TWO ESTIMATES FOR THE JAMES TYPE CONSTANT

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ABSTRACT. Recently, many recent studies have focused on the James type constant introduced by Takahashi. In this study, we have presented two inequalities on the James type constant, and shown that the inequalities can be used to estimate the values of James type constant for some concrete spaces. In particular, for any $\tau \geq 0$, we have calculated the exact values of James type constant, $J_{X,t}(\tau)$ of $l_1 - l_2$, for $t \geq 1$.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we shall assume that X stands for a nontrivial Banach space, i.e, $\dim X \geq 2$. We will use S_X and B_X to denote the unit sphere and unit ball of X , respectively.

A Banach space X is called uniformly non-square in the sense of James if there exists a positive number, $\delta < 2$, such that $\frac{\|x+y\|}{2} \leq \delta$ or $\frac{\|x-y\|}{2} \leq \delta$, whenever $x, y \in S_X$. The non-square or James constant, defined as $J(X) = \sup\{\min(\|x+y\|, \|x-y\|), x, y \in S_X\}$, is introduced to characterize such a concept. Obviously, X is uniformly non-square in the sense of James if and only if $J(X) < 2$ [1].

It can be recalled that the modulus of convexity of a Banach space X is defined for $\varepsilon \in [0, 2]$ as (see [2])

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2}, x, y \in S_X, \|x-y\| = \varepsilon\right\}.$$

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The following relation between the James constant $J(X)$ and the modulus of convexity $\delta_X(\varepsilon)$ is useful to compute $J(X)$ for some concrete Banach spaces:

$$J(X) = \sup\{\varepsilon \in (0, 2) : \delta_X(\varepsilon) \leq 1 - \frac{\varepsilon}{2}\}.$$

The von Neumann-Jordan constant, introduced by Clarkson [3], is defined as

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\right\}.$$

In order to better calculate von Neumann-Jordan constant, the following function $\gamma_X(t) : [0, 1] \rightarrow [1, 4]$ is introduced by (see [4])

$$\gamma_X(t) = \sup\left\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2} : x \in S_X, y \in S_X\right\}.$$

Recently, as a generalization of the James constant, Takahashi [5] introduced the James type constant

$$J_{X,t}(\tau) = \sup\{\mu_t(\|x+\tau y\|, \|x-\tau y\|) : x, y \in S_X\},$$

where $\tau \geq 0, -\infty \leq t < +\infty$. Here, we denote $\mu_t(a, b) = (\frac{a^t+b^t}{2})^{\frac{1}{t}}$ ($t \neq 0$) and $\mu_0(a, b) = \lim_{t \rightarrow 0} \mu_t(a, b) = \sqrt{ab}$ for two positive numbers a and b , respectively. It is well known that $\mu_t(a, b)$ is nondecreasing and $\mu_{-\infty}(a, b) = \lim_{t \rightarrow -\infty} \mu_t(a, b) = \min(a, b)$. Therefore, $J(X) = J_{X,-\infty}(1)$. It is obvious that the James type constants include some known constants, such as Alonso-Llorens-Fuster's constant $T(X)$ [6], Baronti-Casini-Papini's constant $A_2(X)$ [7], Gao's constant $E(X)$ [8] and Yang-Wang's modulus $\gamma_X(t)$ [4]. These constants are defined by $T(X) = J_{X,0}(1)$, $A_2(X) = J_{X,1}(1)$, $E(X) = 2J_{X,2}^2(1)$ and $\gamma_X(t) = J_{X,2}^2(t)$. As mentioned in [5], we could calculate the Jordan-von Neumann type constant $C_t(X)$ using $J_{X,t}(\tau)$, where $C_t(X)$ is defined as

$$C_t(X) = \sup\left\{\frac{J_{X,t}(\tau)^2}{1+\tau^2} : 0 \leq \tau \leq 1\right\}.$$

The following are some properties of the constant $J_{X,t}(\tau)$ [5, 9, 10]:

- (1) If $-\infty \leq t_1 \leq t_2 < \infty$, then $J_{X,t_1}(\tau) \leq J_{X,t_2}(\tau)$ for any $\tau \geq 0$.
- (2) Let $t \geq 1$; then, for any Banach space X , the following conditions are equivalent:
 - (a) X is uniformly non-square.
 - (b) $J_{X,t}(\tau) < 1 + \tau$ for all $\tau > 0$.
 - (c) $J_{X,t}(\tau_0) < 1 + \tau_0$ for some $\tau_0 > 0$.

- (3) Let X be $l_\infty - l_1$ space. If $0 \leq \tau \leq 1$, then

$$J_{X,t}(\tau) = \begin{cases} (\frac{1+(1+\tau)^t}{2})^{\frac{1}{t}}, & t \geq 1, \\ 1 + \frac{\tau}{2}, & t \leq 1. \end{cases}$$

(4) Let $1 \leq t \leq p \leq \infty$, $2 \leq p$ and $0 \leq \tau \leq 1$. Then

$$J_{X,t}(\tau) = 1 + 2^{-\frac{1}{p}}\tau = 1 + \rho_X(\tau),$$

where X is $l_\infty - l_p$ space and $\rho_X(\tau)$ is the modulus of smoothness of X .

In this study, we have presented two inequalities on the James type constant, and shown that the inequalities can be used to estimate the values of James type constant for some concrete spaces. In particular, for any $\tau \geq 0$, we have calculated the exact values of James type constant, $J_{X,t}(\tau)$ of $l_1 - l_2$, for $t \geq 1$.

2. MAIN RESULTS

As $J_{X,t}(\tau) = \tau J_{X,t}(\frac{1}{\tau})$ is valid for any $\tau > 0$, we only considered the case $0 \leq \tau \leq 1$.

Theorem 2.1. *Let $t_2 \geq t_1 \geq 1$ and $0 \leq \tau \leq 1$. Then, for any Banach space X ,*

$$J_{X,t_1}^{t_2}(\tau) \leq J_{X,t_2}^{t_2}(\tau) \leq \frac{(1+\tau)^{t_2} + [2J_{X,t_1}^{t_1}(\tau) - (1+\tau)^{t_1}]^{\frac{t_2}{t_1}}}{2}. \quad (2.1)$$

In particular, we have that $J_{X,t_1}(\tau) = 1 + \tau$ if and only if $J_{X,t_2}(\tau) = 1 + \tau$.

Proof. Let $0 \leq \tau \leq 1$. As the function $f(x) = x^{\frac{t_2}{t_1}}$ is a convex function for $t_2 \geq t_1 \geq 1$, the first inequality is evident by

$$\left(\frac{\|x + \tau y\|^{t_1} + \|x - \tau y\|^{t_1}}{2}\right)^{\frac{1}{t_1}} \leq \left(\frac{\|x + \tau y\|^{t_2} + \|x - \tau y\|^{t_2}}{2}\right)^{\frac{1}{t_2}}.$$

To prove the second inequality, it is sufficient to show that the following inequality holds for any $x, y \in S_X$

$$\|x + \tau y\|^{t_2} + \|x - \tau y\|^{t_2} \leq (1+\tau)^{t_2} + [2J_{X,t_1}^{t_1}(\tau) - (1+\tau)^{t_1}]^{\frac{t_2}{t_1}}. \quad (2.2)$$

Let $u = \max\{\|x + \tau y\|, \|x - \tau y\|\}$ and $v = \min\{\|x + \tau y\|, \|x - \tau y\|\}$, then $1 \leq u \leq 1 + \tau$. By having in mind that $\|x + \tau y\|^{t_1} + \|x - \tau y\|^{t_1} = u^{t_1} + v^{t_1} \leq 2J_{X,t_1}^{t_1}(\tau)$, we easily have

$$\|x + \tau y\|^{t_2} + \|x - \tau y\|^{t_2} = u^{t_2} + v^{t_2} \leq u^{t_2} + [2J_{X,t_1}^{t_1}(\tau) - u^{t_1}]^{\frac{t_2}{t_1}} =: g(u).$$

First, as the function $g(u)$ satisfies $g(1+\tau) = g([2J_{X,t_1}^{t_1}(\tau) - (1+\tau)^{t_1}]^{\frac{1}{t_1}})$, and by

$$g'(u) = t_2 u^{t_1-1} [u^{t_2-t_1} - (2J_{X,t_1}^{t_1}(\tau) - u^{t_1})^{\frac{t_2-t_1}{t_1}}],$$

we have that it is decreasing on $[(2J_{X,t_1}^{t_1}(\tau) - (1+\tau)^{t_1})^{\frac{1}{t_1}}, J_{X,t_1}(\tau)]$ and increasing on $[J_{X,t_1}(\tau), 1+\tau]$, (2.2) holds if $u \in [(2J_{X,t_1}^{t_1}(\tau) - (1+\tau)^{t_1})^{\frac{1}{t_1}}, 1+\tau]$. On the other hand, if $1 \leq u \leq [2J_{X,t_1}^{t_1}(\tau) - (1+\tau)^{t_1}]^{\frac{1}{t_1}}$, then

$$\|x + \tau y\|^{t_2} + \|x - \tau y\|^{t_2} \leq (1+\tau)^{t_2} + [2J_{X,t_1}^{t_1}(\tau) - (1+\tau)^{t_1}]^{\frac{t_2}{t_1}}.$$

Thus, we obtain the inequality (2.2). □

Remark 2.2. Under the condition presented in Theorem 2.1, we have

$$J_{X,t_1}^{t_2}(\tau) \leq J_{X,t_2}^{t_2}(\tau) \leq 2^{\frac{t_2}{t_1}-1} J_{X,t_1}^{t_2}(\tau).$$

By (2.1), we have the following:

Corollary 2.3. [10] *Let $t \geq 1$ and $0 \leq \tau \leq 1$. Then, for any Banach space X ,*

$$[1 + \rho_X(\tau)]^t \leq J_{X,t}^t(\tau) \leq \frac{(1 + \tau)^t + [2(1 + \rho_X(\tau)) - (1 + \tau)]^t}{2},$$

where $\rho_X(\tau)$ is the modulus of smoothness of X . In particular, we have that $\rho_X(\tau) = \tau$ if and only if $J_{X,t}(\tau) = 1 + \tau$.

In [5], the author obtained the following results:

(i) If $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for any $-\infty \leq t \leq q$ and each $\tau \geq 0$

$$J_{l_p,t}(\tau) = (1 + \tau^p)^{\frac{1}{p}}. \quad (2.3)$$

(ii) If $2 \leq p \leq +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for any $-\infty \leq t \leq p$ and each $\tau \geq 0$

$$J_{l_p,t}(\tau) = \left(\frac{|1 + \tau|^p + |1 - \tau|^p}{2} \right)^{\frac{1}{p}}. \quad (2.4)$$

As an application of (2.1) and (2.4), we can simply calculate the exact values of James type constant of l_p for $2 \leq p \leq t$, which have been obtained recently in [9].

Example 2.4. (l_p space). If $2 \leq p \leq t$, then for any $\tau \geq 0$

$$J_{l_p,t}(\tau) = \left(\frac{(1 + \tau)^t + |1 - \tau|^t}{2} \right)^{\frac{1}{t}}.$$

Proof. We may assume that $\tau \in [0, 1]$. By (2.4), we have $2J_{l_p,p}^p(\tau) = (1 + \tau)^p + (1 - \tau)^p$. Now applying (2.1), we can get

$$\begin{aligned} J_{l_p,t}^t(\tau) &\leq \frac{(1+t)^t + [2J_{l_p,p}^p(\tau) - (1+\tau)^p]^{\frac{t}{p}}}{2} \\ &= \frac{(1+\tau)^t + (1-\tau)^t}{2}. \end{aligned}$$

On the other hand, by taking $x = y \in S_X$, we can obtain

$$J_{l_p,t}^t(\tau) \geq \frac{(1 + \tau)^t + (1 - \tau)^t}{2}.$$

Remark 2.5. If $1 < p \leq 2$, and $q \leq t < \infty$, then (2.3) and (2.1) can imply

$$\begin{aligned} J_{l_p,t}^t(\tau) &\leq \frac{(1+t)^t + [2J_{l_p,q}^q(\tau) - (1+\tau)^q]^{\frac{t}{q}}}{2} \\ &= \frac{(1+\tau)^t + [2(1+\tau^p)^{\frac{q}{p}} - (1+\tau)^q]^{\frac{t}{q}}}{2} \\ &\leq 2^{\frac{t}{q}-1} (1 + \tau^p)^{\frac{t}{p}}. \end{aligned}$$

For $l_2 - l_1$ space, we have computed the exact value of $J_{X,t}(1)$ (see [11]). Now, as an application of (2.1), we have

Example 2.6. ($l_2 - l_1$ space) Let $t \geq 1, \tau \geq 0$ and $X = \mathbb{R}^2$ with the norm defined by

$$\|x\| = \|(x_1, x_2)\| = \begin{cases} \|x\|_2, & x_1 x_2 \geq 0, \\ \|x\|_1, & x_1 x_2 \leq 0. \end{cases}$$

Then

$$J_{X,t}(\tau) = \left(\frac{(1+\tau^2)^{\frac{t}{2}} + (1+\tau)^t}{2} \right)^{\frac{1}{t}}. \quad (2.5)$$

Proof. We only considered the case $0 \leq \tau \leq 1$. If $t \geq 2$, by [8], we have $\gamma_X(\tau) = 1 + \tau + \tau^2$, i.e., $J_{X,2}^2(\tau) = 1 + \tau + \tau^2$. Now by applying Theorem 2.1, we have

$$\begin{aligned} J_{X,t}^t(\tau) &\leq \frac{(1+\tau)^t + [2J_{X,2}^2(\tau) - (1+\tau)^2]^{\frac{t}{2}}}{2} \\ &= \frac{(1+\tau)^t + [2(1+\tau+\tau^2) - (1+\tau)^2]^{\frac{t}{2}}}{2} \\ &= \frac{(1+\tau)^t + (1+\tau^2)^{\frac{t}{2}}}{2}. \end{aligned}$$

On the other hand, by taking $x = (1, 0), y = (0, 1)$, we have $\|x + \tau y\| = (1 + \tau^2)^{\frac{1}{2}}, \|x - \tau y\| = 1 + \tau$, so $J_{X,t}^t(\tau) \geq \frac{(1+\tau)^t + (1+\tau^2)^{\frac{t}{2}}}{2}$. Therefore, (2.5) is valid for $t \geq 2$.

To prove (2.5) is also valid for $t \geq 1$, we first give the following Lemma.

Lemma 2.7. Let $a, b, c, d \geq 0$ with $a^2 + b^2 = c^2 + d^2 = 1$, and $\tau \in [0, 1]$ such that $a - c\tau \geq 0, d\tau - b \geq 0$. Then

$$\sqrt{(a + c\tau)^2 + (b + d\tau)^2} + a - c\tau + d\tau - b \leq \sqrt{1 + \tau^2} + 1 + \tau. \quad (2.6)$$

Proof. We may assume that $\tau \in (0, 1], a \geq b$, and $d \geq c$.

Let $L(b, c) = \sqrt{(a + c\tau)^2 + (b + d\tau)^2} + a - c\tau + d\tau - b = \sqrt{1 + \tau^2 + 2\tau(ac + bd)} + a - c\tau + d\tau - b$, where $a = \sqrt{1 - b^2}$ and $d = \sqrt{1 - c^2}$.

(1). If $L(b, c)$ attains its maximum at some $(b_0, c_0) \in (0, \frac{1}{\sqrt{2}}) \times (0, \frac{1}{\sqrt{2}})$.

Then $L_b(b_0, c_0) = L_c(b_0, c_0) = 0$. But by a simple compute, we have

$$\begin{aligned} L_b(b_0, c_0) &= \tau \frac{d_0 - b_0 a_0^{-1} c_0}{\sqrt{1 + \tau^2 + 2\tau(a_0 c_0 + b_0 d_0)}} - 1 - b_0 a_0^{-1} \\ &\leq \frac{\tau}{\sqrt{1 + \tau^2 + 2\tau(a_0 c_0 + b_0 d_0)}} - 1 - b_0 a_0^{-1} < 0, \end{aligned}$$

and

$$\begin{aligned} L_c(b_0, c_0) &= \tau \frac{a_0 - b_0 c d_0^{-1} c_0}{\sqrt{1 + \tau^2 + 2\tau(a_0 c_0 + b_0 d_0)}} - \tau - \tau c_0 d_0^{-1} \\ &\leq \tau \left\{ \frac{1}{\sqrt{1 + \tau^2 + 2\tau(a_0 c_0 + b_0 d_0)}} - 1 - c_0 d_0^{-1} \right\} < 0, \end{aligned}$$

where $a_0^2 + b_0^2 = c_0^2 + d_0^2 = 1$.

Hence, This is a contradiction.

(2). If $c = 0$, and $0 \leq b \leq \frac{1}{\sqrt{2}}$. Then it is easy to see that $f(b) =: \sqrt{1 + \tau^2 + 2b\tau} + \sqrt{1 - b^2} + \tau - b$ is decreasing and

$$L(b, c) = \sqrt{1 + \tau^2 + 2b\tau} + a + \tau - b \leq \sqrt{1 + \tau^2} + 1 + \tau.$$

(3). If $b = 0$, and $0 \leq c \leq \frac{1}{\sqrt{2}}$. Then it is easy to see that $g(c) =: \sqrt{1 + \tau^2 + 2c\tau} + 1 - c\tau + \sqrt{1 - c^2\tau}$ is also decreasing and

$$L(b, c) = \sqrt{1 + \tau^2 + 2c\tau} + 1 - c\tau + d\tau \leq \sqrt{1 + \tau^2} + 1 + \tau.$$

(4). If $c = d = \frac{1}{\sqrt{2}}$, and $0 \leq b \leq \frac{1}{\sqrt{2}}$. Then

$$L(b, c) \leq 1 + \tau + a - b \leq 2 + \tau \leq \sqrt{1 + \tau^2} + 1 + \tau.$$

(5). If $a = b = \frac{1}{\sqrt{2}}$, and $0 \leq c \leq \frac{1}{\sqrt{2}}$. Then

$$L(b, c) \leq 1 + \tau + (d - c)\tau \leq 1 + 2\tau \leq \sqrt{1 + \tau^2} + 1 + \tau.$$

Therefore (2.6) is valid. $\square$

Now we continue proving (2.5). First, let us show that the following inequality is valid for any $x, y \in S_{l_1-l_2}$.

$$\|x + \tau y\| + \|x - \tau y\| \leq \sqrt{1 + \tau^2} + 1 + \tau.$$

In fact, by the convexity of norm, we only need to show this inequality is valid for any $x, y \in \text{ext}(S_{l_1-l_2})$, where $\text{ext}(S_{l_1-l_2})$ denotes the set of extreme points of $S_{l_1-l_2}$. Because $\text{ext}(S_{l_1-l_2}) = \{(x_1, x_2) : x_1^2 + x_2^2 = 1, x_1 x_2 \geq 0\}$, so we may assume that $x = (a, b), y = (c, d)$, where $a, b, c, d \geq 0$ and such that $a^2 + b^2 = c^2 + d^2 = 1$.

(1). If $(a - c\tau)(b - d\tau) \geq 0$. Then

$$\begin{aligned} \|x + \tau y\| + \|x - \tau y\| &= \|x + \tau y\|_2 + \|x - \tau y\|_2 \\ &\leq \sqrt{2(\|x + \tau y\|_2^2 + \|x - \tau y\|_2^2)} \\ &= 2\sqrt{1 + \tau^2} \\ &\leq \sqrt{1 + \tau^2} + 1 + \tau. \end{aligned}$$

(2). If $(a - c\tau)(b - d\tau) \leq 0$. We may assume that $a - c\tau > 0$ and $b - d\tau \leq 0$. Then by using Lemma 2.7, we also have

$$\begin{aligned} \|x + \tau y\| + \|x - \tau y\| &= \|x + \tau y\|_2 + \|x - \tau y\|_1 \\ &= \sqrt{(a + c\tau)^2 + (b + d\tau)^2} + a - c\tau + d\tau - b \\ &\leq \sqrt{1 + \tau^2} + 1 + \tau. \end{aligned}$$

Thus by taking $x = (1, 0)$ and $y = (0, 1)$, we have $2J_{l_1-l_2,1}(\tau) = \sqrt{1 + \tau^2} + 1 + \tau$.

Therefore, (2.5) is also true for $t \geq 1$ by (2.1) and $J_{X,t}^t(\tau) \geq \frac{(1+\tau)^t + (1+\tau^2)^{\frac{t}{2}}}{2}$ as above. $\square$

As a generalization of $C_{NJ}(l_1 - l_2) = \frac{3}{2}$, now (2.5) implies

Corollary 2.8. *Let $t \geq 1$, and X be $l_1 - l_2$ space, then*

$$C_t(X) = \left(\frac{1 + 2^{\frac{t}{2}}}{2}\right)^{\frac{2}{t}}.$$

By the modulus of convexity of a Banach space X , we have the following inequality to estimate the James type constants.

Theorem 2.9. *Let X be uniformly non-square. Then*

$$J_{X,t}(\tau)^t \leq \max_{J(X) \leq \varepsilon \leq 2} \frac{[1 - \tau + \tau\varepsilon]^t + [1 + \tau - 2\tau\delta_X(\varepsilon)]^t}{2}.$$

Proof. For any $x, y \in S_X$ and $0 \leq \tau \leq 1$, we have $\|x \pm \tau y\| \leq \tau\|x \pm y\| + 1 - \tau$. So,

$$\frac{\|x + \tau y\|^t + \|x - \tau y\|^t}{2} \leq \frac{[\tau\|x + y\| + 1 - \tau]^t + [\tau\|x - y\| + 1 - \tau]^t}{2}. \quad (2.7)$$

By X is a uniformly non-square space, we have $J(X) < 2$. If $\max\{\|x + y\|, \|x - y\|\} \leq J(X)$ for $x, y \in S_X$ as above, then (2.7) implies

$$\frac{\|x + \tau y\|^t + \|x - \tau y\|^t}{2} \leq [1 - \tau + \tau J(X)]^t.$$

Now suppose that $\max\{\|x + y\|, \|x - y\|\} \geq J(X)$, we may assume that $\varepsilon =: \|x - y\| \geq J(X)$. Then by applying $\|x + y\| \leq 2(1 - \delta_X(\varepsilon))$ and (2.7), we have

$$\frac{\|x + \tau y\|^t + \|x - \tau y\|^t}{2} \leq \max_{J(X) \leq \varepsilon \leq 2} \frac{[1 - \tau + \tau\varepsilon]^t + [1 + \tau - 2\tau\delta_X(\varepsilon)]^t}{2}.$$

On the other hand, $\delta_X(J(X)) = 1 - \frac{J(X)}{2}$ is valid for $J(X) < 2$, hence

$$[1 - \tau + \tau J(X)]^t \leq \max_{J(X) \leq \varepsilon \leq 2} \frac{[1 - \tau + \tau\varepsilon]^t + [1 + \tau - 2\tau\delta_X(\varepsilon)]^t}{2}.$$

Therefore, this completes the proof. $\square$

In [9], we have computed the constant $J_{X,t}(1)$ of the space $X_{\lambda,p}$. As an application of Theorem 2.9, now we take account of the James type constant $J_{X,t}(\tau)$ of this space.

Lemma 2.10. [9] *For $2 \leq p < +\infty$ and $\lambda \geq 1$, let $X_{\lambda,p}$ be the space l_p with the norm $|x|_{\lambda,p} = \max\{\|x\|_p, \lambda\|x\|_\infty\}$, then,*

(i) if $\lambda \leq 2^{\frac{1}{p}}$, we have

$$\delta_{X_{\lambda,p}}(\epsilon) = \begin{cases} 0, & 0 \leq \epsilon \leq 2(\lambda^p - 1)^{\frac{1}{p}}, \\ 1 - \lambda(1 - (\frac{\epsilon}{2\lambda})^p)^{\frac{1}{p}}, & 2(\lambda^p - 1)^{\frac{1}{p}} \leq \epsilon \leq 2, \end{cases}$$

(ii) if $\lambda \geq 2^{\frac{1}{p}}$, $\delta_{X_{\lambda,p}}(\epsilon) = 0$, where $0 \leq \epsilon \leq 2$.

Example 2.11. For $2 \leq p < +\infty$ and $\lambda \geq 1$, let $X_{\lambda,p}$ be the space l_p with the norm $|x|_{\lambda,p} = \max\{\|x\|_p, \lambda\|x\|_\infty\}$.

(i) If $\lambda < 2^{\frac{1}{p}}$, and $t \leq p + \frac{(p-1)(1-\tau)}{2\tau}$,

$$J_{X,t}(\tau) \leq 1 - \tau + 2^{1-\frac{1}{p}}\lambda\tau; \quad (2.8)$$

(ii) if $\lambda < 2^{\frac{1}{p}}$, and $t \geq p + \frac{(p-1)(1-\tau)}{2\tau(\lambda^p-1)^{\frac{1}{p}}}$,

$$J_{X,t}(\tau) \leq 2^{-\frac{1}{t}}\{[1 + \tau]^t + [1 - \tau + 2\tau(\lambda^p - 1)^{\frac{1}{p}}]^t\}^{\frac{1}{t}}. \quad (2.9)$$

(iii) if $\lambda \geq 2^{\frac{1}{p}}$, $J_{X,t}(\tau) = 1 + \tau$.

Proof. Because if $\lambda \geq 2^{\frac{1}{p}}$, then X is not a uniformly non-square space, so (iii) is valid. Now we prove (i) and (ii). Let $\lambda < 2^{\frac{1}{p}}$, then $J(X) = \lambda 2^{1-\frac{1}{p}}$. By $\varepsilon \geq J(X)$ implying $\varepsilon \geq 2(\lambda^p - 1)^{\frac{1}{p}}$ and Lemma 2.10, we have

$$\begin{aligned} J_{X,t}^t(\tau) &\leq \max_{J(X) \leq \varepsilon \leq 2} \frac{[1-\tau+\tau\varepsilon]^t + [1-\tau+2\lambda\tau(1-(\frac{\varepsilon}{2\lambda})^p)^{\frac{1}{p}}]^t}{2} \\ &= \max_{2^{-\frac{1}{p}} \leq x \leq \frac{1}{\lambda}} \frac{[1-\tau+2\lambda\tau x]^t + [1-\tau+2\lambda\tau(1-x^p)^{\frac{1}{p}}]^t}{2}. \end{aligned}$$

For $2^{-\frac{1}{p}} \leq x \leq \frac{1}{\lambda}$, letting $g(x) = [1-\tau+2\lambda\tau x]^t + [1-\tau+2\lambda\tau(1-x^p)^{\frac{1}{p}}]^t$, then we can compute that

$$g'(x) = 2t\lambda\tau x^{p-1}[x^{1-p}(1-\tau+2\lambda\tau x)^{t-1} - y^{1-p}(1-\tau+2\lambda\tau y)^{t-1}], \quad (2.10)$$

where $y = (1-x^p)^{\frac{1}{p}}$. Taking $h(u) = u^{1-p}(1-\tau+2\lambda\tau u)^{t-1}$, we have

$$h'(u) = u^{-p}(1-\tau+2\lambda\tau u)^{t-2}[2\tau\lambda(t-p)u - (p-1)(1-\tau)].$$

(i) If $t \leq p + \frac{(p-1)(1-\tau)}{2\tau}$ then $h'(u) \leq 0$ on $(0, \frac{1}{\lambda}]$. Hence $h(x) - h(y) \leq 0$ by $0 \leq y \leq 2^{-\frac{1}{p}} \leq x \leq \frac{1}{\lambda}$, and $g(x)$ is decreasing on $[2^{-\frac{1}{p}}, \frac{1}{\lambda}]$ by (2.10). Therefore (2.8) is obtained.

(ii) If $t \geq p + \frac{(p-1)(1-\tau)}{2\tau(\lambda^p-1)^{\frac{1}{p}}}$, then $h'(u) \geq 0$ on $[\frac{(\lambda^p-1)^{\frac{1}{p}}}{\lambda}, \frac{1}{\lambda}]$. By $2^{-\frac{1}{p}} \leq x \leq \frac{1}{\lambda}$, and $x^p + y^p = 1$, we have $\frac{(\lambda^p-1)^{\frac{1}{p}}}{\lambda} \leq y \leq x \leq \frac{1}{\lambda}$. So, $h(x) - h(y) \geq 0$ and $g(x)$ is increasing on $[2^{-\frac{1}{p}}, \frac{1}{\lambda}]$ by (2.10). Therefore (2.9) is also obtained. $\square$

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