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CLOSED RANGE AND FREDHOLM PROPERTIES OF UPPER-TRIANGULAR OPERATOR MATRICES

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Abstract. The closed range and Fredholm properties of the upper-triangular operator matrix $M = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ are studied, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces. It is shown that the range $\mathcal{R}(M)$ of M is closed if and only if the following statements hold:
(i) $\mathcal{R}(P_{\mathcal{R}(A)^\perp} C|_{\mathcal{N}(B)})$ is closed,
(ii) $\mathcal{R}(A) + \mathcal{R}(P_{\overline{\mathcal{R}(A)}} C|_{\mathcal{N}(P_{\mathcal{R}(A)^\perp} C|_{\mathcal{N}(B)})}) = \overline{\mathcal{R}(A)}$,
(iii) $\mathcal{R}(B^*) + \mathcal{R}(P_{\mathcal{N}(B)^\perp} C^*|_{\mathcal{R}(P_{\mathcal{R}(A)^\perp} C|_{\mathcal{N}(B)})^\perp}) = \overline{\mathcal{R}(B^*)}$,
where $P_{\mathcal{G}}$ denotes the orthogonal projection onto $\mathcal{G}$ along $\mathcal{G}^\perp$. Moreover, the analogues for the Fredholmness of M are further presented.

1. Introduction and preliminaries

The closedness of the range of a linear operator is one of the most basic problems in operator theory, and was shown to be very useful in various areas of mathematics and its applications. For example, many practical problems can be described as the equation $Tx = y$ with T being a linear operator between normed linear spaces; while the stability of the underlying systems is closely connected with the closedness of the range of T (see, e.g., [16]).

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable infinite dimensional complex Hilbert spaces. We use $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ to denote the set of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, and abbreviate $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ by $\mathcal{B}(\mathcal{H}_1)$ when $\mathcal{H}_1 = \mathcal{H}_2$. For $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$

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and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, the upper-triangular operator matrix

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{1.1}$$

has been extensively studied. The spectrum and related problems of M were considered, for example, in [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 17] and the references therein. In [10], the authors discussed the closedness of the partial operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, when the diagonal elements $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$ are given. The similar results

$$\bigcap_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_m(M_C) = \{\lambda \in \sigma_m(A) : n(B-\lambda) < \infty\} \cup \{\lambda \in \sigma_m(B) : d(A-\lambda) < \infty\},$$

$$\bigcup_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_m(M_C) = \sigma_m(A) \cup \sigma_m(B) \cup \{\lambda \in \mathbb{C} : n(B-\lambda) = \infty = d(A-\lambda)\}$$

are independently and almost simultaneously considered in [11]. In [7], the perturbations of left and right essential spectra of M_C are given by

$$\begin{aligned} \bigcap_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_{le}(M_C) &= \sigma_{le}(A) \cup \{\lambda \in \sigma_m(B) : d(A-\lambda) < \infty\} \\ &\quad \cup \{\lambda \in \rho_m(B) : d(A-\lambda) < \infty, n(B-\lambda) = \infty\}, \\ \bigcap_{C\in\mathcal{B}(H_2,H_1)} \sigma_{re}(M_C) &= \sigma_{re}(A) \cup \{\lambda \in \sigma_m(A) : n(B-\lambda) < \infty\} \\ &\quad \cup \{\lambda \in \rho_m(B) : n(B-\lambda) < \infty, d(A-\lambda) = \infty\}. \end{aligned}$$

The purpose of this paper is to characterize the closed range and Fredholm properties of the upper-triangular operator matrix M defined as in (1.1). We describe these properties using the particular block structure of M and the properties of its operator entries. It should be mentioned that every bounded linear operator between two Hilbert spaces can be rewritten as an operator matrix form based on the orthogonal decomposition of Hilbert spaces, and one way to study an operator is to see it as an operator matrix with simpler operator entries.

Finally, we introduce some notations and terminologies. Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, and let $\mathcal{G}$ be a linear subspace of a Hilbert space. Then, the closure and orthogonal complement of $\mathcal{G}$ are denoted by $\overline{\mathcal{G}}$ and $\mathcal{G}^\perp$, respectively. Write $P_\mathcal{G}$ for the orthogonal projection onto $\mathcal{G}$ along $\mathcal{G}^\perp$ (when $\mathcal{G}$ is closed) and $T|_\mathcal{G}$ for the restriction of T to $\mathcal{G}$. Also, we use $\mathcal{N}(T)$ and $\mathcal{R}(T)$ to denote the null space and range of T, respectively. The symbol $n(T)$ represents the nullity of T which is equal to $\dim \mathcal{N}(T)$, and $d(T)$ stands for the deficiency of T which is equal to $\dim \mathcal{R}(T)^\perp$.

As usual, we say T is left (resp. right) invertible if there exists an operator $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $ST = I_{\mathcal{H}_1}$ (resp. $TS = I_{\mathcal{H}_2}$), and its left (resp. right) inverse is denoted by T_l^{-1} (resp. T_r^{-1}). If T is both left invertible and right invertible, then T is invertible, and in this case its inverse T^{-1} clearly satisfies the relation $T_l^{-1} = T^{-1} = T_r^{-1}$. It is well known that T is left invertible if and only if T is bounded below, and if and only if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is closed; T is right invertible if and only if T is surjective, i.e., $\mathcal{R}(T) = \mathcal{H}_2$ (see [14]). In particular, if T is left (resp. right) invertible, then the operator $P_{\mathcal{R}(T)}T : \mathcal{H}_1 \to \mathcal{R}(T)$ (resp. $T|_{\mathcal{N}(T)^\perp} : \mathcal{N}(T)^\perp \to \mathcal{H}_2$) is invertible. Also, T is left invertible if and only if its adjoint T^* is right invertible.

Recall that we say the operator T^+ is a Moore-Penrose inverse of T in $\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)$, if it solves the following system of operator equations

$$TT^+T = T, \quad T^+TT^+ = T^+,$$
$$(TT^+)^* = TT^+, \quad (T^+T)^* = T^+T.$$

Note that T is Moore-Penrose invertible if and only if its range $\mathcal{R}(T)$ is closed, and its Moore-Penrose inverse T^+ can be determined uniquely (see [4]). For $T \in \mathcal{B}(\mathcal{H}_1)$, the Moore-Penrose spectrum $\sigma_m(T)$ of T is then defined by

$$\sigma_m(T) = \{\lambda \in \mathbb{C} : \mathcal{R}(T-\lambda) \textit{ is not closed}\}.$$

The set $\rho_m(T)$ consists of the complex numbers λ such that $\mathcal{R}(T-\lambda)$ is closed. Clearly, $\rho_m(T) = \mathbb{C} \setminus \sigma_m(T)$. By the Banach closed range theorem (see [14]), we know that $\lambda \in \sigma_m(T)$ if and only if $\bar{\lambda} \in \sigma_m(T^*)$.

Let $\mathcal{R}(T)$ be closed. Then, the operator T is said to be left Fredholm (or upper semi-Fredholm), if $n(T) < \infty$; while if $d(T) < \infty$, we say T is a right Fredholm (or lower semi-Fredholm) operator. If T is both left Fredholm and right Fredholm, then it is Fredholm. For $T \in \mathcal{B}(\mathcal{H}_1)$, the sets

$$\sigma_{le}(T) = \{\lambda \in \mathbb{C} : T-\lambda \textit{ is not left Fredholm}\},$$
$$\sigma_{re}(T) = \{\lambda \in \mathbb{C} : T-\lambda \textit{ is not right Fredholm}\},$$
$$\sigma_{e}(T) = \{\lambda \in \mathbb{C} : T-\lambda \textit{ is not Fredholm}\}$$

are called the left essential spectrum, right essential spectrum and essential spectrum, respectively. In view of the Fredholm alternative theorem, we have that $\lambda \in \sigma_{re}(T)$ if and only if $\bar{\lambda} \in \sigma_{le}(T^*)$ (see [15]).

2. Closedness of range

In this section, we consider the closedness of the range of upper-triangular operator matrices. We first review some basic results.

Lemma 2.1. (*see* [15]) *For $A \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$, the following statements hold.*

(i) *Let $T \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ be of finite rank. Then, $\mathcal{R}(A+T)$ is closed if and only if $\mathcal{R}(A)$ is closed.*

(ii) *Let $S \in \mathcal{B}(\mathcal{H}_3,\mathcal{H}_4)$ with $n(S) < \infty$ and let $T \in \mathcal{B}(\mathcal{H}_2,\mathcal{H}_3)$ be invertible, where $\mathcal{H}_3$ and $\mathcal{H}_4$ are Hilbert spaces. If $\mathcal{R}(STA)$ is closed, then $\mathcal{R}(A)$ is also closed.*

Lemma 2.2. (*see* [1]) *Let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$ and $C \in \mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)$ with $\overline{\mathcal{R}(A)} = \mathcal{H}_1$. If the range $\mathcal{R}(M)$ of the upper-triangular operator matrix M is closed, then $\mathcal{R}(B)$ is closed.*

Theorem 2.3. *Let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$ and $C \in \mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)$. Then, the range $\mathcal{R}(M)$ of the upper-triangular operator matrix M is closed if and only if the following statements are fulfilled:*

(i) $\mathcal{R}(P_{\mathcal{R}(A)^\perp}C|_{\mathcal{N}(B)})$ *is closed;*

(ii) $\mathcal{R}(A) + \mathcal{R}(P_{\overline{\mathcal{R}(A)}}C|_{\mathcal{N}(P_{\mathcal{R}(A)^\perp}C|_{\mathcal{N}(B)})}) = \overline{\mathcal{R}(A)}$;

(iii) $\mathcal{R}(B^*) + \mathcal{R}(P_{\mathcal{N}(B)^\perp}C^*|_{\mathcal{R}(P_{\mathcal{R}(A)^\perp}C|_{\mathcal{N}(B)})^\perp}) = \overline{\mathcal{R}(B^*)}$.

Proof. For notational convenience, we write

$$\begin{gathered} C_4 = P_{\mathcal{R}(A)^\perp} C|_{\mathcal{N}(B)}, \\ C_{22} = P_{\overline{\mathcal{R}(A)}} C|_{\mathcal{N}(C_4)}, \quad C_{32} = P_{\mathcal{R}(C_4)^\perp} C|_{\mathcal{N}(B)^\perp}. \end{gathered} \tag{2.1}$$

Under the orthogonal decompositions

$$\mathcal{N}(A)^\perp \oplus \mathcal{N}(A) = \mathcal{H}_1 = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp, \ \mathcal{N}(B)^\perp \oplus \mathcal{N}(B) = \mathcal{H}_2 = \overline{\mathcal{R}(B)} \oplus \mathcal{R}(B)^\perp,$$

the upper-triangular operator matrix $M \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ admits the following block representation:

$$M = \begin{pmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \overline{\mathcal{R}(B)} \\ \mathcal{R}(B)^\perp \end{pmatrix},$$

where

$$\begin{gathered} A_1 = P_{\overline{\mathcal{R}(A)}} A|_{\mathcal{N}(A)^\perp}, \quad B_1 = P_{\overline{\mathcal{R}(B)}} B|_{\mathcal{N}(B)^\perp}, \\ C_1 = P_{\overline{\mathcal{R}(A)}} C|_{\mathcal{N}(B)^\perp}, \quad C_2 = P_{\overline{\mathcal{R}(A)}} C|_{\mathcal{N}(B)}, \quad C_3 = P_{\mathcal{R}(A)^\perp} C|_{\mathcal{N}(B)^\perp}. \end{gathered} \tag{2.2}$$

Evidently, $\mathcal{R}(M)$ is closed if and only if $\mathcal{R}(M_1)$ is closed, where

$$M_1 = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & B_1 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \overline{\mathcal{R}(B)} \end{pmatrix}.$$

To complete the proof, it suffices to show that $\mathcal{R}(M_1)$ is closed if and only if the conditions (i), (ii) and (iii) hold.

Assume that $\mathcal{R}(M_1)$ is closed. By Lemma 2.2, we know that $\mathcal{R}(M_2)$ is closed, where

$$M_2 = \begin{pmatrix} C_4 & C_3 \\ 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A)^\perp \\ \overline{\mathcal{R}(B)} \end{pmatrix}.$$

From the factorization

$$M_2 = \begin{pmatrix} I_{\mathcal{R}(A)^\perp} & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} I_{\mathcal{R}(A)^\perp} & C_3 \\ 0 & I_{\mathcal{N}(B)^\perp} \end{pmatrix} \begin{pmatrix} C_4 & 0 \\ 0 & I_{\mathcal{N}(B)^\perp} \end{pmatrix}$$

and Lemma 2.1, it follows that the range of $\begin{pmatrix} C_4 & 0 \\ 0 & I_{\mathcal{N}(B)^\perp} \end{pmatrix}$ is closed. This implies that $\mathcal{R}(C_4)$ is closed, i.e., (i) is proven. In view of

$$\mathcal{N}(C_4) \oplus \mathcal{N}(C_4)^\perp = \mathcal{N}(B), \ \mathcal{R}(A)^\perp = \mathcal{R}(C_4) \oplus \mathcal{R}(C_4)^\perp,$$

we may further write $M_i \, (i = 1, 2)$ as the following new block forms:

$$M_1 = \begin{pmatrix} A_1 & C_1 & C_{21} & C_{22} \\ 0 & C_{31} & C_{41} & 0 \\ 0 & C_{32} & 0 & 0 \\ 0 & B_1 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(C_4)^\perp \\ \mathcal{N}(C_4) \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(C_4) \\ \mathcal{R}(C_4)^\perp \\ \overline{\mathcal{R}(B)} \end{pmatrix}, \tag{2.3}$$

$$M_2 = \begin{pmatrix} C_{41} & 0 & C_{31} \\ 0 & 0 & C_{32} \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(C_4)^\perp \\ \mathcal{N}(C_4) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(C_4) \\ \mathcal{R}(C_4)^\perp \\ \overline{\mathcal{R}(B)} \end{pmatrix}, \tag{2.4}$$

where

$$C_{21} = C_2|_{\mathcal{N}(C_4)^\perp}, \quad C_{31} = P_{\mathcal{R}(C_4)}C_3, \quad C_{41} = P_{\mathcal{R}(C_4)}C_4|_{\mathcal{N}(C_4)^\perp}. \tag{2.5}$$

Clearly, C_{41} is invertible. Applying Lemma 2.2 to (2.4), we see that $\mathcal{R}(\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right))$ is closed, which together with the injectiveness of B_1 shows that the column operator $\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right)$ is left invertible. Thus, the following factorization

$$EFM_1 = M_3 \tag{2.6}$$

holds, where

$$M_3 = \begin{pmatrix} A_1 & 0 & 0 & C_{22} \\ 0 & 0 & C_{41} & 0 \\ 0 & C_{32} & 0 & 0 \\ 0 & B_1 & 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} I_{\overline{\mathcal{R}(A)}} & -C_{21}C_{41}^{-1} & 0 \\ 0 & I_{\mathcal{R}(C_4)} & 0 \\ 0 & 0 & I_{\mathcal{R}(C_4)^\perp \oplus \overline{\mathcal{R}(B)}} \end{pmatrix}, F = \begin{pmatrix} I_{\overline{\mathcal{R}(A)}} & 0 & -C_1\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right)_l^{-1} \\ 0 & I_{\mathcal{R}(C_4)} & -C_{31}\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right)_l^{-1} \\ 0 & 0 & I_{\mathcal{R}(C_4)^\perp \oplus \overline{\mathcal{R}(B)}} \end{pmatrix}.$$

Since E and F are both invertible on $\overline{\mathcal{R}(A)} \oplus \mathcal{R}(C_4) \oplus \mathcal{R}(C_4)^\perp \oplus \overline{\mathcal{R}(B)}$, the closedness of $\mathcal{R}(M_1)$ is equivalent to that of $\mathcal{R}(M_3)$. The fact that

$$\mathcal{R}(M_3) = \mathcal{R}((A_1 \ \ C_{22})) \oplus \mathcal{R}(C_{41}) \oplus \mathcal{R}(\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right)) \tag{2.7}$$

is closed indicates that $\mathcal{R}((A_1 \ \ C_{22}))$, $\mathcal{R}(C_{41})$ and $\mathcal{R}(\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right))$ are all closed. Since $\mathcal{R}((A_1 \ \ C_{22})) = \mathcal{R}(A) + \mathcal{R}(C_{22})$ and $\mathcal{R}(A)$ is dense in $\overline{\mathcal{R}(A)}$, we immediately have $\mathcal{R}(A) + \mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$, i.e., (ii) is valid. By the Banach closed range theorem, $\mathcal{R}(\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right))$ is closed if and only if $\mathcal{R}((C_{32}^* \ \ B_1^*))$ is closed. Since $B^* = \left(\begin{smallmatrix} B_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} B_1^* & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we have $\mathcal{R}(B_1^*) = \mathcal{R}(B^*)$. This together with

$$\mathcal{R}((C_{32}^* \ \ B_1^*)) = \mathcal{R}(C_{32}^*) + \mathcal{R}(B_1^*) = \mathcal{R}(B_1^*) + \mathcal{R}(P_{\mathcal{N}(B)^\perp}C^*|_{\mathcal{R}(P_{\mathcal{R}(A)^\perp}C|_{\mathcal{N}(B)})^\perp})$$

implies the desired relation (iii).

Conversely, assume that (i), (ii) and (iii) are valid. By (i), the operator matrix M_1 possesses the block expression (2.3). The condition (iii) and the relation $\mathcal{R}(B_1^*) = \mathcal{R}(B^*)$ imply that $\mathcal{R}((C_{32}^* \ \ B_1^*))$ is closed, which shows that $\mathcal{R}(\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right))$ is also closed. While the condition (ii) ensure that the row operator $(A_1 \ \ C_{22})$ is surjective, and its range $\mathcal{R}((A_1 \ \ C_{22}))$ is naturally closed. Note that C_{41} is invertible and the column operator $\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right)$ is left invertible. Thus, we still have the factorization (2.6). Therefore, we need only prove the closedness of $\mathcal{R}(M_3)$ to accomplish the proof, which is trivial by the relation (2.7). □

In what follows, we adopt the notations defined as in the proof of Theorem 2.3, i.e., in (2.1), (2.2) and (2.5). We now give an example illustrating Theorem 2.3.

Example 2.4. Let $\mathcal{H}_1 = \mathcal{H}_2 = \ell^2$. Define the operators $A \in \mathcal{B}(\ell^2)$, $B \in \mathcal{B}(\ell^2)$ and $C \in \mathcal{B}(\ell^2)$ by $Ax = (x_1, 0, \frac{x_2}{2}, 0, \frac{x_3}{3}, 0, \cdots)$, $Bx = (x_1, x_3, x_5, \cdots)$, $Cx = (0, x_2, x_4, 0, x_6, 0, x_8, 0, \cdots)$ for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$. Consider $M = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \in \mathcal{B}(\ell^2 \oplus \ell^2)$.

Obviously, $\mathcal{R}(A)$ is not closed and $\mathcal{R}(B)$ is closed. A direct calculation shows that

$$\overline{\mathcal{R}(A)} = \{x = (x_1, 0, x_3, 0, x_5, 0, \cdots) : \sum_{k=1}^{\infty} |x_{2k-1}|^2 < \infty\},$$
$$\mathcal{R}(A)^{\perp} = \mathcal{N}(B) = \{x = (0, x_2, 0, x_4, 0, x_6, \cdots) : \sum_{k=1}^{\infty} |x_{2k}|^2 < \infty\}.$$

Then, we have that $C_4 x = (0, x_2, 0, 0, \cdots)$ for $x = (0, x_2, 0, x_4, 0, x_6, \cdots) \in \mathcal{N}(B)$ and $C_{22} x = (0, 0, x_4, 0, x_6, 0, x_8, 0, \cdots)$ for $x = (0, 0, 0, x_4, 0, x_6, 0, x_8, \cdots) \in \mathcal{N}(C_4)$. Clearly, $\mathcal{R}(A) + \mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$, $\mathcal{R}(C_4)$ and $\mathcal{R}(B^*)$ are closed. Thus, by Theorem 2.3, the range $\mathcal{R}(M)$ of M is closed.

In [10], the authors considered the closedness of the range $\mathcal{R}(M_C)$ of the partial operator matrix M_C. Using Theorem 2.3, we shall give two slightly different statements.

Corollary 2.5. *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then, there exists an operator $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that the range $\mathcal{R}(M_C)$ of the partial operator matrix M_C is closed if and only if the following statements are fulfilled:*

$$\begin{cases} n(B) = \infty, & \textit{if } \mathcal{R}(A) \textit{ is not closed and } \mathcal{R}(B) \textit{ is closed;} \\ d(A) = \infty, & \textit{if } \mathcal{R}(A) \textit{ is closed and } \mathcal{R}(B) \textit{ is not closed;} \\ n(B) = \infty = d(A), & \textit{if none of } \mathcal{R}(A) \textit{ and } \mathcal{R}(B) \textit{ is closed.} \end{cases}$$

Proof. Taking $C = 0$ will immediately yield that the range of M_C is closed, if both $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. Thus, to complete the proof, it suffices to consider the other three incompatible cases. Without loss of generality, we only prove the case when $\mathcal{R}(A)$ is not closed and $\mathcal{R}(B)$ is closed.

If there exists an operator $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $\mathcal{R}(M_C)$ is closed, then $\mathcal{R}(A) + \mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$ by Theorem 2.3. This immediately implies that $\mathcal{R}(C_{22})$ is an infinite dimensional subspace of $\overline{\mathcal{R}(A)}$, since $\mathcal{R}(A)$ is not closed. Obviously, $n(B) = \infty$. Conversely, let $n(B) = \infty$. In view of the closedness of $\mathcal{R}(B)$, it suffices to show that $\mathcal{R}(A) + \mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$ and $\mathcal{R}(C_4)$ is closed for some $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Clearly, there exists a closed subspace $\mathcal{N}$ of $\mathcal{N}(B)$ such that $\mathcal{N}(B) = \mathcal{N} \oplus \mathcal{N}^{\perp}$ with $\dim \mathcal{N} = d$ $(0 \le d \le d(A))$ and $\dim \mathcal{N}^{\perp} = \infty$. Since $\mathcal{R}(A)$ is not closed, $\dim \overline{\mathcal{R}(A)} = \infty$. Let $\{e_k\}_{k=1}^{d}$, $\{f_k\}_{k=1}^{d(A)}$, $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ be orthogonal bases of $\mathcal{N}$, $\mathcal{R}(A)^{\perp}$, $\mathcal{N}^{\perp}$ and $\overline{\mathcal{R}(A)}$, respectively. Thus, we may define the isometric operators U_1 from $\mathcal{N}$ into $\mathcal{R}(A)^{\perp}$ and U_2 from $\mathcal{N}^{\perp}$ onto $\overline{\mathcal{R}(A)}$ by

$$U_1 e_i = f_i, \text{ for } i = 1, 2, \cdots, d,$$
$$U_2 g_i = h_i, \text{ for } i = 1, 2, \cdots .$$

Take

$$C = \begin{pmatrix} 0 & 0 & U_2 \\ 0 & U_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B)^{\perp} \\ \mathcal{N} \\ \mathcal{N}^{\perp} \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R} \\ \mathcal{R}^{\perp} \end{pmatrix},$$

where $\mathcal{R}(A)^{\perp} = \mathcal{R} \oplus \mathcal{R}^{\perp}$ with $\dim \mathcal{R}$ being a d-dimensional closed subspace of $\mathcal{R}(A)^{\perp}$. In this case,

$$C_4 = \begin{pmatrix} U_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N} \\ \mathcal{N}^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{R} \\ \mathcal{R}^{\perp} \end{pmatrix}, \quad C_{22} = U_2.$$

Hence, $\mathcal{R}(C_4) = \mathcal{R}$ is closed and C_{22}, as an isometric operator from $\mathcal{N}(C_4)(= \mathcal{N}^{\perp})$ to $\overline{\mathcal{R}(A)}$, is clearly surjective, i.e., $\mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$. Obviously, we also have $\mathcal{R}(A) + \mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$. The proof is finished. □

Corollary 2.6. (*see* [10]) *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then, the range $\mathcal{R}(M_C)$ of the partial operator matrix M_C is closed for every $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ if and only if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are both closed, and at least one of $d(A)$ and $n(B)$ is finite.*

Proof. Assume that the range $\mathcal{R}(M_C)$ of M_C is closed for every $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Taking $C = 0$, we then have the range $\mathcal{R}(\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right))$ is closed, which implies that the ranges $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are both closed. So, by Theorem 2.3, we further see that the range $\mathcal{R}(C_4)$ is closed for every $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Thus, we claim that at least one of $d(A)$ and $n(B)$ is finite. Otherwise, $d(A) = \infty = n(B)$, and then we can conveniently find an operator $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $\mathcal{R}(C_4)$ is not closed.

Conversely, assume that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, and at least one of $d(A)$ and $n(B)$ is finite. Note that C_4 is an operator from $\mathcal{N}(B)$ into $\mathcal{R}(A)^{\perp}$ for every $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Therefore, C_4 must be a finite rank operator, and its range is obviously closed for every $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. By Theorem 2.3, the conclusion is established. □

Corollary 2.7. (*see* [11]) *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then,*

$$\bigcap_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_m(M_C) = \{\lambda \in \sigma_m(A) \colon n(B-\lambda) < \infty\} \cup \{\lambda \in \sigma_m(B) \colon d(A-\lambda) < \infty\}.$$

Proof. Replacing A by $A - \lambda$ and B by $B - \lambda$, we see that the desired result directly follows from Corollary 2.5. □

Corollary 2.8. (*see* [11]) *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then,*

$$\bigcup_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_m(M_C) = \sigma_m(A) \cup \sigma_m(B) \cup \{\lambda \in \mathbb{C} : \, n(B-\lambda) = \infty = d(A-\lambda)\}.$$

Proof. Replacing A by $A - \lambda$ and B by $B - \lambda$, we see that the desired result follows from Corollary 2.6 immediately. □

3. Fredholmness

This section is devoted to the Fredholmness of upper-triangular operator matrices. Note that the notations are defined as in (2.1), (2.2) and (2.5).

Theorem 3.1. *Let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Then, the upper-triangular operator matrix M is a left Fredholm operator if and only if the following statements are fulfilled:*

(i) *$\mathcal{R}(C_4)$ is closed;*
(ii) *$\mathcal{R}(A) + \mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$;*
(iii) *$\mathcal{R}(B^*) + \mathcal{R}({C_{32}}^*) = \overline{\mathcal{R}(B^*)}$;*
(iv) *$n(A) < \infty$, $n(C_{22}) < \infty$ and $\dim \mathcal{R}(A) \cap \mathcal{R}(C_{22}) < \infty$.*

Proof. By Theorem 2.3, the closeness of $\mathcal{R}(M)$ is equivalent to the first three conditions. So, we may assume that they are always satisfied.

From the proof of Theorem 2.3, it follows that the operator matrix M has the block representation

$$M = \begin{pmatrix} A_1 & 0 & C_1 & C_{21} & C_{22} \\ 0 & 0 & C_{31} & C_{41} & 0 \\ 0 & 0 & C_{32} & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(C_4)^\perp \\ \mathcal{N}(C_4) \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(C_4) \\ \mathcal{R}(C_4)^\perp \\ \overline{\mathcal{R}(B)} \\ \mathcal{R}(B)^\perp \end{pmatrix}. \tag{3.1}$$

Then, we further have

$$M = \begin{pmatrix} A_1 & 0 & C_1 & C_{21} & C_{221} & 0 \\ 0 & 0 & C_{31} & C_{41} & 0 & 0 \\ 0 & 0 & C_{32} & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(C_4)^\perp \\ \mathcal{N}(C_{22})^\perp \\ \mathcal{N}(C_{22}) \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(C_4) \\ \mathcal{R}(C_4)^\perp \\ \overline{\mathcal{R}(B)} \\ \mathcal{R}(B)^\perp \end{pmatrix}.$$

Here, $C_{221} = C_{22}|_{\mathcal{N}(C_{22})^\perp}$, C_{41} is invertible and $\left(\begin{smallmatrix} C_{32} \\ B_1 \end{smallmatrix}\right)$ is left invertible. Thus, the null space $\mathcal{N}(M)$ of the upper-triangular operator matrix M is given by

$$\begin{aligned} \mathcal{N}(M) &= \mathcal{N}(A) \oplus \mathcal{N}((A_1 \quad C_{22})) \\ &= \mathcal{N}(A) \oplus \mathcal{N}(C_{22}) \oplus \{\left(\begin{smallmatrix} A_1^{-1}y \\ -C_{221}^{-1}y \end{smallmatrix}\right) : y \in \mathcal{R}(A) \cap \mathcal{R}(C_{22})\}, \end{aligned}$$

from which the theorem follows right away. □

The following is the dual result of Theorem 3.1.

Theorem 3.2. *Let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Then, the upper-triangular operator matrix M is a right Fredholm operator if and only if the following statements are fulfilled:*
(i) *$\mathcal{R}(C_4)$ is closed;*
(ii) *$\mathcal{R}(A) + \mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$;*
(iii) *$\mathcal{R}(B^*) + \mathcal{R}({C_{32}}^*) = \overline{\mathcal{R}(B^*)}$;*
(iv) *$d(B) < \infty$, $d(C_{32}) < \infty$ and $\dim \mathcal{R}(B^*) \cap \mathcal{R}(C_{32}^*) < \infty$.*

As a direct consequence of Theorems 3.1 and 3.2, we have

Theorem 3.3. *Let $A \in \mathcal{B}(\mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2)$ and $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Then, the upper-triangular operator matrix M is a Fredholm operator if and only if the following statements are fulfilled:*
(i) *$\mathcal{R}(C_4)$ is closed;*
(ii) *$\mathcal{R}(A) + \mathcal{R}(C_{22}) = \overline{\mathcal{R}(A)}$;*
(iii) *$\mathcal{R}(B^*) + \mathcal{R}({C_{32}}^*) = \overline{\mathcal{R}(B^*)}$;*
(iv) *$n(A) < \infty$, $n(C_{22}) < \infty$ and $\dim \mathcal{R}(A) \cap \mathcal{R}(C_{22}) < \infty$;*
(v) *$d(B) < \infty$, $d(C_{32}) < \infty$ and $\dim \mathcal{R}(B^*) \cap \mathcal{R}(C_{32}^*) < \infty$.*

Corollary 3.4. (*see* [7]) *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then, there exists an operator $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that the partial operator matrix M_C is a left Fredholm operator if and only if A is a left Fredholm operator and the following statements are fulfilled:*

$$\begin{cases} d(A) = \infty, & if\ \mathcal{R}(B)\ is\ not\ closed; \\ n(B) < \infty\ or\ n(B) = \infty = d(A), & if\ \mathcal{R}(B)\ is\ closed. \end{cases}$$

Proof. Assume that there exists an operator $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that the partial operator matrix M_C is a left Fredholm operator. Evidently, A is a left fredholm operator. By Corollary 2.5, we know that $d(A) = \infty$ if $\mathcal{R}(B)$ is not closed. If $\mathcal{R}(B)$ is closed, then replacing $\overline{\mathcal{R}(A)}$ by $\mathcal{R}(A)$ and $\overline{\mathcal{R}(B)}$ by $\mathcal{R}(B)$, we still have the relation (3.1). In this case, A_1 and B_1 are clearly invertible. Thus, we deduce that

$$\begin{pmatrix} C_{41} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(C_4)^\perp \\ \mathcal{N}(C_4) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(C_4) \\ \mathcal{R}(C_4)^\perp \end{pmatrix}$$

is a left Fredholm operator. Therefore, $n(C_4) < \infty$, which immediately implies $n(B) < \infty$ or $n(B) = d(A) = \infty$.

Conversely, assume that A is a left Fredholm operator. If $\mathcal{R}(B)$ is not closed and $d(A) = \infty$, then there exist infinite dimensional closed subspaces Ω and $\Omega^\perp$ such that $\mathcal{R}(A)^\perp = \Omega \oplus \Omega^\perp$, and hence taking

$$C = \begin{pmatrix} 0 & 0 \\ 0 & T \\ S & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \Omega \\ \Omega^\perp \end{pmatrix}$$

will establish the desired result, where $S : \mathcal{N}(B)^\perp \to \Omega^\perp$ and $T : \mathcal{N}(B) \to \Omega$ are both invertible operators. Indeed, $C_{32} = S$ and $C_4 = \begin{pmatrix} T \\ 0 \end{pmatrix}$, which imply that $\mathcal{N}(C_4) = \{0\}$, $\mathcal{R}(C_4) = \Omega$ and $\begin{pmatrix} C_{32} \\ B_1 \end{pmatrix} : \mathcal{N}(B)^\perp \to \begin{pmatrix} \Omega^\perp \\ \overline{\mathcal{R}(B)} \end{pmatrix}$ is left invertible. So, $n(C_{22}) = 0 = \dim \mathcal{R}(A) \cap \mathcal{R}(C_{22})$, and $\begin{pmatrix} C_{32}^* & B_1^* \end{pmatrix}$ is surjective, i.e., $\mathcal{R}(C_{32}^*) + \mathcal{R}(B_1^*) = \overline{\mathcal{R}(B^*)}$. Thus, the assumptions of Theorem 3.1 are all satisfied. If $\mathcal{R}(B)$ is closed and $n(B) = \infty = d(A)$, we take $C = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{pmatrix}$, where $S : \mathcal{N}(B) \to \mathcal{R}(A)^\perp$ is left invertible. One can verify the left Fredholmness of M_C similarly. The case when $\mathcal{R}(B)$ is closed and $n(B) < \infty$ is trivial. □

The following is the dual result of Corollary 3.4.

Corollary 3.5. (*see* [7]) *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then, there exists an operator $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that the partial operator matrix M_C is a right Fredholm operator if and only if B is a right Fredholm operator and the following statements are fulfilled:*

$$\begin{cases} n(B) = \infty, & if\ \mathcal{R}(A)\ is\ not\ closed; \\ d(A) < \infty\ or\ n(B) = \infty = d(A), & if\ \mathcal{R}(A)\ is\ closed. \end{cases}$$

The two results below are obvious, and their proofs are omitted.

Corollary 3.6. (*see* [7]) *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Then, there exists an operator $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that the partial operator matrix M_C is a Fredholm operator if and only if the following statements are fulfilled:*

(i) *A is a left Fredholm operator;*
(ii) *B is a right Fredholm operator;*
(iii) $n(B) = \infty = d(A)$, *or* $n(B)$ *and* $d(A)$ *are both finite.*

Corollary 3.7. *Let* $A \in \mathcal{B}(\mathcal{H}_1)$ *and* $B \in \mathcal{B}(\mathcal{H}_2)$. *Then, the partial operator matrix* M_C *is a (left or right) Fredholm operator for every* $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ *if and only if A and B are both (left or right) Fredholm operators.*

Similar to the discussions in Section 2, we have

Corollary 3.8. (*see* [7]) *Let* $A \in \mathcal{B}(\mathcal{H}_1)$ *and* $B \in \mathcal{B}(\mathcal{H}_2)$. *Then,*

$$\begin{aligned}\bigcap_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_{le}(M_C) &= \sigma_{le}(A) \cup \{\lambda \in \sigma_m(B) : d(A-\lambda) < \infty\} \\ &\quad \cup \{\lambda \in \rho_m(B) : d(A-\lambda) < \infty, n(B-\lambda) = \infty\}, \\ \bigcap_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_{re}(M_C) &= \sigma_{re}(A) \cup \{\lambda \in \sigma_m(A) : n(B-\lambda) < \infty\} \\ &\quad \cup \{\lambda \in \rho_m(B) : n(B-\lambda) < \infty, d(A-\lambda) = \infty\}, \\ \bigcap_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_{e}(M_C) &= \sigma_{le}(A) \cup \sigma_{re}(B) \cup \{\lambda \in \mathbb{C} : n(B-\lambda) < \infty, d(A-\lambda) = \infty\} \\ &\quad \cup \{\lambda \in \mathbb{C} : n(B-\lambda) = \infty, d(A-\lambda) < \infty\}.\end{aligned}$$

Corollary 3.9. *Let* $A \in \mathcal{B}(\mathcal{H}_1)$ *and* $B \in \mathcal{B}(\mathcal{H}_2)$. *Then,*

$$\bigcup_{C\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)} \sigma_*(M_C) = \sigma_*(A) \cup \sigma_*(B), \ \textit{where} \ * \in \{le, re, e\}.$$

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