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BLOCH–ORLICZ FUNCTIONS WITH HADAMARD GAPS

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Abstract. In this paper, we give a sufficient and necessary condition for an analytic function $f(z)$ on the unit disc $\mathbb{D}$ with Hadamard gaps, that is, for $f(z)=\sum_{k=1}^{\infty} a_k z^{n_k}$ where $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \in \mathbb{N}$, belongs to the Bloch–Orlicz space $\mathcal{B}^{\varphi}$. As an application of our results, the compactness of composition operator are discussed.

1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ be the class of all holomorphic functions on $\mathbb{D}$. Recall that the Bloch-type space, for example, the α-Bloch space, for $\alpha > 0$, denote as $\mathcal{B}^{\alpha}$, consists of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{\alpha} := \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha}|f'(z)| < \infty.$$

The α-Bloch space is introduced and studied by numerous authors. For general theory of α-Bloch functions see [13]. Recently, a different class of Bloch-type space defined on $\mathbb{D}$ is studied by many authors, where the typical weight function, $w(z) = 1-|z|^2 (z \in \mathbb{D})$, is replaced by bounded continuous positive function μ (see [4, 9]). More precisely, a function $f \in H(\mathbb{D})$ is called a μ-Bloch function, denoted as $f \in \mathcal{B}^{\mu}$, if

$$\|f\|_{\mu} := \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

Clearly, if $\mu(z) = w(z)^{\alpha}$ with $\alpha > 0$, $\mathcal{B}^{\mu}$ is just the α-Bloch space. It is readily seen that $\mathcal{B}^{\mu}$ is a Banach space with the norm $\|f\|_{\mathcal{B}^{\mu}} := |f(0)| + \|f\|_{\mu}$.

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The weighted Bloch space appears in 1991, when Brown and Shields [3] showed that an analytic function μ is a multiplier on the Bloch space $\mathcal{B}$ if and only if $\mu \in \mathcal{B}^{\log}$. The logarithmic Bloch-type space with $\mu(z) = w(z)^{\alpha} \ln^{\beta} \frac{e}{w(z)}$, $\alpha > 0$ and $\beta \geq 0$, is studied by Stević [14].

By using a Young's functions, the Bloch–Orlicz space is introduced by Ramos–Fernández (see [11, 12]). More precisely, let $\varphi : [0, \infty) \to [0, \infty)$ be an $\mathcal{N}$-function, that is, φ is a strictly increasing convex function such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \frac{t}{\varphi(t)} = \lim_{t\to 0} \frac{\varphi(t)}{t} = 0$. The Bloch–Orlicz space associated with the function φ, denoted as $\mathcal{B}^{\varphi}$, is the class of all analytic functions f in $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)\varphi(\lambda|f'(z)|) < \infty, \tag{1.1}$$

for some $\lambda > 0$ depending on f. We can observe that when $\varphi(t) = t$ with $t \geq 0$, it gets back to the Bloch space $\mathcal{B}$. Furthermore, we can suppose, without loss of generality, that φ^{-1} is continuously differentiable on $(0, \infty)$. In fact, if φ^{-1} is not differentiable everywhere, we set the function

$$\psi(t) = \int_0^t \frac{\varphi(x)}{x} dx (t \geq 0),$$

then ψ is differentiable, whence ψ^{-1} is differentiable everywhere on $(0, \infty)$. Furthermore, since φ is a strictly increasing convex function satisfying $\varphi(0) = 0$, the function$\frac{\varphi(t)}{t}, t > 0$ is increasing and

$$\varphi(t) \geq \psi(t) \geq \int_{t/2}^t \frac{\varphi(x)}{x} dx \geq \varphi\left(\frac{t}{2}\right)$$

for all $t > 0$. Hence, $\mathcal{B}^{\varphi} = \mathcal{B}^{\psi}$. By the convexity of φ, it is not hard to see that the Minkowski's functional

$$\|f\|_{\varphi} = \inf\left\{k > 0 : S_{\varphi}\left(\frac{f'}{k}\right) \leq 1\right\}, \tag{1.2}$$

defines a seminorm for $\mathcal{B}^{\varphi}$, which, in this case, is known as Luxemburg's seminorm, where

$$S_{\varphi}(f) := \sup_{z\in\mathbb{D}} (1 - |z|^2)\varphi(|f(z)|).$$

In fact, it can be shown that $\mathcal{B}^{\varphi}$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^{\varphi}} = |f(0)| + \|f\|_{\varphi}.$$

Let the Green's function of $\mathbb{D}$ be defined as $g(z, a) = \log \frac{1}{|\sigma_a(z)|}$, where $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ is the automorphism of $\mathbb{D}$ interchanging the points zero and $a \in \mathbb{D}$. Let $p > 0$, $q > -2$ and let $K : [0, \infty) \to [0, \infty)$ be right continuous and nondecreasing function. We say a function f analytic in $\mathbb{D}$, belongs to $Q_K(p, q)$, if

$$\|f\|^p_{Q_K(p,q)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z, a))\, dA(z) < \infty. \tag{1.3}$$

Here and elsewhere dA stands for the Euclidean area element.

The space $Q_K(p,q)$, equipped with the norm $\|f\| = |f(0)| + \|f\|_{Q_K(p,q)}$ for $p \geq 1$, is a Banach space (see [16]). In fact, the definition of $Q_K(p,q)$ is strongly motivated by the studying of spaces Q_p, $F(p,q,s)$ and Q_K, where Q_K and $F(p,q,s)$ are different natural generalizations of Q_p (see [15, 1]). Namely, $Q_K(2,0) = Q_K$ and $Q_K(p,q) = F(p,q,s)$ when $K(t) = t^s$ (see [6]). In particular, if $K(t) = t$, then Q_K coincides with $BMOA$, the space of analytic functions in the Hardy space H^1 with boundary values of bounded mean oscillation (see [2]). Moreover, the spaces $Q_K(p,q)$ and $Q_{K,0}(p,q)$ are subsets of spaces $\mathcal{B}^\alpha$ and $\mathcal{B}^\alpha_0$, respectively, when $\alpha = (q+2)/p$.

Let ϕ be a holomorphic self-mapping of $\mathbb{D}$, the symbol ϕ induces a linear composition operator $C_\phi(f) = f \circ \phi$ from $H(\mathbb{D})$ into itself. The boundedness and compactness of the operator $C_\phi : X \to Y$, where X and Y are some normalized spaces of analytic functions in the unit disc, have attracted many authors interest (see [14, 19]).

We say that an analytic function f on the unit disc $\mathbb{D}$ has Hadamard gaps if $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$, where $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \in \mathbb{N}$. Results on analytic functions with Hadamard gaps, see, e.g., [5, 7, 8, 10].

The following result for Hadamard gaps, on α-Bloch space, is obtained by Yamashita (see [18]).

Theorem A: Assume that f is an analytic function on $\mathbb{D}$ with Hadamard gaps. Then for $\alpha > 0$, the following two propositions hold:

(1) $f \in \mathcal{B}^\alpha$ if and only if $\lim_{k\to\infty} \sup |a_k| n_k^{1-\alpha} < \infty$.

(2) $f \in \mathcal{B}^\alpha_0$ if and only if $\lim_{k\to\infty} \sup |a_k| n_k^{1-\alpha} = 0$.

By applying Theorem A, Xiao in [17] obtains the following result.

Theorem B: Let $\alpha \in (0,\infty)$. Then there exist two functions $f_1, f_2 \in \mathcal{B}^\alpha$ such that

$$|f_1'(z)| + |f_2'(z)| \geq \frac{C}{(1-|z|^2)^\alpha}, \quad z \in \mathbb{D}.$$

In view of the fundamental importance of the Hadamard gaps in function space, we consider the nature posed problem: how it looks like in Bloch–Orlicz space $\mathcal{B}^\varphi$? This is logical since the Bloch–Orlicz space is the most natural generalization of the Bloch-type space. In this paper, we establish the Bloch–Orlicz extension of the Hardmard gaps, and the boundedness and compactness of the composition operator C_ϕ from $\mathcal{B}^\varphi$ into $Q_K(p,q)$.

Throughout this paper, positive constant is denoted by C, and it may have different value at different place.

The paper is organized as follows. In section 2, we introduce some lemmas and give the Hadamard gaps in Bloch–Orlicz space $\mathcal{B}^\varphi$. Some applications of our results are presented in section 3.

2. Hadamard gap series belonging to $\mathcal{B}^{\varphi}$

First, we give the following Lemma.

Lemma 2.1. *For any $f \in \mathcal{B}^{\varphi}\backslash\{0\}$, the following relation*

$$S_{\varphi}\left(\frac{f'}{\|f\|_{\varphi}}\right) \leq 1 \tag{2.1}$$

holds. The inequality (2.1) allows us to obtain that

$$|f'(z)| \leq \varphi^{-1}\left(\frac{1}{1-|z|^2}\right)\|f\|_{\varphi}, \tag{2.2}$$

for all $f \in \mathcal{B}^{\varphi}$ and for all $z \in \mathbb{D}$.

The inequality (2.2) implies that the evaluation functional defined as $T_z(f) := F(z)$, where $z \in \mathbb{D}$ is fixed and $f \in \mathcal{B}^{\varphi}$, is continuous on $\mathcal{B}^{\varphi}$. In fact, for $z \in \mathbb{D}$ fixed and any $f \in \mathcal{B}^{\varphi}$, we have

$$\begin{aligned} |T_z(f)| = |f(z)| &\leq |f(0)| + \int_{[0,z]} |f'(s)||ds| \\ &\leq \left(1 + \int_0^1 \varphi^{-1}\left(\frac{1}{1-|z|^2t^2}\right)dt\right)\|f\|_{\mathcal{B}^{\varphi}}. \end{aligned}$$

From the definition of the Luxemburg seminorm and the expression (2.1), we have that

$$S_{\varphi}(f') \leq 1 \Longleftrightarrow \|f\|_{\varphi} \leq 1.$$

for any $f \in \mathcal{B}^{\varphi}$.

Also, as an easy consequence of (2.1), we have that the Bloch–Orlicz space is isometrically equal to the μ-Bloch space, where

$$\mu(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{1-|z|^2}\right)}, \tag{2.3}$$

with $z \in \mathbb{D}$. Thus for any $f \in \mathcal{B}^{\varphi}$, we have

$$\|f\|_{\varphi} = \sup_{z\in\mathbb{D}} \mu(z)|f'(z)|. \tag{2.4}$$

Furthermore, if $\varphi(t) = t^p$ with $p > 1$ and $t \geq 0$, then the Bloch–Orlicz space coincides with the α-Bloch space when $\alpha = \frac{1}{p} \in (0, 1)$.

Remark 1. Throughout this paper, we use $\|f\|_{\varphi}$ or $\|f\|_{\mu}$ to evaluate the norm of function $f \in \mathcal{B}^{\varphi} = \mathcal{B}^{\mu}$. Where μ is the weight defined in (2.3).

The following result is proved in [11], hence we omit it's proof.

Lemma 2.2. *Let $a \in \mathbb{D}$ be fixed. Then exists a holomorphic function $f_a \in H(\mathbb{D})$ such that*

$$\varphi(|f_a(z)|) = \frac{1-|a|^2}{|1-\bar{a}z|^2},$$

for all $z \in \mathbb{D}$.

Remark 2. It is clear that for any $a \in \mathbb{D}$, the function

$$g_a(z) = \int_0^z f_a(s)ds$$

with $z \in \mathbb{D}$ and f_a is the function found in Lemma 2.2 belongs to the space $\mathcal{B}^\varphi$.

We need the following lemma in proving the main results of this paper.

Lemma 2.3. *Assume $a > 0$, $\alpha \in (0, 1/a)$, μ is a positive nonincreasing bounded continuous function, such that*

$$\lim_{x\to\infty} \frac{x\left(\frac{1}{\mu\left(1-\frac{1}{x}\right)}\right)'}{\frac{1}{\mu\left(1-\frac{1}{x}\right)}} = 0 \tag{2.5}$$

and

$$\lim_{x\to\infty} x\left(\frac{1}{\mu(1-\frac{1}{x})}\right) = \infty. \tag{2.6}$$

Then, there is a positive constant C independent of α such that

$$\int_a^\infty \frac{e^{-\alpha t}}{\mu\left(1-\frac{1}{t}\right)}dt \le \frac{C}{\alpha\mu(1-\alpha)}. \tag{2.7}$$

Proof. We have

$$\begin{aligned}\int_a^\infty \frac{e^{-\alpha t}}{\mu\left(1-\frac{1}{t}\right)}dt &= \int_a^{1/\alpha} \frac{e^{-\alpha t}}{\mu\left(1-\frac{1}{t}\right)}dt + \int_{1/\alpha}^\infty \frac{e^{-\alpha t}}{\mu\left(1-\frac{1}{t}\right)}dt \\ &\le \int_a^{1/\alpha} \frac{dt}{\mu\left(1-\frac{1}{t}\right)} + \frac{1}{\mu(1-\alpha)}\int_{1/\alpha}^\infty e^{-\alpha t}dt \\ &= \int_a^{1/\alpha} \frac{dt}{\mu\left(1-\frac{1}{t}\right)} + \frac{1}{e\alpha\mu(1-\alpha)}.\end{aligned} \tag{2.8}$$

Using the integration by parts we obtain

$$\begin{aligned}\int_a^{1/\alpha} \frac{dt}{\mu\left(1-\frac{1}{t}\right)} &= t\cdot\frac{1}{\mu\left(1-\frac{1}{t}\right)}\Big|_a^{1/\alpha} - \int_a^{1/\alpha} t\cdot\left(\frac{1}{\mu\left(1-\frac{1}{t}\right)}\right)'dt \\ &= \frac{1}{\alpha\mu(1-\alpha)} - \frac{a}{\mu\left(1-\frac{1}{a}\right)} - \int_a^{1/\alpha} t\cdot\left(\frac{1}{\mu\left(1-\frac{1}{t}\right)}\right)'dt \\ &\sim \frac{1}{\alpha\mu(1-\alpha)} \quad as \quad \alpha \longrightarrow 0.\end{aligned} \tag{2.9}$$

Where (2.9) follows the L'Hopital rule and the assumption (2.5) and (2.6) as follows

$$\lim_{x\to\infty}\frac{\int_a^x t\left(\frac{1}{\mu\left(1-\frac{1}{t}\right)}\right)'dt}{x\left(\frac{1}{\mu\left(1-\frac{1}{x}\right)}\right)}=\lim_{x\to\infty}\frac{x\left(\frac{1}{\mu\left(1-\frac{1}{x}\right)}\right)'}{x\left(\frac{1}{\mu\left(1-\frac{1}{x}\right)}\right)'+\frac{1}{\mu(1-\frac{1}{x})}}=0.$$

From (2.8) and (2.9) and the continuity of the first integral in (2.9) with α, inequality (2.7) follows. □

Now we give an estimation for coefficients of a holomorphic function $\mathcal{B}^{\varphi}$.

Theorem 2.4. *Assume that* $f(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathcal{B}^{\varphi}$, *then* $\lim_{n\to\infty} n\mu\Big(1-\frac{1}{n}\Big)|a_n|<\infty$.

Proof. Applying the Cauchy integral formula to the first derivative of f, we have

$$a_n=\frac{1}{2\pi in}\int_{|\xi|=r}\frac{f'(\xi)}{\xi^n}d\xi=\frac{1}{2\pi n}\int_0^{2\pi}f'(re^{i\theta})r^{1-n}e^{i(1-n)\theta}d\theta.$$

By the definition of the space $\mathcal{B}^{\varphi}$, combine with (2.2) and (2.3), it follows that

$$|a_n|\le\frac{1}{2\pi n}\int_0^{2\pi}|f'(re^{i\theta})|r^{1-n}d\theta\le\frac{r^{1-n}\varphi^{-1}\left(\frac{1}{1-r^2}\right)}{n}\|f\|_{\varphi}=\frac{r^{1-n}}{n\mu(r)}\|f\|_{\varphi}.\tag{2.10}$$

By choosing $r=1-\frac{1}{n}$, $n\ge 2$ in (2.10), we obtain

$$|a_n|\le\frac{\left(1-\frac{1}{n}\right)^{1-n}}{n\mu\left(1-\frac{1}{n}\right)}\|f\|_{\varphi}.\tag{2.11}$$

Multiplying (2.11) by $n\mu\left(1-\frac{1}{n}\right)$, and letting $n\to\infty$ in such obtained inequality, the result follows. □

Remark 3. Since $\left(1-\frac{1}{n}\right)^{1-n}=\left(1+\frac{1}{n-1}\right)^{n-1}<e,\quad n\ge 2$. By Theorem 2.4, it follows that $\lim_{n\to\infty}\sup n\mu\left(1-\frac{1}{n}\right)|a_n|\le e\|f\|_{\varphi}$, for every $f\in\mathcal{B}^{\varphi}$.

Theorem 2.5. *Assume that* $f(z)=\sum_{k=1}^{\infty}a_k z^{nk}\in H(\mathbb{D})$, *where* n_k *is a sequence such that* $\frac{n_{k+1}}{n_k}\ge\lambda>1$, $k\in\mathbb{N}$. *Let* $\varphi:[0,\infty)\to[0,\infty)$ *be an* $\mathcal{N}$*-function, the function* μ *satisfies all the condition of Lemma 2.3, and*

$$\mu(t)=\frac{1}{\varphi^{-1}\left(\frac{1}{1-t^2}\right)},\tag{2.12}$$

such that

$$\lim_{k\to\infty}\inf\frac{\mu\left(1-\frac{1}{n_k}\right)}{\mu\left(1-\frac{1}{n_{k+1}}\right)}=q>1\tag{2.13}$$

and

$$\mu\Big(1-\ln\big(\frac{1}{|z|}\big)\Big)\sim C\mu(|z|) \quad as \quad |z|\to 0, \tag{2.14}$$

for a positive constant C. Then the condition

$$\limsup_{k\to\infty} n_k\mu\Big(1-\frac{1}{n_k}\Big)|a_k|<\infty \tag{2.15}$$

implies that $f\in\mathcal{B}^{\varphi}$.

Proof. Assume $\limsup_{k\to\infty} n_k\mu\left(1-\frac{1}{n_k}\right)|a_k|<\infty$, we have

$$|zf'(z)|=\Big|\sum_{k=1}^{\infty}a_kn_kz^{n_k}\Big|\le C\sum_{k=1}^{\infty}\frac{|z|^{n_k}}{\mu\left(1-\frac{1}{n_k}\right)},$$

and consequently

$$\frac{|zf'(z)|}{1-|z|}\le C\sum_{n=1}^{\infty}\left(\sum_{n_k\le n}\frac{1}{\mu\left(1-\frac{1}{n_k}\right)}\right)|z|^n, \tag{2.16}$$

for some positive constant C.

By (2.13), and μ is nonincreasing for $n_k\le n\le n_{k+1}$, we have

$$\frac{1}{\mu\left(1-\frac{1}{n_l}\right)}<\frac{C}{\mu\left(1-\frac{1}{n}\right)}\left(\frac{2}{1+q}\right)^{k-l},$$

for every $l=1,\ 2,\ \cdots,\ k$, and some $C>0$. This along with the fact $\frac{2}{1+q}<2$, implies that

$$\sum_{n_k\le n}\frac{1}{\mu\left(1-\frac{1}{n_k}\right)}\le\frac{C}{\mu\left(1-\frac{1}{n}\right)}. \tag{2.17}$$

Together with (2.16) and (2.17) it follows that

$$\frac{zf'(z)}{1-|z|}\le C\sum_{n=1}^{\infty}\frac{|z|^n}{\mu\left(1-\frac{1}{n}\right)}. \tag{2.18}$$

Since the function

$$g_x(t)=\frac{x^t}{\mu\left(1-\frac{1}{t}\right)}=\frac{e^{-t\ln\frac{1}{x}}}{\mu\left(1-\frac{1}{t}\right)},$$

is decreasing in t, for sufficiently large t and each $x\in(0,1)$, we have

$$\sum_{n=1}^{\infty}\frac{|z|^n}{\mu\left(1-\frac{1}{n}\right)}\sim\int_0^{\infty}\frac{e^{-t\ln\frac{1}{|z|}}}{\mu\left(1-\frac{1}{t}\right)}dt. \tag{2.19}$$

Note that the function $\mu(t)$ satisfies condition of Lemma 2.3, we obtain

$$\int_e^{\infty}\frac{e^{t\ln\frac{1}{|z|}}}{\mu\left(1-\frac{1}{t}\right)}\le C\frac{1}{\ln\frac{1}{|z|}\mu\left(1-\ln\frac{1}{|z|}\right)}, \tag{2.20}$$

as $|z| \to 1$. Using the asymptotic relation

$$\ln \frac{1}{|z|} \sim 1 - |z| \quad as \quad |z| \to 1 - 0,$$

and take (2.14) in (2.20), the result easily follows. □

Now we extend Theorem B into Bloch–Orlicz space $\mathcal{B}^\varphi$.

Theorem 2.6. *There exist two functions $f, g \in \mathcal{B}^\varphi$, such that for each $z \in \mathbb{D}$,*

$$|f'(z)| + |g'(z)| \geq C\varphi^{-1}\left(\frac{1}{1-|z|^2}\right), \tag{2.21}$$

for some positive constant C.

Proof. Since $\mathcal{B}^\varphi$ is isometrically equal to $\mathcal{B}^\mu$, where $\mu(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{1-|z|^2}\right)}$, with $z \in \mathbb{D}$, then (2.21) equals to

$$|f'(z)| + |g'(z| \geq \frac{C}{\mu(|z|)}. \tag{2.22}$$

Next, we are going to prove (2.22) holds.

Set $f(z) = \varepsilon z + \sum_{j=1}^{\infty} \frac{z^{q^j}}{\mu\left(1-\frac{1}{q^j}\right)q^j}$ for $z \in \mathbb{D}$, where q is a large natural number and ε is positive and small sufficiently. Applying Theorem 2.4 with

$$a_j = \left(\mu\big(1 - \frac{1}{q^j}\big)q^j\right)^{-1}, \quad n_j = q^j,$$

it is easy to check $f \in \mathcal{B}^\varphi$. We first show that

$$|f'(z)| \geq \frac{C}{\mu(|z|)}$$

if $1 - q^{-k} \leq |z| \leq 1 - q^{-(k+1/2)}$ for $K \in \mathbb{N}$. For any $z \in \mathbb{D}$ we have

$$\begin{aligned}
|f'(z)| &= \left|\varepsilon + \sum_{j=1}^{\infty} \frac{z^{q^j-1}}{\mu\left(1-\frac{1}{q^j}\right)}\right| \\
&\geq \frac{|z|^{q^k-1}}{\mu\left(1-\frac{1}{q^k}\right)} - \left(\varepsilon + \sum_{j=1}^{k-1} \frac{|z|^{q^j-1}}{\mu\left(1-\frac{1}{q^j}\right)}\right) - \sum_{j=k+1}^{\infty} \frac{|z|^{q^j-1}}{\mu\left(1-\frac{1}{q^j}\right)} \\
&= \frac{1}{|z|}(I_1 - I_2 - I_3) \geq I_1 - I_2 - I_3
\end{aligned}$$

if $I_1 - I_2 - I_3 > 0$. Since $(1 - q^{-k})^{q^k} \leq |z|^{q^K} \leq \left((1 - q^{(-k+1/2)})^{q^{k+1/2}}\right)^{q^{-1/2}}$, we have $\frac{1}{3} \leq |z|^{q^k} \leq \left(\frac{1}{2}\right)^{q-1/2}$ for q large enough. We have $\frac{1}{3} \leq |z|^{q^k} \leq (\frac{1}{2})^{q-1/2}$, and hence $I_1 \geq \frac{1}{3\mu\left(1-\frac{1}{q^k}\right)}$.

On the other hand, for large enough q, by (2.5) it follows that

$$I_2 \le \varepsilon \frac{1}{\mu\left(1-\frac{1}{q^k}\right)} + \frac{1}{\mu\left(1-\frac{1}{q^k}\right)} \sum_{j=1}^{k-1} \frac{\mu\left(1-\frac{1}{q^{j+1}}\right)}{\mu\left(1-\frac{1}{q^j}\right)} \cdot \frac{\mu\left(1-\frac{1}{q^{j+2}}\right)}{\mu\left(1-\frac{1}{q^{j+1}}\right)} \cdots \frac{\mu\left(1-\frac{1}{q^k}\right)}{\mu\left(1-\frac{1}{q^{k-1}}\right)}$$

$$\le \epsilon \frac{1}{\mu\left(1-\frac{1}{q^k}\right)} + \frac{1}{\mu\left(1-\frac{1}{q^k}\right)} \sum_{j=1}^{k-1} \frac{1}{q^{k-j}} \le \frac{1}{\mu\left(1-\frac{1}{q^k}\right)} \left[\varepsilon + \frac{1}{q-1}\right].$$

We also have

$$I_3 \le \frac{|z|^{q^{k+1}}}{\mu\left(1-\frac{1}{q^k}\right)} \sum_{j=k+1}^{\infty} \frac{\mu\left(1-\frac{1}{q^k}\right)}{\mu\left(1-\frac{1}{q^j}\right)} |z|^{q^j - q^{k+1}}$$

$$\le \frac{|z|^{q^{k+1}}}{\mu\left(1-\frac{1}{q^k}\right)} \sum_{j=k+1}^{\infty} \frac{\mu\left(1-\frac{1}{q^k}\right)}{\mu\left(1-\frac{1}{q^{k+1}}\right)} \frac{\mu\left(1-\frac{1}{q^{k+1}}\right)}{\mu\left(1-\frac{1}{q^{k+z}}\right)} \cdots \frac{\mu\left(1-\frac{1}{q^{j-1}}\right)}{\mu\left(1-\frac{1}{q^j}\right)} |z|^{q^j - q^{k+1}}$$

$$\le \frac{|z|^{q^{k+1}}}{\mu\left(1-\frac{1}{q^k}\right)} \sum_{j=k+1}^{\infty} q(q)^{(i-(k+1))} |z|^{(q^j - q^{k+1})}$$

$$\le \frac{|z|^{q^{k+1}}}{\mu\left(1-\frac{1}{q^k}\right)} \sum_{s=0}^{\infty} q\left(q|z|^{(q^{k+2}-q^{k+1})}\right)^s = \frac{(|z|^{qk})^q}{\mu\left(1-\frac{1}{q^k}\right)} \frac{q}{1-q(|z|^{qk})^{q^2-q}}$$

$$= \frac{1}{\mu\left(1-\frac{1}{q^k}\right)} \frac{q(|z|^q k)^q}{1-q(|z|^{qk})^{q^2-q}} \le \frac{1}{\mu\left(1-\frac{1}{q^k}\right)} \frac{q2^{-q^{1/2}}}{1-q2^{-(q^{3/2}-q^{1/2})}}.$$

By the estimates above, then

$$|f'(z)| \ge \frac{1}{\mu(1-\frac{1}{q^k})} \left(\frac{1}{3} - \varepsilon - \frac{1}{q-1} - \frac{q2^{-q^{1/2}}}{1-q2^{-q^{3/2}-q^{1/2}}}\right)$$
$$\ge \frac{C}{\mu(1-\frac{1}{q^k})},$$

for q large enough, ε sufficiently small, and $k \in \mathbb{N}$. By inequality (2.13), inequality (2.17) and

$$1 - q^{-k} \le |z| \le 1 - q^{-(k+1/2)} \le 1 - q^{-(k+1)},$$

together yield

$$|f'(z)| \ge \frac{C}{\mu\left(1-\frac{1}{q^k}\right)} \ge \frac{C}{C_q \mu\left(1-\frac{1}{q^{k+1}}\right)} \ge \frac{C}{\mu(|z|)}.$$

Similarly, the function

$$g(z) = \sum_{j=1}^{\infty} \frac{z^{n_j}}{\mu\left(1-\frac{1}{q^{j+1/2}}\right)},$$

where n_j is the integer closest to $q^{j+1/2}$, satisfies $|g'(z)| \geq \frac{C}{\mu(|z|)}$, when $1 - q^{-(k+1/2} \leq |z| \leq 1 - q^{-(k+1)}$ for any $k \in \mathbb{N}$. So (2.22) holds for $1 - q^{-1} \leq |z| \leq 1$.

Since $f'(0) \neq 0$ and the functions f' and g' only have a finite number of zeroes in the disc $|z| \leq 1 - q^{-1}$. The function g can be replaced by $g_\theta(z) = g(e^{i\theta}z)$ for an appropriate θ to prevent f' and g' have common zeroes. Hence we get a pair of function satisfying (2.22), so (2.21) holds and the proof is completed. □

3. Some applications to composition operators

In this section, some applications of our result are presented. Firstly, we give the boundedness of the composition operators from $C_\phi : \mathcal{B}^\varphi \to Q_K(p,q)$.

Theorem 3.1. *Assume ϕ is an analytic self-map of $\mathbb{D}$. Let $p > 0$, $q > -2$, and let K be an nonnegative and nondecreasing function in $[0, \infty)$. Then $C_\phi : \mathcal{B}^\varphi \to Q_K(p,q)$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left[\varphi^{-1}\left(\frac{1}{1-|\phi(z)|^2}\right)\right]^p |\phi'(z)|^p (1-|z|^2)^q g(z,a)dA(z) < \infty. \tag{3.1}$$

Proof. First assume $C_\phi : \mathcal{B}^\varphi \to Q_K(p,q)$ is bounded. Let f and g be the functions constructed in Theorem 2.6. Using these functions and by some simple calculation, we obtain

$$\begin{aligned}
\infty &> \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (|(f\circ\phi)'(z)|^p + |(g\circ\phi)'(z)|^p)(1-|z|^2)^q K(g(z,a))dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (|f'(\phi)(z)|^p + |g'(\phi)(z)|^p)(1-|z|^2)^q |\phi'(z)|^p K(g(z,a))dA(z) \\
&\geq 2^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (|f'(\phi)(z)| + |g'(\phi)(z)|)^p (1-|z|^2)^q |\phi'(z)|^p K(g(z,a))dA(z) \\
&\geq C \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[\phi^{-1}\left(\frac{1}{1-|\phi(z)|^2}\right)\right]^p |\phi'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z),
\end{aligned}$$

from which (3.1) follows.

Now assume that (3.1) holds, then for an $f \in \mathcal{B}^\varphi$ we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f\circ\phi)'(z)|^p |(1-|z|^2)^q K(g(z,a))dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f'(\phi)(z)|^p |\phi'(z)|^p |(1-|z|^2)^q K(g(z,a))dA(z) \\
&\leq \|f\|_\varphi^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[\phi^{-1}(\frac{1}{1-|\phi(z)|^2})\right]^p |\phi'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z).
\end{aligned}$$

Hence, it follows by (3.1) that C_ϕ is a bounded operator from $\mathcal{B}^\varphi \to Q_K(p,q)$. □

Another application of our result is the following theorem regards to the compactness of composition operators $C_\phi : \mathcal{B}^\varphi \to Q_K(p,q)$.

Theorem 3.2. *Assume ϕ is an analytic self-map of $\mathbb{D}$. Let $p > 0$, $q > -2$, and let K be an nonnegative and nondecreasing function in $[0, \infty)$. Then $C_\phi : \mathcal{B}^\varphi \to$*

$Q_K(p,q)$ is compact if and only if $\phi \in Q_K(p,q)$, and

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{\Omega_r}\Big[\varphi^{-1}\big(\frac{1}{1-|\phi(z)|^2}\big)^p|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))\Big]dA(z)=0, \quad (3.2)$$

where $\Omega_r=\{z\in\mathbb{D},|\phi(z)|>r\}$ for $0<r<1$.

Proof. We assume $\phi \in Q_K(p,q)$ and (3.2) holds. Assume $\{f_n\}_{n\in N}$ be a bounded sequence in $\mathcal{B}^\varphi$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. For simplicity, assume $\|f_n\|_\varphi \le 1$. By (3.2) for any given $\varepsilon>0$, there exists $r\in(0,1)$ such that

$$\sup_{a\in\mathbb{D}}\int_{\Omega_r}\left[\varphi^{-1}\left(\frac{1}{1-|\phi(z)|^2}\right)^p|\phi'(z)|(1-|z|^2)^qK(g(z,a))\right]dA(z)<\varepsilon.$$

Since $f_n\to 0$ uniformly on compact subsets of $\mathbb{D}$, it implies that $f_n'\to 0$ uniformly on compact sets of $\mathbb{D}$. For above ε, there exists $N\in\mathbb{N}$ such that $n>N$ and $|f_n'(z)|<\varepsilon$ for $|z|\le r$. Hence

$$\begin{aligned}
&\int_{\mathbb{D}}|(f_n\circ\phi)'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
&=\left\{\int_{\Omega_r}+\int_{\mathbb{D}\setminus\Omega_r}\right\}|f_n'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
&\le \|f_n\|_\varphi^p\int_{\Omega_r}\Big(\varphi^{-1}\big(\frac{1}{1-|\phi(z)|^2}\big)\Big)^p|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
&+\varepsilon^p\int_{\mathbb{D}}|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
&\le \varepsilon+\varepsilon^p\|\phi\|^p_{Q_K(p,q)}.
\end{aligned}$$

Then, it follows that

$$\|f_n\circ\phi\|_{Q_K(p,q)}\to 0 \quad as \quad n\to\infty.$$

Thus $C_\phi:\mathcal{B}^\varphi\to Q_K(p,q)$ is compact.

To prove the converse, suppose that there exist an $\varepsilon_0>0$ such that

$$\sup_{a\in\mathbb{D}}\int_{\Omega_r}\Big[\varphi^{-1}\big(\frac{1}{1-|\phi(z)|^2}\big)^p|\phi'(z)|(1-|z|^2)^qK(g(z,a))\Big]dA(z)\ge\varepsilon_0,$$

for any $r\in(0,1)$. Then given a sequence of real numbers $\{r_n\}\subset(0,1)$ such that $r_n\to 1$ as $n\to\infty$, we can find a sequence $\{z_n\}\subset\mathbb{D}$ such that $|\phi(z_n)|>r_n$ and

$$\int_{|\varphi(z_n)|>r_n}\Big[\varphi^{-1}\big(\frac{1}{1-|\omega_n|^2}\big)^p|\phi'(z_n)|(1-|z_n|^2)^qK(g(z_n,a))\Big]dA(z_n)\ge\frac{1}{2}\varepsilon_0, \quad (3.3)$$

where $\omega_n=\phi(z_n)$.

By taking a subsequence, if necessary, we may suppose that $\omega_n\to\omega_0\in\partial\mathbb{D}$. Now for $n\in\mathbb{N}$ and $z\in\mathbb{D}$, we set

$$g_n(z)=\int_0^z f_{\omega_n}(s)ds, \quad (3.4)$$

where f_{ω_n} is the function in Lemma 2.2 with $a = \omega_n$. We can see that $\{g_n\}$ is a bounded sequence in $\mathcal{B}^{\varphi}$. Furthermore since

$$|g_n'(z)| \leq \varphi^{-1}((1-|z|^2))^{-2}(1-|\omega_n|^2), \tag{3.5}$$

φ^{-1} is an increasing continuous function satisfying $\varphi^{-1}(0) = 0$ and

$$|g_n(z)| \leq \int_0^z |g_n'(s)||ds|,$$

for all $z \in \mathbb{D}$. We know that $\{g_n\}$ is a sequence converging to 0 uniformly on compact subsets of $\mathbb{D}$, and satisfying

$$\begin{aligned}\|C_\phi(g_n)\|_{Q_k(p,q)} &\geq \int_{|\phi(z_n)|>r_n} |g_n'(\omega_n)|^p|\phi'(z_n)|^p(1-|z_n|^2)^qK(g(z_n,a))dA(z_n)\\ &= \int_{|\phi(z_n)|>r_n} \varphi^{-1}\Big(\frac{1}{1-|\omega_n|^2}\Big)^p|\phi'(z_n)|^p(1-|z_n|^2)^qK(g(z_n,a))dA(z_n)\\ &\geq \frac{1}{2}\varepsilon_0 > 0.\end{aligned}$$

Where we have used the fact

$$|g_n'(\omega_n)| = \varphi^{-1}\left(\frac{1}{1-|\omega_n|^2}\right).$$

Therefore $C_\phi : \mathcal{B}^{\varphi} \to Q_K(p,q)$ is not a compact operator. This completes the proof of the theorem. □

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