Annals of Functional Analysis
Ann. Funct. Anal. 6 (2015), no. 3, 1–7
http://doi.org/10.15352/afa/06-3-1
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

INNER-OUTER FACTORIZATION ON $\mathcal{Q}_p$ SPACES

RUISHEN QIAN[1] AND SONGXIAO LI[2*]

Communicated by K. Zhu

Abstract. It is well known that every function in Hardy space can be factorized into an inner function and outer function. Since the factorization is unique, if we fix a function in Hardy space, inner and outer factors must be control by each other. In this note, we give an inner-outer factorization on $\mathcal{Q}_p$ spaces and some subspace of $\mathcal{Q}_p$ spaces, where $0 < p < 1$.

1. Introduction

We denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{D}$ and its boundary $\{z \in \mathbb{C} : |z| = 1\}$ by $\partial\mathbb{D}$. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space H^p is the set of $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual, H^∞ is the set of $f \in H(\mathbb{D})$ with $\sup_{z\in\mathbb{D}} |f(z)| < \infty$ (See [5]).

Let $0 < p < \infty$. The $\mathcal{Q}_p$ space is the set of $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Q}_p} = |f(0)| + \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^p dA(z)\right)^{\frac{1}{2}} < \infty,$$

where g denotes the Green function given by

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|}, \quad z, a \in \mathbb{D}, z \neq a,$$

$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, $dA(z) = \frac{1}{\pi} dxdy$. If $p = 1$, $\mathcal{Q}_1 = BMOA$, the space of analytic functions in the Hardy space $H^1(\mathbb{D})$ whose boundary functions have bounded

Date: Received: Sep. 2, 2014; Accepted: Sep. 18, 2014.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46E15; Secondary 30H25.
Key words and phrases. $BMOA$ space, inner function, $\mathcal{Q}_p$ space.

mean oscillation (see, for example [14, 18]). When $p > 1$, $\mathcal{Q}_p$ spaces coincide with the Bloch space. For more information on $\mathcal{Q}_p$ spaces, we refer to [17, 20, 21].

Let $0 < q < \infty$ and $-1 < \alpha < \infty$. The A^q_α space is the set of $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^q (1-|z|^2)^\alpha dA(z) < \infty.$$

For $1 \leq q < \infty$ and $0 < s < 1$, the Besov space B^s_q is the set of functions $f \in L^q(\partial\mathbb{D})$ such that

$$\int_{-\pi}^{\pi} \frac{dt}{|t|^{sq+1}} \int_{\partial\mathbb{D}} |f(e^{it}\zeta) - f(\zeta)|^q dm(\zeta) < \infty.$$

The analytic subspace $AB^s_q = B^s_q \cap H^q$ is the set of functions $f \in H^q$ such that

$$\|f\|_{AB^s_q} = |f(0)|^q + \left(\int_{\mathbb{D}} |f'(z)|^q (1-|z|)^{(1-s)q-1} dA(z) \right)^{\frac{1}{q}} < \infty.$$

We refer the reader to [2, 3, 4, 10].

An $f \in H(\mathbb{D})$ is said to be an inner function if it is bounded and has boundary values of modulus 1 almost everywhere on $\partial\mathbb{D}$. If θ is an inner function, for $0 < \epsilon < 1$, define the level set of order ϵ of θ as follows.

$$\Omega(\theta, \epsilon) = \{z \in \mathbb{D} : |\theta(z)| < \epsilon\}.$$

For more information about inner function, we refer to [1, 12, 15, 16, 19].

We say that $g \in H(\mathbb{D})$ is an outer function if there exists a positive function h with $\log h \in L^1(\partial\mathbb{D})$ and a complex number C of modulus 1 such that

$$g(z) = C \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log h(e^{it}) \frac{e^{it}+z}{e^{it}-z} dt \right).$$

Moreover, the boundary values of g satisfy $h(\zeta) = |g(\zeta)|$ for almost all $\zeta \in \partial\mathbb{D}$.

It is well known that every $f \in H^p$ has a factorization θg, where θ is an inner function and g is an outer function. If we fix an $f \in H^p$, there must be some relationship between θ and g, since the factorization is unique. Dyakonov gave many interesting theorems on inner-outer factorization and characterized the modulus of analytic functions in the disc whose boundary values belong to certain smoothness classes. For many results concern this topic, we refer to [6, 7, 9, 11]. The following theorem can be found in [7, Theorem 1].

Theorem A. *If $f \in BMOA$ and θ is an inner function, then the following conditions are equivalent:*

(1) $f\overline{\theta} \in BMO$;
(2) $f\theta \in BMOA$;
(3) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;
(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0 < \epsilon < 1$;
(5) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

Before we state next theorem, we need to give the definition of $\mathcal{Q}_p(\partial\mathbb{D})$. Let $0 < p < \infty$. The $\mathcal{Q}_p(\partial\mathbb{D})$ space is the set of $f \in L^2(\partial\mathbb{D})$ such that

$$\sup_{I\subseteq\partial\mathbb{D}} |I|^{-p} \int_I\int_I \frac{|f(\zeta)-f(\eta)|^2}{|\zeta-\eta|^{2-p}}|d\zeta||d\eta| < \infty.$$

In this paper, if we control the inner factor, in some sense, we first extend Theorem A from the $BMOA$ space to $\mathcal{Q}_p$ spaces, $0 < p \le 1$.

Theorem 1. *Let $0 < p \le 1$. If $f \in \mathcal{Q}_p$ and $\theta \in \mathcal{Q}_p$ is an inner function, then the following conditions are equivalent:*

(1) $f\overline{\theta} \in \mathcal{Q}_p(\partial\mathbb{D})$;
(2) $f\theta \in \mathcal{Q}_p$;
(3) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;
(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0 < \epsilon < 1$;
(5) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

Let $M(X)$ denote the space of multipliers of X. The next theorem was another main theorem in [7, Theorem 6].

Theorem B. *If $f \in M(BMOA)$ and θ is an inner function, then the following conditions are equivalent:*

(1) $f\theta \in M(BMOA)$;
(2) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| \log\frac{1}{1-|z|} < \infty$, *for every* ϵ, $0 < \epsilon < 1$;
(3) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| \log\frac{1}{1-|z|} < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

Using Theorem 1, we also have the following theorem.

Theorem 2. *Let $0 < p \le 1$. If $f \in M(\mathcal{Q}_p)$ and $\theta \in \mathcal{Q}_p$ is an inner function, then the following conditions are equivalent:*

(1) $f\theta \in M(\mathcal{Q}_p)$;
(2) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| \log\frac{1}{1-|z|} < \infty$, *for every* ϵ, $0 < \epsilon < 1$;
(3) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| \log\frac{1}{1-|z|} < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

Using the idea as Theorems 1 and 2, we can also get the similar result on $AB_q^s \cap \mathcal{Q}_p$.

Theorem 3. *Let $1 \le q < \infty$, $0 < p \le 1$ and $0 < s < 1/q$. If $f \in \mathcal{Q}_p \cap AB_q^s$ and $\theta \in \mathcal{Q}_p \cap AB_q^s$ is an inner function, then the following conditions are equivalent:*

(1) $f\overline{\theta} \in \mathcal{Q}_p(\partial\mathbb{D}) \cap B_q^s$;
(2) $f\theta \in \mathcal{Q}_p \cap AB_q^s$;
(3) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;
(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0 < \epsilon < 1$;
(5) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0 < \epsilon < 1$.

Throughout this paper, for two functions f and g, $f \lesssim g$ means that there is a positive constant C such that $f \le Cg$.

2. Proofs of Main Results

To prove our main results in this paper, we need some lemmas which will be stated as follows.

Lemma A. [20] *Let* $0<p<1$. $f\in\mathcal{Q}_p$ *if and only if*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\left(\int_{\partial\mathbb{D}}|f(\zeta)|^2d\mu_z(\zeta)-|f(z)|^2\right)\frac{(1-|\varphi_a(z)|^2)^p}{(1-|z|^2)^2}dA(z)<\infty,$$

where $d\mu_z(\zeta)=\frac{1-|z|^2}{|\zeta-z|^2}\frac{|d\zeta|}{2\pi}$.

Before state the next lemma, we first recall some properties of the system Γ_ϵ of the so-called Carleson curves associated with θ and ϵ, see [14, Chapter VIII] and [8]. $\Gamma_\epsilon=\cup_i\gamma_i$ is a countable union of simple closed rectifiable curves γ_i in $\overline{\mathbb{D}}$ with the following properties:

1. (1) The curves γ_i have pairwise disjoint interiors; for each of them one has $l(\gamma_i\cap\partial\mathbb{D})=0$, where $l(.)$ denotes length.
2. (2) Arc length measure $|dz|$ on $\Gamma_\epsilon\cap\mathbb{D}$ is a Carleson measure.
3. (3) For $z\in\Gamma_\epsilon\cap\mathbb{D}$ we have $\eta<|\theta(z)|<\epsilon$, where $\eta(\epsilon)$ is some positive constant depending on ϵ. Moreover, $\Gamma_\epsilon\cap\mathbb{D}\subseteq\Omega(\theta,\epsilon)$.

Lemma B. [8] *Let* $1\le q<\infty$ *and* $s>0$. *Suppose that* $f\in H^2$ *and* θ *is an inner function. If*

$$\int_{\Gamma_\epsilon}\frac{|f(z)|^q|dz|}{(1-|z|^2)^{sq}}<\infty,\quad 0<\epsilon<1,$$

then $P_-(\overline{\theta}f)\in B_q^s$. *Here* P_- *denoted by the orthogonal projection from* $L^2(\partial\mathbb{D})$ *onto* $L^2(\partial\mathbb{D})\ominus H^2$.

Lemma C. [19] *Let* $1\le q<\infty$ *and* $0<s<1$. *Suppose that* $f\in B_q^s$ *and* $u\in H^\infty$. *Then the following are equivalent:*

1. (1) $f\overline{u}\in B_q^s$;
2. (2) $fu\in B_q^s$;
3. (3) $P_-(\overline{u}f)\in B_q^s$.

Here P_- *denoted by the orthogonal projection from* $L^2(\partial\mathbb{D})$ *onto* $L^2(\partial\mathbb{D})\ominus H^2$.

Lemma D. [19] *Let* $1\le q<\infty$, $q-2<\alpha<q-1$ *and* $0<\epsilon<1$. *Suppose that* θ *is an inner function and* $B_{\theta,\epsilon}$ *is its associated interpolating Blaschke product, then the following are equivalent:*

1. (1) $\theta'\in A^q_\alpha$;
2. (2) $B'_{\theta,\epsilon}\in A^q_\alpha$;
3. (3) *If* $\{a_k\}_{k=0}^\infty$ *is the sequence of zeros of* $B_{\theta,\epsilon}$, *then*

$$\sum_{k=0}^\infty(1-|a_k|^2)^{\alpha-q+2}<\infty;$$

4. (4)

$$\int_{\Gamma_\epsilon}\frac{|f(z)|^q|dz|}{(1-|z|^2)^{q-\alpha-1}}<\infty.$$

Now we are in a position to prove our main results.

Proof of Theorem 1. Since $\mathcal{Q}_1=BMOA$ and $\theta\in H^\infty\subseteq BMOA$, it is only to prove the case of $p\in(0,1)$.

(1) $\Leftrightarrow$ (2). Suppose that $f \in \mathcal{Q}_p$ and θ is an inner function. From Theorem 2.1 of [13], we have $f\theta \in \mathcal{Q}_p$ if and only if

$$\sup_{I\subseteq\partial\mathbb{D}} |I|^{-p} \int_I \int_I \frac{|f(\zeta)\theta(\zeta) - f(\eta)\theta(\eta)|^2}{|\zeta - \eta|^{2-p}} |d\zeta||d\eta| < \infty.$$

Noting that

$$f(\zeta)\theta(\zeta) - f(\eta)\theta(\eta) = (f(\zeta) - f(\eta))\theta(\zeta) + f(\eta)(\theta(\zeta) - \theta(\eta)),$$

we can deduce that $f\theta \in \mathcal{Q}_p$ if and only if

$$\sup_{I\subseteq\partial\mathbb{D}} |I|^{-p} \int_I \int_I \frac{|f(\eta)|^2|\theta(\zeta) - \theta(\eta)|^2}{|\zeta - \eta|^{2-p}} |d\zeta||d\eta| < \infty,$$

and if and only if $f\overline{\theta} \in \mathcal{Q}_p(\partial\mathbb{D})$.

(2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3). If $f \in \mathcal{Q}_p \subseteq BMOA$, $f\theta \in \mathcal{Q}_p \subseteq BMOA$. From Theorem A, we easily to obtain the desired result.

(3) $\Rightarrow$ (2). From Lemma A, we see that $\theta \in \mathcal{Q}_p$ if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(1 - |\theta(z)|^2\right) \frac{(1 - |\varphi_a(z)|^2)^p}{(1 - |z|^2)^2} dA(z) < \infty.$$

Suppose $\theta \in \mathcal{Q}_p$, $f \in \mathcal{Q}_p$. To prove $f\theta \in \mathcal{Q}_p$, we only need to prove

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2|\theta'(z)|^2(1 - |\varphi_a(z)|^2)^p dA(z) < \infty.$$

Applying the well known Schwarz lemma and (3) of Theorem A, we obtain that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2|\theta'(z)|^2(1 - |\varphi_a(z)|^2)^p dA(z) \\
\leq &\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 \left(\frac{1 - |\theta(z)|^2}{1 - |z|^2}\right)^2 (1 - |\varphi_a(z)|^2)^p dA(z) \\
\leq &\left(\sup_{z\in\mathbb{D}} |f(z)|^2(1 - |\theta(z)|^2)\right) \\
&\qquad \times \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{1 - |\theta(z)|^2}{(1 - |z|^2)^2}(1 - |\varphi_a(z)|^2)^p dA(z) < \infty.
\end{aligned}$$

The proof is complete.

Proof of Theorem 2. (1) $\Rightarrow$ (2). Let $f \in M(\mathcal{Q}_p)$ and $\theta \in \mathcal{Q}_p$ be an inner function. For any $g \in \mathcal{Q}_p$, we have $fg\theta \in \mathcal{Q}_p$ by the assumption. From Theorem 1, we know that $fg\theta \in \mathcal{Q}_p$ if and only if

$$\sup_{z\in\Omega(\theta,\epsilon)} |f(z)g(z)| < \infty$$

for every ϵ, $0 < \epsilon < 1$.

For any $a \in \Omega(\theta, \epsilon)$, we define

$$g_a(z) = \log\left(\frac{a}{|a|} - z\right).$$

Clearly, $g_a \in \mathcal{Q}_p$ and notice the fact that $\|\cdot\|_{\mathcal{Q}_p}$ is Möbius invariant, hence, $\|g_a\|_{\mathcal{Q}_p}$ is independent of a. Thus, for any $a \in \Omega(\theta, \epsilon)$,

$$|f(a)g_a(a)| < \infty.$$

Since

$$|\log(1-|a|)| = |Re \log\left(\frac{a}{|a|} - a\right)| \leq |\log\left(\frac{a}{|a|} - a\right)| = |g_a(a)|,$$

we easily get the desired result by the arbitrary of a.

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Let $g \in \mathcal{Q}_p \subseteq \mathcal{B}$. Using the fact that

$$|g(z)| \lesssim \log\frac{1}{1-|z|}\|g\|_{\mathcal{B}} \leq \log\frac{1}{1-|z|}\|g\|_{\mathcal{Q}_p},$$

we have

$$|f(z)g(z)| \lesssim \log\frac{1}{1-|z|}|f(z)|\|g\|_{\mathcal{Q}_p}$$

for $f \in M(\mathcal{Q}_p)$. Therefore,

$$\sup_{z\in\Omega(\theta,\epsilon)} |f(z)g(z)| \lesssim \sup_{z\in\Omega(\theta,\epsilon)} \log\frac{1}{1-|z|}|f(z)| < \infty,$$

for some ϵ, $0 < \epsilon < 1$. From Theorem 1, we deduce that

$$fg\theta \in \mathcal{Q}_p.$$

Hence, $f\theta \in M(\mathcal{Q}_p)$. The proof is complete.

Proof of Theorem 3.(1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). Combine with Theorem 1 and Lemma C, similarly to the proof of Theorem 1, we can easily get the desired result.

(5) $\Rightarrow$ (2). From Theorem 1, we have $f\theta \in \mathcal{Q}_p$. Using Lemmas B and C, it is only to prove

$$\int_{\Gamma_\epsilon} \frac{|f(z)|^q|dz|}{(1-|z|^2)^{sq}} < \infty, \quad 0 < \epsilon < 1.$$

If $\theta \in \mathcal{Q}_p \cap AB_q^s \subseteq AB_q^s$, then $\theta' \in A^q_{(1-s)q-1}$. Thus, by Lemma D, we have

$$\int_{\Gamma_\epsilon} \frac{|dz|}{(1-|z|^2)^{sq}} < \infty.$$

Using the fact that $\Gamma_\epsilon \cap \mathbb{D} \subseteq \Omega(\theta, \epsilon)$, we deduce that

$$\int_{\Gamma_\epsilon} \frac{|f(z)|^q|dz|}{(1-|z|^2)^{sq}} \leq \left(\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)|\right)^q \int_{\Gamma_\epsilon} \frac{|dz|}{(1-|z|^2)^{sq}} < \infty.$$

The proof is complete.

Acknowledgement. The second author was partially supported by the Macao Science and Technology Development Fund(No.083/2014/A2).

References

1. P. Ahern, *The mean modulus of derivative of an inner function*, Indiana Univ. Math. J. **28** (1979), 311–347.
2. N. Arcozzi, D. Blasi and J. Pau, *Interpolating sequences on analytic Besov type spaces*, Indiana Univ. Math. J. **58** (2009), 1281–1318.
3. D. Blasi and J. Pau, *A characterization of Besov type spaces and applications to Hankel type operators*, Michigan Math. J. **56** (2008), 401-417.
4. B. Böe, *A norm on the holomorphic Besov space*, Proc. Amer. Math. Soc. **131** (2003), 235–241.
5. P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
6. K. Dyakonov, *Multiplicative structure in weighted BMOA spaces*, J. Anal. Math. **75** (1998), 85–104.
7. K. Dyakonov, *Division and multiplication by inner functions and embedding theorems for star-invariant subspaces*, Amer. J. Math. **115** (1993), 881–902.
8. K. Dyakonov, *Smooth functions in the range of a Hankel operator*, Indiana Univ. Math. J. **43** (1994), 805–838.
9. K. Dyakonov, *Equivalent norms on Lipschitz-type spaces of holomorphic functions*, Acta Math. **178** (1997), 143–167.
10. K. Dyakonov, *Besov spaces and outer functions*, Michigan Math. J. **45** (1998), 143–157.
11. K. Dyakonov, *Holomorphic functions and quasiconformal mappings with smooth moduli*, Adv. Math. **187** (2004), 146–172.
12. K. Dyakonov, *Self-improving behaviour of inner functions as multipliers*, J. Funct. Anal. **240** (2006), 429–444.
13. M. Essen and J. Xiao, *Some results on $\mathcal{Q}_p$ spaces,* $0 < p < 1$, J. Reine. Angew. Math. **485** (1997), 173–195.
14. J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
15. A. Gluchoff, *On inner functions with derivative in Bergman spaces*, Illinois J. Math. **31** (1987), 518–527.
16. H. Kim, *Derivatives of Blaschke products*, Pacific. J. Math. **114** (1984), 175–190.
17. A. Nicolau and J. Xiao, *Bounded functions in Möbius invariant Dirichlet-type spaces*, J. Funct. Anal. **150** (1997), 383–425.
18. J. Ortega and J. Fàbrega, *Pointwise multipliers and Corona type decomposition in BMOA*, Ann. Inst. Fourier (Grenoble) **46** (1996), 111–137.
19. J. Peláez, *Inner functions as improving multipliers*, J. Funct. Anal. **255** (2008), 1403–1418.
20. J. Xiao, *Holomorphic $\mathcal{Q}$ Classes*, Lecture Notes in Mathematics, 1767, Springer-Verlag, Berlin, 2001.
21. J. Xiao, *Geometric $\mathcal{Q}_p$ Functions*, Birkhäuser Verlag, Basel-Boston-Berlin, 2006.
22. K. Zhu, *Operator Theory in Function Spaces*, American Mathematical Society, Providence, RI, 2007.

[1] School of Mathematics and Computation Science, Lingnan Normal University, Guangdong Zhanjiang 524048, P. R. China.
E-mail address: qianruishen@sina.cn

[2] Faculty of Information Technology, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau.
E-mail address: jyulsx@163.com