for } n\in\mathbb{N}.$$

Set

$$f := \sum_{i=1}^{\infty} f_n.$$

Calculation shows that

$$\begin{aligned}\int_0^\infty \varphi\big(f^*(s)\big)w(s)\,\mathrm{d}s &= \sum_{k=1}^{\infty}\int_{t_{k-1}}^{t_k}\varphi\Big(\varphi^{-1}\Big(\Big(2^k\int_{t_{k-1}}^{t_k} w\Big)^{-1}\Big)\Big)w(s)\,\mathrm{d}s\\ &= \sum_{k=1}^{\infty}2^{-k} = 1.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_0^\infty \varphi\big(E_2 f^*(s)\big)w(s)\,\mathrm{d}s &= 2\int_0^\infty \varphi\big(f^*(t)\big)w(2t)\,\mathrm{d}t\\ &\geq \int_{t_{k-1}}^{t_k}\frac{w(2t)}{2^k W(t_k)-W(t_{k-1})}\,\mathrm{d}t\\ &\geq \frac{W(2t_k)-W(2t_{k-1})}{2^k(W(t_k)-W(t_{k-1}))} \geq c2^k\end{aligned}$$

for some $c > 0$ and all $k \in \mathbb{N}$. Therefore, $E_2 f^* \notin L^\varphi_w$, which implies that $\Lambda_{\varphi,w}$ is not a linear set. On the other hand, if $t_k \downarrow 0$, then we may suppose that

$$W(t_{n-1}) \geq 2W(t_n), \qquad W(2t_{n-1}) \geq 2W(2t_n) \qquad \text{and} \qquad \int_{t_n}^{t_{n-1}} w(s)\,\mathrm{d}s \downarrow .$$

Now we find f_n with disjointed supports such that

$$f_n^* = \chi_{(0,t_{n-1}-t_n)}\varphi^{-1}\Big(\Big(2^n\int_{t_n}^{t_{n-1}} w\Big)^{-1}\Big).$$

We have

$$\begin{aligned}\|f\|_{\Lambda_{\varphi,w}} &= \sum_{n=2}^{\infty}\int_{t_n}^{t_{n-1}}\frac{w(t)}{2^n(W(t_{n-1})-W(t_n))}\,\mathrm{d}t\\ &= \sum_{n=2}^{\infty}2^{-n} = \frac{1}{2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|E_2 f^*\|_{L^\varphi_w} &= 2\sum_{n=2}^{\infty}\int_{t_n}^{t_{n-1}}\frac{w(2t)}{2^{-n}(W(t_{n-1})-W(t_n))}\\ &\geq 2\int_{t_n}^{t_{n-1}}\frac{w(2t)}{2^{-n}(W(t_{n-1})-W(t_n))} \geq 2^n,\end{aligned}$$

whence $E_2 f^* \notin L^\varphi_w$. This shows that $\Lambda_{\varphi,w}$ is not linear, which is a contradiction. The proof is complete. $\square$

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