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A MAX VERSION OF PERRON–FROBENIUS THEOREM FOR NONNEGATIVE TENSOR

HAMID REZA AFSHIN*, ALI REZA SHOJAEIFARD

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Abstract. In this paper we generalize the max algebra system of nonnegative matrices to the class of nonnegative tensors and derive its fundamental properties. If $\mathbb{A} \in \Re_+^{[m,n]}$ is a nonnegative essentially positive tensor such that satisfies the condition class NC, we prove that there exist $\mu(\mathbb{A})$ and a corresponding positive vector x such that $\max\limits_{1 \leq i_2 \cdots i_m \leq n} \{a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}\} = \mu(\mathbb{A}) x_i^{m-1}$, $i = 1, 2, \cdots, n$. This theorem, is well known as the max algebra version of Perron–Frobenius theorem for this new system.

1. Introduction

The algebraic system max algebra provide an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimization, mathematical physics, DNA analysis and etcetera. The usefulness of max algebras arises from a fact that these non-linear problems become linear when described in the max algebra language. The max eigenproblem is well studied and there are important explicit applications of it in solving the problems mentioned above. In particular, there exists significant analogy with the usual Perron–Frobenius theory. For a recent reference focussing specifically on the Perron–Frobenius Theorem for the max algebra, see [1], wherein several proofs of this fundamental result were presented.

Tensors are increasingly ubiquitous in various areas of applied, computational, and industrial mathematics and have wide applications in data analysis and mining, information science, signal/image processing, computational biology,

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* Corresponding author.
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and so on, see the workshop report [6] and references therein. A tensor can be regarded as a higher order generalization of a matrix, which takes the form

$$\mathbb{A} = (a_{i_1,\cdots,i_m}), \quad a_{i_1,\cdots,i_m} \in \Re, \ \ 1 \leq \ i_1, \cdots, i_m \ \leq n,$$

where $\Re$ is a real field. Such a multi-array $\mathbb{A}$ is said to be an mth order n-dimensional square real tensor with n^m entries $a_{i_1,\cdots,i_m}$. In this regard, a vector is a first order tensor and a matrix is a second order tensor. Tensors of order more than two are called higher order tensors. Many important ideas, notions, and results have been successfully extended from matrices to higher order tensors. Among these, in particular, are the notions and certain basic algebraic and geometric properties of rank, eigenvalue, eigenvector, see [8]. Nonnegative tensors have attracted more and more attention because they share some intrinsic properties with those of the nonnegative matrices. One of those properties is the Perron–Frobenius theorem on eigenvalues. In [2], Chang et al. generalized the Perron–Frobenius theorem for nonnegative matrices to irreducible nonnegative tensors. In [3], Friedland et al. generalized the Perron–Frobenius theorem to weakly irreducible nonnegative tensors. Further generalization of the Perron–Frobenius theorem to nonnegative tensors can be found in [10].

Now, the question arises is it possible to define max algebra system for nonnegative tensors as a generalization of max algebra on nonnegative matrices? Can we describe the analogue of the Perron–Frobenius theory for the system of max algebra on nonnegative tensors, as a generalization of max version theory which is proved in [1]?, in this paper we show the answer is affirmative.

The paper is organized as follows. In Section 2 the fundamental concept of max algebra system and tensors are given briefly for readers. In Section 3 the max algebra system is generalized to nonnegative tensors, also the new version of the Perron–Frobenius theory for some class of nonnegative tensors is given.

We first add a comment on the notation that is used. Vectors are written as $(x, y, \cdots)$, matrices correspond to $(A, B, \cdots)$ and tensors are written as $(\mathbb{A}, \mathbb{B}, \cdots)$. The entry with row index i and column index j in a matrix A, i.e. $(A)_{ij}$ is symbolized by a_{ij} (also $(\mathbb{A})_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m}$). $\Re$ and $\mathcal{C}$ represents the real and complex field, respectively. For each nonnegative integer n, denote $[n] = \{1, 2, \cdots, n\}$. $\Re^n_+ \left(\Re^n_{++}\right)$ denotes the cone $\{x \in \Re^n \ : \ x_i \geq (>) \ 0, \ i = 1, \cdots, n\}$.

2. Preliminaries

2.1. **Max algebra system.** In this section we give the basic definition of the max algebra. The max algebra we consider here is the set $\Re_+$ of nonnegative real numbers, where for $a, b \in \Re_+$ the sum $a \oplus b$ is defined as $\max\{a, b\}$ and the product is defined as the usual product ab. For vectors $x = (x_i)$, $y = (y_i)$ in $\Re^n_+$ and $c \in \Re_+$ the vectors $x \oplus y = (\max\{x_i, y_i\})$ and $cx = (cx_i)$ are defined entrywise. The sum $A \oplus B$ of two matrices is defined analogously. If $A = (a_{ik})$ is a nonnegative n-by-n matrix then the map

$$x \in \Re_+ \Rightarrow A \otimes x \in \Re_+$$

where $(A \otimes x)_i = \max_k a_{ik} x_k,\ i = 1, \cdots, n$ is linear in the sense given above, namely for all $x, y \in \Re_+^n,\ c \in \Re_+$

$$A \otimes (x \oplus y) = (A \otimes x) \oplus (A \otimes y)\,,\ A \otimes (cx) = c\,(A \otimes x)\,.$$

The max-product $C = (c_{il}) = A \otimes B$ of two n-by-n nonnegative matrices $A = (a_{ik})$ and $B = (b_{kl})$ is defined by $c_{il} = \max_k a_{ik} b_{kl}, i, l = 1, 2, \cdots, n$. The weighted directed graph $G(A)$ associated with A has vertex set $\{1, 2, \cdots, n\}$ and an edge (i, j) from vertex i to vertex j with weight a_{ij} if and only if $a_{ij} > 0$. A path $L(i_1, i_2, \cdots, i_{k+1})$ of length k is a sequence of k edges

$$(i_1, i_2)\,,\ (i_2, i_3)\,,\ \cdots\,,\ (i_k, i_{k+1})\,.$$

The weight of a path $L(i_1, i_2, \cdots, i_{k+1})$, as denoted by $w(L(i_1, i_2, \cdots, i_{k+1}))$ or simply by $w(L)$, is defined by

$$w(L(i_1, i_2, \cdots, i_{k+1})) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_{k+1}}.$$

A circuit C of length k is a path $L(i_1, i_2, \cdots, i_{k+1})$ with $i_{k+1} = i_1$, where $i_1, i_2, \cdots, i_{k+1}$ are distinct. Associated with this circuit C is the circuit geometric mean known as $w(C) = (a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1})^{\frac{1}{k}}$. The maximum circuit geometric mean in $G(A)$ is denoted by $\mu(A)$. Note that we also consider empty circuits, namely, circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero. In the literature, the maximum circuit geometric mean $\mu(A)$ has been studied extensively, and it is known that $\mu(A)$ is the largest max eigenvalue of A. Moreover, if A is irreducible, then $\mu(A)$ is the unique eigenvalue and every eigenvector is positive.

Definition 2.1. [1] Let A be an $n \times n$ nonnegative matrix. We say that λ is a max eigenvalue of A if there exists a nonzero, nonnegative vector x such that $A \otimes x = \lambda x$. We refer to x as a corresponding max eigenvector.

Theorem 2.2. *Let A be an $n \times n$ nonnegative, irreducible matrix. Then there exists a positive vector x such that $A \otimes x = \mu(A)\, x$.*

Proof. See [1]. □

2.2. **Basic definition of tensor.** In this subsection, we will cover some fundamental notions and properties on tensors. A tensor can be regarded as a higher order generalization of a matrix, which takes the form

$$\mathbb{A} = (a_{i_1, \cdots, i_m})\,, \quad a_{i_1, \cdots, i_m} \in \Re, \quad 1 \leq\ i_1, \cdots, i_m\ \leq n,$$

where $\Re$ is a real field. Such a multi-array $\mathbb{A}$ is said to be an mth order n-dimensional square real tensor with n^m entries $a_{i_1, \cdots, i_m}$. In this regard, a vector is a first order tensor and a matrix is a second order tensor. Tensors of order more than two are called higher order tensors. An mth order n-dimensional tensor $\mathbb{A}$ is called nonnegative if $a_{i_1 i_2 \cdots i_m} \geq 0$. We denote the set of all nonnegative mth order n-dimensional tensors by $\Re_+^{[m,n]}$. For a vector $x = (x_1, \cdots, x_n)^T$, let $\mathbb{A}x^{m-1}$

be a vector in $\Re^n$ whose ith component is defined as the following:

$$\left(\mathbb{A}x^{m-1}\right)_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}, \tag{2.1}$$

and let $x^{[m]} = \left(x_1^m, \cdots, x_n^m\right)^T$.

Definition 2.3. [8] A pair $(\lambda, x) \in \mathcal{C} \times (\mathcal{C}^n \backslash \{0\})$ is called an eigenvalue and an eigenvector of $\mathbb{A} \in \Re^{[m,n]}$, if they satisfy

$$\mathbb{A}x^{m-1} = \lambda x^{[m-1]}. \tag{2.2}$$

Definition 2.4. [9] Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and order $k \geq 1$), dimension n tensor, respectively. The product $\mathbb{AB}$ is defined to be the following tensor $\mathbb{C}$ of order $(m-1)(k-1)+1$ and dimension n:

$$c_{i\alpha_1\cdots\alpha_{m-1}} = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} b_{i_2\alpha_1}\cdots b_{i_m\alpha_{m-1}},$$

where $\left(i \in [n],\ \alpha_1, \cdots, \alpha_{m-1} \in [n]^{k-1}\right)$.

It is easy to check from the definition that $I_n\mathbb{A} = \mathbb{A} = \mathbb{A}I_n$, where I_n is the identity matrix of order n. When $k = 1$ and $\mathbb{B} = x \in \mathcal{C}^n$ is a vector of dimension n, then $(m-1)(k-1)+1 = 1$. Thus $\mathbb{AB} = \mathbb{A}x$ is still a vector of dimension n, and we have

$$(\mathbb{A}x)_i = (\mathbb{AB})_i = c_i = \sum_{i_2\cdots i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} = \left(\mathbb{A}x^{m-1}\right)_i,$$

thus we have $\mathbb{A}x^{m-1} = \mathbb{A}x$. So the first application of the tensor product defined above is that now $\mathbb{A}x^{m-1}$ can be simply written as $\mathbb{A}x$.

Definition 2.5. [5] A tensor $\mathbb{A} \in \Re^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \cdots, n\}$ such that

$$a_{i_1,\cdots,i_m} = 0,\ \ \forall i_1 \in I,\ \ \forall i_2, \cdots, i_m \notin I,$$

If $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

The following definition was first introduced by Friedland et. al. [3].

Lemma 2.6. *Let $\mathbb{A} \in \Re_+^{[m,n]}$ be an irreducible tensor. Then for every $1 \leq i \leq n$*

$$\sum_{i_2\cdots i_m=1}^{n} a_{ii_2\cdots i_m} > 0.$$

3. Main results

In this section we define the max algebra system on tensors and our interest will be in describing the analogue of the Perron–Frobenius theory for this new system, referred to as the max version of the theory.

Definition 3.1. The max algebraic addition ($\oplus$) and multiplication ($\otimes$) are defined as follows:
(i) Suppose that $\mathbb{A}, \mathbb{B} \in \Re_+^{[m,n]}$ then we have $\mathbb{A} \oplus \mathbb{B} \in \Re_+^{[m,n]}$ and

$$(\mathbb{A} \oplus \mathbb{B})_{i_1 \cdots i_m} = a_{i_1 \cdots i_m} \oplus b_{i_1 \cdots i_m} = \max\left(a_{i_1 \cdots i_m}, b_{i_1 \cdots i_m}\right). \tag{3.1}$$

(ii) Suppose that $\mathbb{A} \in \Re_+^{[m,n]}$ and $\mathbb{B} \in \Re_+^{[k,n]}$ where $m \geq 2\,, k \geq 1$ then we have $\mathbb{A} \otimes \mathbb{B} \in \Re_+^{[(m-1)(k-1)+1,n]}$ and

$$\begin{aligned}(\mathbb{A} \otimes \mathbb{B})_{i\alpha_1 \cdots \alpha_{m-1}} &= \bigoplus_{i_2 \cdots i_m = 1}^{n} a_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}} \\ &= \max_{1 \leq i_2 \cdots i_m \leq n} \left\{a_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}\right\},\end{aligned} \tag{3.2}$$

where $i \in \{1, \cdots, n\}$, $\alpha_1, \cdots, \alpha_{m-1} \in [n]^{k-1}$. In particular for $x \in \Re_+^n$ we have

$$(\mathbb{A} \otimes x)_i = \max_{1 \leq i_2 \cdots i_m \leq n} \left\{a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}\right\}.$$

Example 3.2. Let $\mathbb{A}$ and $\mathbb{B}$ be third-order two-dimensional tensors of the following form:

$$\begin{array}{llll} a_{111} = 1 & a_{121} = 2 & a_{112} = 1 & a_{122} = 2 \\ a_{211} = 2 & a_{221} = 1 & a_{212} = 2 & a_{222} = 1, \end{array}$$

$$\begin{array}{llll} b_{111} = 2 & b_{121} = 0 & b_{112} = 4 & b_{122} = 1 \\ b_{211} = 0 & b_{221} = 3 & b_{212} = 1 & b_{222} = 0, \end{array}$$

if $\mathbb{C} = \mathbb{A} \otimes \mathbb{B}$, then for example $c_{12112} = 24$.
If $x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ then $(\mathbb{A} \otimes x) = \begin{pmatrix} 50 \\ 40 \end{pmatrix}$.

Theorem 3.3. *The max algebraic addition ($\oplus$) and multiplication ($\otimes$) have the following properties:*
(i)*Let $\mathbb{A} \in \Re_+^{[m,n]}$ then $I_n \otimes \mathbb{A} = \mathbb{A} = \mathbb{A} \otimes I_n$ where I_n is a identity matrix.*
(ii)$(\lambda \mathbb{A}) \otimes \mathbb{B} = \lambda (\mathbb{A} \otimes \mathbb{B})$ *where λ be a nonnegative number.*
(iii)$\mathbb{A} \otimes (\lambda \mathbb{B}) = \lambda^{m-1} (\mathbb{A} \otimes \mathbb{B})$ *where λ be a nonnegative number.*
(iv) *Let $\mathbb{A}_1$, $\mathbb{A}_2 \in \Re_+^{[m,n]}$ and $\mathbb{B} \in \Re_+^{[k,n]}$ then $(\mathbb{A}_1 \oplus \mathbb{A}_2) \otimes \mathbb{B} = (\mathbb{A}_1 \otimes \mathbb{B}) \oplus (\mathbb{A}_2 \otimes \mathbb{B})$.*
(v) *Let A be an $n \times n$ matrix and $\mathbb{B}_1, \mathbb{B}_2 \in \Re_+^{[k,n]}$ then $A \otimes (\mathbb{B}_1 \oplus \mathbb{B}_2) = (A \otimes \mathbb{B}_1) \oplus (A \otimes \mathbb{B}_2)$. (Note that in general when A is not a matrix, then the right distributivity doesn't hold.)*

Proof. The proof of **(i)**, **(ii)** and **(iii)** is trivial. We also have

$$\begin{aligned}&((\mathbb{A}_1 \oplus \mathbb{A}_2) \otimes \mathbb{B})_{i\alpha_1 \cdots \alpha_{m-1}} = \max_{1 \leq i_2, \cdots, i_m \leq n} \left((\mathbb{A}_1 \oplus \mathbb{A}_2)_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}\right) \\ &= \max_{1 \leq i_2, \cdots, i_m \leq n} \left(\max\left((\mathbb{A}_1)_{ii_2 \cdots i_m}, (\mathbb{A}_2)_{ii_2 \cdots i_m}\right) b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}\right) \\ &= \max_{1 \leq i_2, \cdots, i_m \leq n} \left(\max\left((\mathbb{A}_1)_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}, (A_2)_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}\right)\right) \\ &= \max\left(\max_{1 \leq i_2, \cdots, i_m \leq n} \left((\mathbb{A}_1)_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}, (\mathbb{A}_2)_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}\right)\right) \\ &= \max\left(\max_{1 \leq i_2, \cdots, i_m \leq n} (\mathbb{A}_1)_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}, \max_{1 \leq i_2, \cdots, i_m \leq n} (\mathbb{A}_2)_{ii_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}}\right) \\ &= ((\mathbb{A}_1 \otimes \mathbb{B}) \oplus (\mathbb{A}_2 \otimes \mathbb{B}))_{i\alpha_1 \cdots \alpha_{m-1}}.\end{aligned}$$

Thus the proof of **(iv)** is complete. The proof of **(v)** is similar. $\square$

Now we use a method similar with the proof of Theorem 1.1 in [9] to show the associative law.

Theorem 3.4. *Let* $\mathbb{A}$ *(and* $\mathbb{B}$*,* $\mathbb{C}$*) be an order* $m+1$ *(and order* $k+1$*, order* $r+1$*), dimension* n *tensor, respectively. Then we have*

$$\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}) = (\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C}.$$

Proof. For $\beta_1, \cdots, \beta_m \in ([n]^r)^k$, we write:

$$\beta_1 = \theta_{11} \cdots \theta_{1k}, \ \cdots, \ \beta_m = \theta_{m1} \cdots \theta_{mk} \quad (\theta_{ij} \in [n]^r, \ i = 1, \cdots, m; \ j = 1, \cdots, k).$$

Then we have:

$$\begin{aligned}
(\mathbb{A} \otimes (\mathbb{B} \otimes \mathbb{C}))_{i\beta_1 \cdots \beta_m} &= \max_{1 \le i_1, \cdots, i_m \le n} a_{ii_1 \cdots i_m} \left(\prod_{j=1}^{m} (\mathbb{B} \otimes \mathbb{C})_{i_j \beta_j} \right) \\
&= \max_{1 \le i_1, \cdots, i_m \le n} a_{ii_1 \cdots i_m} \left(\prod_{j=1}^{m} (\mathbb{B} \otimes \mathbb{C})_{i_j \theta_{j1} \cdots \theta_{jk}} \right) \\
&= \max_{1 \le i_1, \cdots, i_m \le n} a_{ii_1 \cdots i_m} \left(\prod_{j=1}^{m} \max_{1 \le t_{j1}, \cdots, t_{jk} \le n} b_{i_j t_{j1} \cdots t_{jk}} \left(c_{t_{j1}\theta_{j1}} \cdots c_{t_{jk}\theta_{jk}} \right) \right) \\
&= \max_{1 \le i_1, \cdots, i_m \le n} a_{ii_1 \cdots i_m} \max_{1 \le t_{jh} \le n (1 \le j \le m; 1 \le h \le k)} \left(\prod_{j=1}^{m} b_{i_j t_{j1} \cdots t_{jk}} \left(c_{t_{j1}\theta_{j1}} \cdots c_{t_{jk}\theta_{jk}} \right) \right).
\end{aligned}$$

On the other hand, for $\alpha_1, \cdots, \alpha_m \in [n]^k$, we write:

$$\alpha_1 = t_{11} \cdots t_{1k}, \cdots, \alpha_m = t_{m1} \cdots t_{mk} \quad (t_{ij} \in [n], \ i = 1, \cdots, m; \ j = 1 \cdots, k).$$

Then we also have:

$$\begin{aligned}
((\mathbb{A} \otimes \mathbb{B}) \otimes \mathbb{C})_{i\beta_1 \cdots \beta_m} &= \max_{\alpha_1, \cdots, \alpha_m \in [n]^k} (\mathbb{A} \otimes \mathbb{B})_{i\alpha_1 \cdots \alpha_m} \left(\prod_{j=1}^{n} \left(c_{t_{j1}\theta_{j1}} \cdots c_{t_{jk}\theta_{jk}} \right) \right) \\
&= \max_{1 \le t_{jh} \le n (1 \le j \le m; 1 \le h \le k)} \max_{1 \le i_1, \cdots, i_m \le n} a_{ii_1 \cdots i_m} \left(\prod_{j=1}^{m} b_{i_j \alpha_j} \right) \left(\prod_{j=1}^{m} c_{t_{j1}\theta_{j1}} \cdots c_{t_{jk}\theta_{jk}} \right) \\
&= \max_{1 \le i_1, \cdots, i_m \le n} a_{ii_1 \cdots i_m} \max_{1 \le t_{jh} \le n (1 \le j \le m; 1 \le h \le k)} \left(\prod_{j=1}^{m} b_{i_j t_{j1} \cdots t_{jk}} \left(c_{t_{j1}\theta_{j1}} \cdots c_{t_{jk}\theta_{jk}} \right) \right).
\end{aligned}$$

Thus the proof is complete. □

Theorem 3.5. *Let* $\mathbb{A}, \mathbb{B}, \in \Re_+^{[m,n]}$ *and* T, S *are both matrices. Then*

$$T \otimes (\mathbb{A} \oplus \mathbb{B}) \otimes S = (T \otimes \mathbb{A} \otimes S) \oplus (T \otimes \mathbb{B} \otimes S).$$

Proof. By the left distributive law and right distributive, the proof is clear. □

Lemma 3.6. *Let* $\mathbb{A}, \mathbb{B}, \in \Re_+^{[m,n]}$ *and* $y \in \Re^n$*. Then*
(i) $(\mathbb{A} \oplus \mathbb{B}) \otimes y = (\mathbb{A} \otimes y) \oplus (\mathbb{B} \otimes y)$.
(ii) $\mathbb{I} \otimes y = \mathbb{I} y = y^{[m-1]}$.

Proof. By Definition 3.1 the assertion is clear. □

Lemma 3.7. *Suppose that $\mathbb{A} \in \Re_+^{[m,n]}$ and P be an $n \times n$ permutation matrix. Then*

$$P\mathbb{A}P^T = P \otimes \mathbb{A} \otimes P^T \tag{3.3}$$

Proof. Let $\sigma \in S_n$ be a permutation on the set $\{1, \cdots, n\}$, $P = P_\sigma = (p_{ij})$ be the corresponding permutation matrix of σ where $p_{ij} = 1$ iff $\sigma(i) = j$. Then

$$(P\mathbb{A})_{i_1 \cdots i_m} = a_{\sigma(i_1) i_2 \cdots i_m} = (P \otimes \mathbb{A})_{i_1 \cdots i_m} = \max_{j_1} a_{j_1 i_2 \cdots i_m} p_{i_1 j_1}.$$

Similarly $\mathbb{A}P = \mathbb{A} \otimes P$. Thus by Theorem 3.4 we have

$$P \otimes \mathbb{A} \otimes P^T = (P \otimes \mathbb{A}) \otimes P^T = (P\mathbb{A}) \otimes P^T = P\mathbb{A}P^T$$

□

Lemma 3.8. *Let $\mathbb{A} \in \Re^{[m,n]}$ be a reducible and P be an $n \times n$ permutation matrix. Then $P\mathbb{A}P^T \in \Re^{[m,n]}$ is a reducible tensor.*

Proof.

$$\begin{aligned}\left(P\mathbb{A}P^T\right)_{i_1 \cdots i_m} &= \sum_{j_1, \cdots, j_m = 1}^{n} a_{j_1 \cdots j_m} p_{i_1 j_1} \left(\left(P^T\right)_{j_2 i_2} \cdots \left(P^T\right)_{j_m i_m}\right) \\ &= \sum_{j_1, \cdots, j_m = 1}^{n} a_{j_1 \cdots j_m} p_{i_1 j_1} p_{i_2 j_2} \cdots p_{i_m j_m} = a_{\sigma(i_1) \cdots \sigma(i_m)}.\end{aligned}$$

Where σ is defined in pervious lemma. Therefore by Definition 2.5 the proof is complete. □

Theorem 3.9. *Suppose that $\mathbb{A} \in \Re_+^{[m,n]}$, $n \geq 2$, irreducible, and y is a nonnegative n-tuple with exactly k positive coordinates, $1 \leq k \leq n-1$, then $(\mathbb{I} \oplus \mathbb{A}) \otimes y$ has more than k positive coordinates.*

Proof. Suppose that k coordinates of y are positive and the others are zero. Let P be a permutation matrix such that the first k coordinates of $x = P \otimes y$ are positive and the others are zero. Since $\mathbb{A}$ is a nonnegative tensor hence the number of zero coordinates in $(\mathbb{I} \oplus \mathbb{A}) \otimes y$ cannot be greater than $n-k$. Because if $y_i > 0$ then

$$((\mathbb{I} \oplus \mathbb{A}) \otimes y)_i = ((\mathbb{I} \otimes y) \oplus (\mathbb{A} \otimes y))_i = \max_i \left\{y_i^{m-1}, (\mathbb{A} \otimes y)_i\right\} \geq y_i^{m-1} > 0.$$

Suppose it is $n-k$. That is if $y_i = 0$ then

$$0 = ((\mathbb{I} \oplus \mathbb{A}) \otimes y)_i = \max_i \left\{y_i^{m-1}, (\mathbb{A} \otimes y)_i\right\} = (\mathbb{A} \otimes y)_i.$$

Therefore $(P \otimes y)_i = 0$ then $(P \otimes (\mathbb{A} \otimes y))_i = 0$. But $x = P \otimes y$ and therefore the assumption that $(\mathbb{I} \oplus \mathbb{A}) \otimes y$ has as many 0's as y is equivalent to the assertion that $\left(\left(P \otimes \mathbb{A} \otimes P^T\right) \otimes x\right)_i = 0$ for $i = k+1, k+2, \cdots, n$. By relation 3.3 $\left(\left(P\mathbb{A}P^T\right) \otimes x\right)_i = 0$ for $i = k+1, k+2, \cdots, n$. Let $\mathbb{B} = (b_{i_1 \cdots i_m}) = P\mathbb{A}P^T$ Then

$$0 = (\mathbb{B} \otimes x)_i = \max_{1 \leq i_2, \cdots, i_m \leq n} \left\{b_{i i_2 \cdots i_m} y_{i_2} \cdots y_{i_m}\right\},$$

for $i = k+1, k+2, \cdots, n$. Therefore $\max\limits_{1 \leq i_2, \cdots, i_m \leq k} \left\{b_{i i_2 \cdots i_m} y_{i_2} \cdots y_{i_m}\right\} = 0$ for $i = k+1, k+2, \cdots, n$. Since for $i_2, \cdots, i_m \in \{1, \cdots, k\}$ we have $y_{i_2}, \cdots, y_{i_m}$ are positive thus $b_{i i_2 \cdots i_m} = 0$ for $i \in \{k+1, \cdots, n\}$ and $i_2, \cdots, i_m \in \{1, \cdots, k\}$.

This means that $\mathbb{B}$ is reducible thus by Lemma 3.8 $\mathbb{A}$ is reducible. and this is a contradiction. $\square$

Definition 3.10. Let $\mathbb{A} \in \Re_+^{[m,n]}$. We say that λ is a max eigenvalue of $\mathbb{A}$ if there exists a nonzero, nonnegative vector x such that $\mathbb{A} \otimes x = \lambda x^{[m-1]}$. We refer to x as a corresponding max eigenvector.

Definition 3.11. For a given $\mathbb{A} = (a_{i_1 \cdots i_m}) \in \Re_+^{[m,n]}$, it is associated to a directed graph $G(\mathbb{A}) = (V, E(\mathbb{A}))$, where $V = \{1, 2, \cdots, n\}$ and a directed edge $(i,j) \in E(\mathbb{A})$ if there exists indices $\{i_2, \cdots, i_m\}$ such that $j \in \{i_2, \cdots, i_m\}$ and $a_{ii_2 \cdots i_m} > 0$. In particular, we have $\sum\limits_{j \in \{i_2, \cdots, i_m\}} a_{ii_2 \cdots i_m} > 0$. A graph is strongly connected if it contains a directed path from i to j and a directed path from j to i for every pair of vertices i, j.

Definition 3.12. ([7]) Suppose that $\mathbb{A}$ is a nonnegative tensor of order m and dimension n. $\mathbb{A}$ is called essentially positive if $\mathbb{A}x \in \Re_{++}^n$ for any nonzero $x \in \Re_+^n$.

It is clear that $\mathbb{A}$ is essentially positive iff for any $i, j \in [n]$, $a_{ij \cdots j} > 0$ holds. Also a nonnegative essentially positive tensor is irreducible (see Theorem (3.2) in [7]).

Definition 3.13. Let $\mathbb{A} \in \Re_+^{[m,n]}$ be an essentially positive tensor. Consider the directed graph $G(\mathbb{A}) = (V, E(\mathbb{A}))$. In this directed graph, k is a simple cycle of length q described by a sequence of distinct integers $i_1, \cdots, i_q \in \{1, \cdots, n\}$. Then with $|k| = q$,

$$\mu(\mathbb{A}) = \max_k \left\{ \left(a_{i_1 i_2 \cdots i_2} \, a_{i_2 i_3 \cdots i_3} \cdots a_{i_q i_1 \cdots i_1} \right)^{\frac{1}{|k|}} \right\}.$$

The following result plays a central role in the proof of the main result of this section.

Lemma 3.14. *Let $\mathbb{A} \in \Re_+^{[m,n]}$ be irreducible tensor and $x \in \Re^n$, $x \geq 0$, $x \neq 0$, $\lambda > 0$ such that $\mathbb{A} \otimes x = \lambda x^{[m-1]}$. Then x is positive.*

Proof. Suppose that $\mathbb{A} \otimes x = \lambda x^{[m-1]}$ where $\mathbb{A} \geq 0$ is irreducible, $x \geq 0$, and $x \neq 0$. Clearly, λ must be nonnegative. Now

$$(\mathbb{I} \oplus \mathbb{A}) \otimes x = (\mathbb{I} \otimes x) \oplus (\mathbb{A} \otimes x) = x^{[m-1]} \oplus (\mathbb{A} \otimes x) = (1 \oplus \lambda) \, x^{[m-1]},$$

thus $(\mathbb{I} \oplus \mathbb{A}) \otimes x = (1 \oplus \lambda) \, x^{[m-1]}$. If x had k zero coordinates, $1 \leq k < n$, then $(1 \oplus \lambda) \, x^{[m-1]}$ would have k zeros as well, whereas by Theorem 3.9, $(\mathbb{I} \oplus \mathbb{A}) \otimes x$ would have less than k zeros. Hence x must be positive. $\square$

Definition 3.15. We define NC to be the set of all $\mathbb{A} \in \Re_+^{[m,n]}$ such that for it, there exist $x \neq 0$, $x \in \Re_+^n$ and $\lambda > 0$ such that $\mathbb{A} \otimes x = \lambda x^{[m-1]}$ and $\left\{ (i,j) : a_{ij \cdots j} x_j^{m-1} = \lambda x_i^{m-1}, \ 1 \leq i, j \leq n \right\}$ has at least a circuit.

Lemma 3.16. *Let $\mathbb{A}$ be essentially positive tensor such that belong to NC, Then $\lambda = \mu(\mathbb{A})$.*

Proof. Since $\mathbb{A}$ is an essentially positive tensor thus it is irreducible and therefore by lemma 3.14, $x > 0$. We then have

$$(\mathbb{A} \otimes x)_i = \max_{1 \le i_2 \cdots i_m \le n} \{a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}\} = \lambda x_i^{m-1}, \quad i = 1, 2, \cdots, n. \tag{3.4}$$

If $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_1)$ is a circuit in $G(\mathbb{A})$, then by (3.4),

$$a_{i_t i_{t+1} \cdots i_{t+1}} x_{i_{t+1}}^{m-1} \le \lambda x_{i_t}^{m-1}, \; t = 1, 2, \cdots, k, \tag{3.5}$$

where $k+1$ is taken to be 1. It follows that $\sqrt[k]{a_{i_1 i_2 \cdots i_2} a_{i_2 i_3 \cdots i_3} \cdots a_{i_k i_1 \cdots i_1}} \le \lambda$, and thus we have shown that $\mu(\mathbb{A}) \le \lambda$ because the circuit is arbitrary. Since $\mathbb{A}$ belong to NC, thus we conclude that $\lambda \le \mu(\mathbb{A})$. therefore the proof is complete. □

Lemma 3.17. *Let $\mathbb{A} \in \Re_+^{[m,n]}$ be essentially positive tensor and suppose $x \in \Re^n$, $x \ge 0$, $x \ne 0$, $\lambda > 0$ such that $\mathbb{A} \otimes x = \lambda x^{[m-1]}$. Then $\mu(\mathbb{A}) \le \lambda$.*

The following result, extends the Perron–Frobenius theorem to essentially positive nonnegative tensors over the max algebra.

Theorem 3.18. *Let $\mathbb{A}$ be essentially positive tensor such that belong to NC. Then there exists a positive vector x such that $\mathbb{A} \otimes x = \mu(\mathbb{A}) \, x^{[m-1]}$.*

Proof. Suppose that $E = \{x \in \Re^n : \; x \ge 0, \; \|x\|_1 = 1\}$ which is a nonempty, compact, and convex set. Now we define the map $f : E \to E$ as

$$f(x) = \frac{(\mathbb{A} \otimes x)^{\frac{1}{m-1}}}{\left(\sum_{i=1}^{n} (\mathbb{A} \otimes x)_i\right)^{\frac{1}{m-1}}}$$

if $x \in E$. Since $\mathbb{A}$ is irreducible thus by Lemma 2.6, $f(x)$ well defined on E. Also f is continuous. By Brouwer's fixed-point theorem, there exists $x_0 \in E$ such that $f(x_0) = x_0$. Therefore

$$(\mathbb{A} \otimes x_0)^{\frac{1}{m-1}} = \left(\sum_{i=1}^{n} (\mathbb{A} \otimes x_0)_i\right)^{\frac{1}{m-1}} x_0$$

hence

$$(\mathbb{A} \otimes x_0) = \left(\sum_{i=1}^{n} (\mathbb{A} \otimes x_0)_i\right) {x_0}^{[m-1]}.$$

By Lemma 3.14 and Lemma 3.16 we have $\mu(\mathbb{A}) = \left(\sum_{i=1}^{n} (\mathbb{A} \otimes x_0)_i\right)$ and $x_0 > 0$ thus the proof is complete. □

Example 3.19. Consider the positive order 3, dimension 2 tensor given by $a_{122} = a_{211} = t > 0$, $a_{ijk} = z > 0$ otherwise, and let $t > z$. Then $\lambda = t$ and $x_1 = x_2$ satisfy in $\mathbb{A} \otimes x = \lambda x^{[2]}$, thus we have $\mu(\mathbb{A}) = t$.

Example 3.20. Consider the positive order 3, dimension 2 tensor given by $a_{121} = a_{221} = 1$ and $a_{ijk} = 0.1$ otherwise. Then $\lambda = 1$ and $x_1 = x_2$ satisfy in $\mathbb{A} \otimes x = \lambda x^{[2]}$, but we have $\mu(\mathbb{A}) = 0.1$.

Remark 3.21. By Theorem 2 in [4], if the function $f:\Re^n_+ \to \Re^n_+$ is homogeneous, monotone and $G(A)$ is strongly connected then f has an eigenvector in $\Re^n_+$. In case $m=2$ Theorem 3.18 is a result of this theorem.

Theorem 3.18 is right for the class of weakly positive tensors which satisfy in definition NC, since for a tensor $\mathbb{A}$ belong to this class we have $a_{ij\cdots j}$ is positive for all $i \neq j$. Now it is very nice if one could generalize Theorem 3.18 to other classes of tensors, it is an unsolved problem.

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Department of Mathematics, Faculty of Mathematical Sciences, Vali-e-Asr University of Rafsanjan, Rafsanjan, Islamic Republic of Iran.

E-mail address: afshin@vru.ac.ir; alireza.shojaei@yahoo.com