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THE STABLE RANK OF C^*-MODULES

MAURICIO ACHIGAR

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Abstract. We prove equality between the Topological Stable Rank and the Bass Stable Rank for finitely generated projective left modules over a unital C^*-algebra. In order to do so, the concept of Stable Rank of a Hilbert module is introduced.

1. Introduction and preliminaries

In the mid 1960s, H. Bass introduced the concept of Stable Rank of a ring A, now refered to as Bass Stable Rank and denoted by $\mathrm{Bsr}(A) \in \mathbb{N}$. In the late 1970s, R. B. Warfield extend this concept defining the Bass Stable Rank for modules over rings. Later, in [4], M. A. Rieffel introduced the notion of Topological Stable Rank for a Banach algebra A, $\mathrm{tsr}(A) \in \mathbb{N}$, as well as for Banach modules over unital Banach algebras. In this work, Rieffel shows that $\mathrm{Bsr}(A) \leq \mathrm{tsr}(A)$ holds for unital Banach Algebras and that $\mathrm{Bsr}(V) \leq \mathrm{tsr}(V)$ holds for finitely generated projective modules V over unital C^*-algebras. In [3], R. H. Herman and L. N. Vaserstein prove that $\mathrm{Bsr}(A) = \mathrm{tsr}(A)$ for a unital C^*-algebra A.

In this article we show that $\mathrm{Bsr}(V) = \mathrm{tsr}(V)$ for finitely generated projective modules V over unital C^*-algebras, using similar techniques to the ones presented in [3]. In order to generalize Herman-Vaserstein's theorem, we introduce the concept of Stable Rank for a Hilbert module. This definition is inspired by the works of Ara and Goodearl [1] and Blackadar [2].

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2. Types of stable ranks

Definition 2.1. Let ${}_AV$ be a left module over a ring A. The set of n-generators of V is defined as

$$\mathrm{Gen}_n(V) = \{(x_1, \dots, x_n) \in V^n : A \cdot x_1 + \dots + A \cdot x_n = V\}.$$

When $n = 1$ we simply write $\mathrm{Gen}(V) = \mathrm{Gen}_1(V)$. We say that an $(n+1)$-generator $(x_1, \dots, x_n, y) \in \mathrm{Gen}_{n+1}(V)$ is *reducible* if there exist $a_1, \dots, a_n \in A$ such that $(x_1 + a_1 \cdot y, \dots, x_n + a_n \cdot y) \in \mathrm{Gen}_n(V)$.

Remark 2.2. The column space V^n can be viewed as a left module over the matrix ring $M_n(A)$. In that case we have

$$\mathrm{Gen}_n({}_AV) = \mathrm{Gen}({}_{M_n(A)}V^n).$$

Definition 2.3. Let ${}_AV$ be a left module over a ring A. The *Bass Stable Rank* of V, denoted $\mathrm{Bsr}(V)$, is defined as the least $n \in \mathbb{N}$ $(n \geq 1)$ such that every $(n+1)$-generator $(x_1, \dots, x_n, y) \in \mathrm{Gen}_{n+1}(V)$ is reducible.

When $V = {}_AA$ and A has a unit, this definition becomes definition 2.1 in [4] and it's due to Bass. For arbitrary V and unital A, the above definition is equivalent to definition [4, Definition 9.1] of Bass stable rank for modules, introduced by Warfield.

Definition 2.4. ([4, Definition 9.3]) Let ${}_AV$ be a left Banach module over a Banach algebra A. The *Topological Stable Rank* of V is defined as

$$\mathrm{tsr}(V) = \min\{n \in \mathbb{N} : \mathrm{Gen}_n(V) \text{ is dense in } V^n\}.$$

3. Stable rank for Hilbert modules

Definition 3.1. Given a right Hilbert module X_B over a unital C^*-algebra B, we consider

$$\mathrm{Um}_n(X) = \{(x_1, \dots, x_n) \in X^n : \textstyle\sum_k \langle x_k, x_k \rangle \in \mathrm{GL}(B)\}.$$

An n-tuple in $\mathrm{Um}_n(X)$ is called *unimodular* tuple. If $n = 1$ we write $\mathrm{Um}(X) = \mathrm{Um}_1(X)$.

Remark 3.2. The column space X^n can be viewed as a right C^*-module over B, and in this case we have

$$\mathrm{Um}_n(X_B) = \mathrm{Um}(X_B^n).$$

Definition 3.3. Let X_B be a right Hilbert module over a unital C^*-algebra B. We define the *Stable Rank* of X_B as

$$\mathrm{sr}(X_B) = \min\{n \in \mathbb{N} : \mathrm{Um}_n(X) \text{ is dense in } X^n\}.$$

Note that if $k \geq \mathrm{sr}(X_B)$ then $\mathrm{Um}_k(X)$ is dense in X^k.

Remark 3.4. For a unital C^*-algebra B, taking $X_B = B_B$ in the previous definition we recover Rieffel's definition of (left) topological stable rank of B [4, Definition 1.4]. Indeed, Lemma 3.7 or Remark 3.8 can be used to see that $\mathrm{Um}_n(B_B) = \mathrm{Lg}_n(B)$, the later being the set of left n-generators of B considered in [4]. Then, $\mathrm{sr}(B_B) = \mathrm{tsr}(B)$ $(= \mathrm{Bsr}(B))$.

Remark 3.5. For projections p, q in a C^*-algebra A, Blackadar ([2]) considered the set

$$\mathrm{Lg}_{(p,q)}(A) = \{x \in pAq : \exists\, y \in qAp \text{ such that } yx = q\}.$$

and used the condition of $\mathrm{Lg}_{(p,q)}(A)$ being dense in pAq. Taking X as the skew corner $X = pAq$ and $B = qAq$ we have, by Lemma 3.7, that $\mathrm{Lg}_{(p,q)}(A) = \mathrm{Um}(X_B)$, and $\mathrm{Lg}_{(p,q)}(A)$ is dense in pAq if and only if $\mathrm{sr}(X_B) = 1$.

Example 3.6. Let A be a unital C^*-algebra and consider the set $M_{n\times m}(A)$ as a right C^*-module over $M_m(A)$ with formal matrix operations. The stable rank of $M_{n\times m}(A)$ is

$$\mathrm{sr}(M_{n\times m}(A)) = \left\lceil \frac{\mathrm{sr}(A) + m - 1}{n} \right\rceil. \tag{3.1}$$

This expression extends the well-known formula for $\mathrm{sr}(M_n(A))$ ([4, theorem 6.1]).

Proof. Firstly, by Lemma 3.7 we know that $\mathrm{Um}(M_{n\times m}(A))$ is the set of left invertible $n \times m$ matrices over A. Then, by [2, Corollary 4.3] we have

$$\mathrm{Um}(M_{(n+1)\times 1}(A)) \text{ is dense} \quad \text{iff} \quad \mathrm{Um}(M_{(n+k)\times k}(A)) \text{ is dense},$$

where "$\mathrm{Um}(M_{r\times s}(A))$ dense" means dense in $M_{r\times s}(A)$. Equivalently,

$$\mathrm{Um}(M_{r\times s}(A)) \text{ is dense} \quad \text{iff} \quad \mathrm{Um}(M_{(r-s+1)\times 1}(A)) \text{ is dense}, \quad \text{for } r \geq s. \tag{3.2}$$

Secondly, we can realize $M_{n\times m}(A)$ as a skew corner of $M_{\bar{n}}(A)$ for $\bar{n}$ large in the following way: $M_{n\times m}(A) \cong pM_{\bar{n}}(A)q$ for $p, q \in M_{\bar{n}}(A)$ diagonal projections of ranks n and m, respectively. Then, by [2, Proposition 3.2.iii] we have

$$\text{If } \mathrm{Um}(M_{n\times m}(A)) \text{ is dense then } n \geq m. \tag{3.3}$$

For $k \in \mathbb{N}$, we have $k \geq \mathrm{sr}(M_{n\times m}(A))$ iff $\mathrm{Um}_k(M_{n\times m}(A))$ is dense. Identifying the column space $M_{n\times m}(A)^k$ with $M_{nk\times m}(A)$ we have $\mathrm{Um}_k(M_{n\times m}(A)) = \mathrm{Um}(M_{n\times m}(A)^k) = \mathrm{Um}(M_{nk\times m}(A))$. If $\mathrm{Um}(M_{nk\times m}(A))$ is dense then $nk \geq m$, by (3.3), and $\mathrm{Um}(M_{(nk-m+1)\times 1}(A))$ is dense, by (3.2). Therefore $nk - m + 1 \geq \mathrm{sr}(A)$ by definition of $\mathrm{sr}(A)$, and then $k \geq \frac{\mathrm{sr}(A)+m-1}{n}$. Conversely, the inequalities imply $nk \geq m$ and the density of $\mathrm{Um}(M_{(nk-m+1)\times 1}(A))$, then by (3.2) $\mathrm{Um}(M_{nk\times m}(A))$ ($= \mathrm{Um}_k(M_{n\times m}(A))$) is dense, and finally $k \geq \mathrm{sr}(M_{n\times m}(A))$. Thus we have shown that $k \geq \mathrm{sr}(M_{n\times m}(A))$ *iff* $k \geq \frac{\mathrm{sr}(A)+m-1}{n}$, which is equivalent to (3.1). $\square$

Lemma 3.7. *Let X_B be a unital Hilbert module. For $x \in X$, the following are equivalent:*

(a) $\langle x, x\rangle \in GL(B)$.
(b) *There exists $y \in X$ such that $\langle y, x\rangle = 1$.*
(b*) *There exists $y \in X$ such that $\langle x, y\rangle = 1$.*
(c) *There exists $y \in X$ such that $\langle y, x\rangle \in GL(B)$.*
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Proof. It suffices to show that (a) is equivalent to (b). If $\langle x, x\rangle \in \mathrm{GL}(B)$ let $b = \langle x, x\rangle$ and $y = x(b^{-1})^*$. Then $\langle y, x\rangle = b^{-1}\langle x, x\rangle = 1$. On the other hand, if $\langle y, x\rangle = 1$ we have $1 = \langle y, x\rangle^*\langle y, x\rangle \leq \|y\|^2\langle x, x\rangle$, so that $\langle x, x\rangle \in \mathrm{GL}(B)$. $\square$

Remark 3.8. Applying Lemma 3.7 to X^n we obtain different expressions for $\mathrm{Um}_n(X)$. For example, using (b) we have

$$\mathrm{Um}_n(X) = \{(x_1, \dots, x_n) \in X^n : \exists\, y_1, \dots, y_n \in X \ / \ \textstyle\sum_k \langle y_k, x_k\rangle = 1\}.$$

Lemma 3.9. *Let X_B be a unital Hilbert module. Then X_B is full if and only if there exists $n \in \mathbb{N}$ such that $Um_n(X) \neq \varnothing$.*

Proof. The module X_B is full if and only if the $*$-ideal $J = \mathrm{span}(\langle X, X\rangle)$ is dense in B, iff $J \cap \mathrm{GL}(B) \neq \varnothing$, iff $1 \in J$ $(= B)$, iff there exists $n \in \mathbb{N}$ and $x_1, \dots, x_n, y_1, \dots, y_n \in X$ such that $\sum_k \langle x_k, y_k\rangle = 1$, iff there exists $n \in \mathbb{N}$ and $x, y \in X^n$ such that $\langle x, y\rangle = 1$, iff there exists $n \in \mathbb{N}$ such that $\mathrm{Um}_n(X) \neq \varnothing$. ☐

Notice that if X_B is not full then $\mathrm{Um}_n(X) = \varnothing$ for all $n \in \mathbb{N}$, and therefore $\mathrm{sr}(X) = \infty$. Thus, throughout this paper we shall consider X_B to be full.

Definition 3.10. A C^*-*correspondence* is a right Hilbert module X_B equipped with a left action $A \to \mathcal{L}(X_B)$ of a C^*-algebra A by adjointable operators. When B has a unit we say that the correspondence is *right-unital*.

Lemma 3.11. *Let ${}_AX_B$ be a full and right-unital C^*-correspondence. Then*

$$Gen_n({}_AX) \subseteq Um_n(X_B) \quad \forall\, n \in \mathbb{N}.$$

Proof. Since $\mathrm{Gen}_n({}_AX) = \mathrm{Gen}({}_{M_n(A)}X^n)$, $\mathrm{Um}_n(X_B) = \mathrm{Um}(X_B^n)$, and the C^*-correspondence ${}_{M_n(A)}X_B^n$ is full and right-unital, we may assume $n = 1$. As X_B is full there exist $x_1, \dots, x_r, y_1, \dots, y_r \in X$ such that $\sum_k \langle x_k, y_k\rangle = 1$. If $x \in \mathrm{Gen}({}_AX)$ let $a_1, \dots, a_r \in A$ be such that $x_k = a_k \cdot x$, for $k = 1, \dots, r$. Then,

$$1 = \textstyle\sum_k \langle x_k, y_k\rangle = \sum_k \langle a_k \cdot x, y_k\rangle = \sum_k \langle x, a_k^* \cdot y_k\rangle = \langle x, \sum_k a_k^* \cdot y_k\rangle,$$

so $x \in \mathrm{Um}(X_B)$. ☐

Definition 3.12. A vector space X is said to be a C^*-*bimodule* whenever it is equipped with compatible left and right Hilbert module structures (that is $x \cdot \langle y, z\rangle_R = \langle x, y\rangle_L \cdot z$, for $x, y, z \in X$) over C^*-algebras A and B, respectively.

Lemma 3.13. *Let ${}_AX_B$ be a right-full and right-unital C^*-bimodule. Then*

$$Um_n(X_B) \subseteq Gen_n({}_AX) \quad \forall\, n \in \mathbb{N}.$$

Proof. Since $\mathrm{Gen}_n({}_AX) = \mathrm{Gen}({}_{M_n(A)}X^n)$, $\mathrm{Um}_n(X_B) = \mathrm{Um}(X_B^n)$ and the C^*-bimodule ${}_{M_n(A)}X_B^n$ is right-full and right-unital, we may assume $n = 1$. Given $x \in \mathrm{Um}(X)$, let $y \in X$ be such that $\langle y, x\rangle_R = 1$. Then, for all $z \in X$ we have $z = z \cdot 1 = z \cdot \langle y, x\rangle_R = \langle z, y\rangle_L \cdot x$. Then $x \in \mathrm{Gen}(X)$. ☐

Proposition 3.14. *Let ${}_AX_B$ be a right-full and right-unital C^*-bimodule. Then*

$$Um_n(X_B) = Gen_n({}_AX) \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad tsr({}_AX) = sr(X_B).$$

4. Stable rank Inequality for C^*-modules

[4, Proposition 9.7] says that $\mathrm{Bsr}(V) \leq \mathrm{tsr}(V)$ for a finitely generated projective left module ${}_AV$ over a unital C^*-algebra A. Inspired by this, we prove a similar result in the C^*-module context, namely, *if X is a right-full and right-unital C^*-bimodule, then $Bsr(X) \leq tsr(X)$.*

Lemma 4.1. *(Warfield Condition,* [4, Propositions 2.2 and 9.2]*) Let ${}_AX_B$ be a right-full and right-unital C^*-bimodule and $x_1, \ldots, x_{n+1} \in X$. The following conditions are equivalent:*

(a) $\exists\ a_1, \ldots, a_n \in A \ /\ (x_1 + a_1 \cdot x_{n+1}, \ldots, x_n + a_n \cdot x_{n+1}) \in Um_n(X)$.

(b) $\exists\ y_1, \ldots, y_{n+1} \in X \ /\ \sum_{k=1}^{n+1} \langle y_k, x_k \rangle_R = 1$ *and* $(y_1, \ldots, y_n) \in Um_n(X)$.

Proof. If $(x_1 + a_1 \cdot x_{n+1}, \ldots, x_n + a_n \cdot x_{n+1}) \in \mathrm{Um}_n(X)$, there exist $y_1, \ldots, y_n \in X$ such that $\sum_{k=1}^{n} \langle y_k, x_k + a_k \cdot x_{n+1} \rangle_R = 1$. Then we have $(y_1, \ldots, y_n) \in \mathrm{Um}_n(X)$ by Lemma 3.7, and

$$\sum_{k=1}^{n} \langle y_k, x_k \rangle_R + \langle \sum_{k=1}^{n} a_k^* \cdot y_k, x_{n+1} \rangle_R = 1.$$

Taking $y_{n+1} = \sum_{k=1}^{n} a_k^* \cdot y_k$ we obtain (b).

Conversely, if condition (b) holds, as $(y_1, \ldots, y_n) \in \mathrm{Um}_n(X)$ there are $z_1, \ldots, z_n \in X$ such that $\sum_{k=1}^{n} \langle y_k, z_k \rangle_R = 1$. Let $a_k = \langle z_k, y_{n+1} \rangle_L$, for $k = 1, \ldots, n$. Then

$$\sum_{k=1}^{n} \langle y_k, x_k + a_k \cdot x_{n+1} \rangle_R = \sum_{k=1}^{n} \langle y_k, x_k \rangle_R + \langle \sum_{k=1}^{n} a_k^* \cdot y_k, x_{n+1} \rangle_R. \tag{4.1}$$

Now, we have

$$\sum_{k=1}^{n} a_k^* \cdot y_k = \sum_{k=1}^{n} \langle y_{n+1}, z_k \rangle_L \cdot y_k = \sum_{k=1}^{n} y_{n+1} \cdot \langle z_k, y_k \rangle_R$$
$$= y_{n+1} \cdot \sum_{k=1}^{n} \langle z_k, y_k \rangle_R = y_{n+1} \cdot 1 = y_{n+1}.$$

Then, the right-hand side of equation (4.1) equals 1 by (b), and (a) holds. □

The proof of the following proposition is analogous to that of [4, Theorem 2.3].

Proposition 4.2. *(*[4, Proposition 9.7]*) Let ${}_AX_B$ be a right-full and right-unital C^*-bimodule. Then*

$$Bsr(X) \leq tsr(X).$$

Proof. Let $n = \mathrm{tsr}(X) = \mathrm{sr}(X)$. Given $(x_1, \ldots, x_{n+1}) \in \mathrm{Gen}_{n+1}({}_AX) = \mathrm{Um}_{n+1}(X_B)$ consider $z_1, \ldots, z_{n+1} \in X$ such that $\sum_{k=1}^{n+1} \langle z_k, x_k \rangle_R = 1$. For $k = 1, \ldots, n$, pick perturbations $\bar{z}_k \simeq z_k$ with $(\bar{z}_1 \ldots, \bar{z}_n) \in \mathrm{Um}_n(X)$ so that

$$d^* := \langle \bar{z}_1, x_1 \rangle_R + \cdots + \langle \bar{z}_n, x_n \rangle_R + \langle z_{n+1}, x_{n+1} \rangle_R \in \mathrm{GL}(B).$$

Then, taking $y_1 = \bar{z}_1 \cdot d^{-1}, \ldots, y_n = \bar{z}_n \cdot d^{-1}, y_{n+1} = z_{n+1} \cdot d^{-1}$ we have $(y_1, \ldots, y_n) = (\bar{z}_1, \ldots, \bar{z}_n) \cdot d^{-1} \in \mathrm{Um}_n(X)$ and $\sum_{k=1}^{n+1} \langle y_k, x_k \rangle_R = 1$. By the previous lemma $(x_1, \ldots, x_{n+1})$ is reducible, and then $\mathrm{Bsr}(X) \leq n$. □

5. Herman-Vaserstien theorem for C^*-modules

Herman–Vaserstein theorem states that for a unital C^*-algebra A, $\mathrm{tsr}(A) \leq \mathrm{Bsr}(A)$. In this section we obtain $\mathrm{tsr}(X) \leq \mathrm{Bsr}(X)$ for a right-full and right-unital Hilbert bimodule X.

Lemma 5.1. *Let X_B be a full and unital Hilbert module. Given $x_1, \ldots, x_n$, $u_1, \ldots, u_r \in X$ such that $\sum_k \langle u_k, u_k \rangle = 1$, and $\varepsilon > 0$, let $b_0 = \sum_i \langle x_i, x_i \rangle$, $b = (1 - \frac{b_0}{\varepsilon})^+$ and $y_k = u_k \cdot b$, for $k = 1, \ldots, r$. Then $(x_1, \ldots, x_n, y_1, \ldots, y_r) \in Um_{n+r}(X_B)$.*

Proof. Let $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_r)$ and $y = (y_1, \ldots, y_r)$, then $y = u \cdot b$ and $b_0 = \langle x, x \rangle$. Consider the commutative C^*-subalgebra $B_0 := C^*(1, b_0) \subseteq B$. Let $c \in B$ the element given by

$$c = \langle y, y \rangle = \langle u \cdot b, u \cdot b \rangle = b^* \langle u, u \rangle b = b^* b = [(1 - \frac{b_0}{\varepsilon})^+]^2.$$

Consequently c and b_0 belong to B_0^+ and do not have common roots. Therefore

$$\langle (x, y), (x, y) \rangle = \langle x, x \rangle + \langle y, y \rangle = b_0 + c \in \mathrm{GL}(B_0) \subseteq \mathrm{GL}(B).$$

That is, $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_r) \in \mathrm{Um}_{n+r}(X_B)$. □

Theorem 5.2. *Let ${}_AX_B$ be a right-full and right-unital C^*-bimodule. Then*

$$Bsr(X) = tsr(X).$$

Proof. By Proposition 4.2 it suffices to show $\mathrm{Bsr}(X) \geq \mathrm{tsr}(X)$. Suppose $\mathrm{Bsr}({}_AX) = n$ and let $x = (x_1, \ldots, x_n) \in X^n$, $\varepsilon > 0$ be given. As X is right-full and right-unital, by Lemma 3.9, there exists $u = (u_1, \ldots, u_r) \in \mathrm{Um}_r(X)$ for suitable $r \in \mathbb{N}$. Replacing u with $u \cdot \langle u, u \rangle_R^{-1/2}$, we may suppose $\langle u, u \rangle_R = 1$. Taking b_0, b and y as in Lemma 5.1 we have that $(x, y) \in \mathrm{Um}_{n+r}(X_B) = \mathrm{Gen}_{n+r}({}_AX)$. Then, as $\mathrm{Bsr}({}_AX) = n$, the generator (x, y) can be reduced r times to an n-generator. Therefore, there exists $a \in M_{n \times r}(A)$ such that $x + a \cdot y \in \mathrm{Gen}_n({}_AX) = \mathrm{Um}_n(X_B)$. Let

$$k > \frac{\|a\|}{\varepsilon}, \quad d = 1 + kb \in B^+ \cap \mathrm{GL}(B) \quad \text{and}$$
$$x' = (x + a \cdot y) \cdot d^{-1} \in \mathrm{Um}_n(X) = \mathrm{Gen}_n(X),$$

where $\|a\|$ is the norm of a as a B-adjointable operator $a \colon X^r \to X^n$.

We have $x - x' = (x \cdot d - x - a \cdot y) \cdot d^{-1} = (x \cdot kb - a \cdot y) \cdot d^{-1}$ and

$$\|x - x'\| \leq \|x \cdot kbd^{-1}\| + \|a \cdot y \cdot d^{-1}\|. \tag{5.1}$$

As $b_0 = \langle x, x \rangle_R$, we have

$$|x \cdot kbd^{-1}|_R^2 = (kbd^{-1})^* \langle x, x \rangle_R (kbd^{-1}) = (kbd^{-1})^* b_0 (kbd^{-1}).$$

Now, as $b = (1 - \frac{b_0}{\varepsilon})^+$, we have $d = 1 + kb \in C^*(1, b_0) \cong C(T)$ which is commutative. Therefore $kbd^{-1} = kb(1 + kb)^{-1} = b(\frac{1}{k} + b)^{-1} \leq 1$ in $C(T)$ and consequently

$(kbd^{-1})^* b_0 (kbd^{-1}) \leq b_0$. Moreover, if $b_0(t) > \varepsilon$ for suitable $t \in T$, then $b(t) = 0$, because $b = (1 - \frac{b_0}{\varepsilon})^+$. Hence $(kbd^{-1})^* b_0 (kbd^{-1}) \leq \varepsilon$ and

$$\|x \cdot kbd^{-1}\| = \||x \cdot kbd^{-1}|_R^2\|^{1/2} \leq \sqrt{\varepsilon}. \tag{5.2}$$

On the other hand, since $y = u \cdot b$ we have

$$\|a \cdot y \cdot d^{-1}\| = \|\frac{a}{k} \cdot u \cdot kbd^{-1}\| \leq \frac{\|a\|}{k}\|u\|\|kbd^{-1}\| < \varepsilon, \tag{5.3}$$

where we have used that $\|a\|/k < \varepsilon$, $\|u\| = 1$ and $\|kbd^{-1}\| \leq 1$.

Thus we can estimate (5.1) using equations (5.2) and (5.3) to get

$$\|x - x'\| < \sqrt{\varepsilon} + \varepsilon.$$

Then, x' can be taken arbitrarily close to x and $x' \in \mathrm{Gen}_n(X)$. Therefore, $\mathrm{Gen}_n(X)$ is dense and $\mathrm{tsr}(X) \leq n$. □

Remark 5.3. If ${}_AX$ is a finitely generated projective module over a unital C^*-algebra A we can make it into a right-full and right-unital C^*-bimodule in the following way. The module ${}_AX$ is a direct summand of A^n for suitable $n \in \mathbb{N}$, and is therefore the range of a (selfadjoint) projection $p \in M_n(A)$. Then we have ${}_AX$ as the submodule ${}_A(A^n p)$ of ${}_AA^n$. As we actually have a Hilbert $A - M_n(A)$ bimodule structrue on A^n (thinking of A^n as a row space and using the usual matrix operations) we obtain, by restriction, an $A - pM_n(A)p$ C^*-bimodule ${}_A(A^n p)_{pM_n(A)p}$, which is right-full and right-unital.

Combining this construction with Theorem 5.2 we have that $\mathrm{Bsr}(X) = \mathrm{tsr}(X)$ for every finitely generated projective left module over a unital C^*-algebra.

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Facultad de Ciencias, Iguá 4225, CP 11400, Montevideo, Uruguay.
E-mail address: achigar@cmat.edu.uy; mauricio.achigar3@gmail.com