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QUADRATIC FOURIER TRANSFORMS

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Dedicated to Professor Tsuyoshi Ando in celebration of his distinguished achievements in Matrix Analysis and Operator Theory

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ABSTRACT. In this paper we shall examine the quadratic Fourier transform which is introduced by the generalized quadratic function for one order parameter in the ordinary Fourier transform. This will be done by analyzing corresponding six subcases of the quadratic Fourier transform within a reproducing kernel Hilbert spaces framework.

1. INTRODUCTION

Thirty years ago, the fourth named-author derived a typical result for a simple integral transform within a problem modeled by the heat equation by applying the theory of reproducing kernels. Namely, it was considered the simple heat equation

$$u_t(x,t) = u_{xx}(x,t)$$

on the domain $\mathcal{D} := \mathbb{R} \times \mathbb{R}_+$ (where $\mathbb{R}_+$ denotes the positive half-line), with the initial condition

$$u_F(x,0) = F(x) \in L_2(\mathbb{R}).$$

Using the Fourier transform, it was obtained a representation of the solution $u(x,t)$, in the form

$$u_F(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi$$

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(at least in the formal sense). Therefore, for any fixed $t > 0$, it was examined the integral transform $F \mapsto u_F$, and characterized the image function $u_F(x,t)$ as follows, simply and in a very natural way:

Proposition 1.1 ([19]). *In the integral transform of $L_2(\mathbb{R})$ functions F, the images $u_F(x,t)$ are extended analytically onto $\mathbb{C}$ in the form $u_F(z,t)$, and the images are characterized by the isometric identities*

$$\int_{\mathbb{R}} |F(\xi)|^2 d\xi = \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |u_F(z,t)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy.$$

From this result and the corresponding derived method, it was expanded some great theory; see the original paper and others [19, 20, 21].

After having passed about thirty years from that knowledge, and due to the present main idea of having a much more general framework by considering global quadratic functions in the exponent of the associated transforms, in this paper we would like to examine the corresponding possibilities of the *quadratic Fourier transform*:

$$\int_{\mathbb{R}} \exp(-ix(a\xi^2 + b\xi + c))F(\xi)d\xi = f(x),$$

for some real constant parameters a, b, c.

It seems to us that the results obtained in this paper may not be derived without using the theory of reproducing kernels. Therefore, one of the purposes of the present work is to show the fundamental power of the theory of reproducing kernels when applied to some general integral transforms. For the related theory of reproducing kernels, see [18, 20, 21]. Anyway, for the reader convenience, in what follows we will revise some general theory for linear mappings in the framework of Hilbert spaces in which our method will be based on.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathcal{H}$-valued function on E. Then, we shall consider the linear transform

$$f(p) = \langle \mathbf{f}, \mathbf{h}(p) \rangle_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{1.1}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued functions on E. In order to investigate the linear mapping (1.1), we consider a positive definite quadratic form function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = \langle \mathbf{h}(q), \mathbf{h}(p) \rangle_{\mathcal{H}} \quad \text{on} \quad E \times E. \tag{1.2}$$

Then, we obtain the following:

(I) The range of the linear mapping (1.1) by $\mathcal{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$ whose characterization is given by the two properties:

$$K(\cdot, q) \in H_K(E)$$

for any $q \in E$, and

$$\langle f(\cdot), K(\cdot, p) \rangle_{H_K(E)} = f(p)$$

for any $f \in H_K(E)$ and for any $p \in E$.

(II) In general, we have the inequality

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}.$$

Here, for any member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathcal{H}$ satisfying

$$f(p) = \langle \mathbf{f}^*, \mathbf{h}(p)\rangle_{\mathcal{H}} \quad \text{on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}.$$

(III) In general, we have the inversion formula in (1.1) in the form

$$f \mapsto \mathbf{f}^* \tag{1.3}$$

in (II) by using the reproducing kernel Hilbert space $H_K(E)$.

However, this inverse transformation (1.3) is, in general, not easy to handle. Consequently, case by case, we need different arguments. When the Hilbert space $\mathcal{H}$ itself is a reproducing kernel Hilbert space, we can apply the Tikhonov regularization method in order to obtain the inversion numerically and sometimes analytically (see [21]).

Let $\{\mathbf{v}_j\}$ be a complete orthonormal basis for $\mathcal{H}$. Then, for

$$v_j(p) = \langle \mathbf{v}_j, \mathbf{h}(p)\rangle_{\mathcal{H}},$$

we have

$$\mathbf{h}(p) = \sum_j \langle \mathbf{h}(p), \mathbf{v}_j\rangle_{\mathcal{H}} \mathbf{v}_j = \sum_j \overline{v_j(p)} \mathbf{v}_j.$$

By setting

$$\overline{\mathbf{h}}(\cdot) = \sum_j v_j(\cdot)\, \mathbf{v}_j,$$

we define

$$\langle f, \overline{\mathbf{h}}(p)\rangle_{H_K} = \sum_j \langle f, v_j\rangle_{H_K} \mathbf{v}_j.$$

Then, we have the following proposition.

Proposition 1.2. *Assume that for $f \in H_K$,*

$$\langle f, \overline{\mathbf{h}}\,\rangle_{H_K} \in \mathcal{H},$$

and for all $p \in E$,

$$\langle f, \langle \mathbf{h}(p), \mathbf{h}(\cdot)\rangle_{\mathcal{H}}\rangle_{H_K} = \langle \langle f, \overline{\mathbf{h}}\,\rangle_{H_K}, \mathbf{h}(p)\rangle_{\mathcal{H}}.$$

Then,

$$\|f\|_{H_K} \leq \|\langle f, \overline{\mathbf{h}}\,\rangle_{H_K}\|_{\mathcal{H}}.$$

If $\{\mathbf{h}(p); p \in E\}$ is complete in $\mathcal{H}$, then the equality always holds.

Furthermore, if:

$$\langle \mathbf{f}_0, \langle f, \overline{\mathbf{h}}\,\rangle_{H_K}\rangle_{\mathcal{H}} = \langle \langle \mathbf{f}_0, \mathbf{h}\rangle_{\mathcal{H}}, f\rangle_{H_K} \quad \textit{for} \quad \mathbf{f}_0 \in N(L)$$

(where L denotes the linear map $L : \mathcal{H} \longrightarrow \mathcal{F}(E)$ associated with (1.1)), then the following identity yields for $\mathbf{f}^*$ defined as in (II) and (III)

$$\mathbf{f}^* = \langle f, \overline{\mathbf{h}}\,\rangle_{H_K}.$$

The structure of the Hilbert space H_K is, in general, very complicated and abstract. In particular, note that the basic assumption $\langle f, \overline{\mathbf{h}} \rangle_{H_K} \in \mathcal{H}$ in Proposition 1.2, is not valid for many analytical problems and we need to consider some delicate treatment for the inversion. In view of this, let us analyse again the possibilities for the linear mapping defined by (1.1). In order to derive a general inversion formula for (1.1) that is widely applicable in analysis, we shall assume that both Hilbert spaces $\mathcal{H}$ and H_K are given as

$$\mathcal{H} = L_2(T, dm), \qquad H_K \subset L_2(E, d\mu),$$

on the sets T and E, respectively. Note that for $dm, d\mu$ measurable sets T, E we assume that they are the Hilbert spaces comprising $dm, d\mu - L_2$ integrable complex-valued functions, respectively. Therefore, we shall consider the integral transform

$$f(p) = \int_T F(t)\overline{h(t,p)}\, dm(t).$$

Here, $h(t,p)$ is a function on $T \times E$, $h(\cdot, p) \in L_2(T, dm)$, and $F \in L_2(T, dm)$. The corresponding reproducing kernel for (1.2) is given by

$$K(p,q) = \int_T h(t,q)\overline{h(t,p)}\, dm(t) \quad \text{on} \quad E \times E.$$

The norm of the reproducing kernel Hilbert space H_K is represented as $L_2(E, d\mu)$. Under these situations, we have the following result (cf. Theorem 5 of [20, Chapter 2]).

Proposition 1.3. *We assume that an approximating sequence $\{E_N\}_{N=1}^{\infty}$ of E satisfies*

(i) $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$,

(ii) $\bigcup_{N=1}^{\infty} E_N = E$,

(iii) $\int_{E_N} K(p,p) d\mu(p) < \infty, \quad N = 1, 2, \ldots$.

Then, for $f \in H_K$ satisfying $\int_{E_N} f(p)h(t,p)d\mu(p) \in L_2(T, dm)$ for any N, the sequence

$$\left\{ \int_{E_N} f(p)h(t,p)d\mu(p) \right\}_{N=1}^{\infty}$$

converges strongly (in the sense of the $L_2(T, dm)$ norm) to F^* satisfying

$$f(p) = \int_T F^*(t)\overline{h(t,p)}\, dm(t)$$

and

$$\|f\|_{H_K(E)} = \|F^*\|_{L_2(T,dm)}.$$

Practically, for many cases, the assumptions in Proposition 1.3 will be satisfied automatically, and so Proposition 1.3 may be applied in all those cases. To construct the inversion formula is – in general – difficult. However, the formulation in Proposition 1.3 may be considered as a natural one. Indeed, this may be realized if we recognize that it was given as the strong convergence in the Hilbert space $L_2(T, dm)$.

In view of all this, we shall approximate the space by taking a finite number of points and by using matrix theory. In order to realize approximately the inner product in the space H_K with some practical sense, we shall consider the natural and general approximate realization of the space H_K.

By taking a finite number of points $\{t_j\}_{j=1}^n$, we set

$$K(t_j, t_{j'}) := a_{jj'}.$$

If the matrix $A :=\| a_{jj'} \|$ is positive definite, then the corresponding norm in H_A comprising the vectors $\mathbf{t} = (t_1, t_2, \cdots, t_n)^{\mathsf{T}}$ is determined by

$$\|\mathbf{t}\|_{H_A}^2 = \mathbf{t}^* \widetilde{A} \mathbf{t},$$

where

$$\widetilde{A} = \overline{A^{-1}} = \|\widetilde{a_{jj'}}\|$$

(see [20, p.250]). Following this idea we developed recently many concrete applications in [5, 8, 11, 9].

Thinking on the dedicatory of the present paper, we feel pertinent to refer at this point some of the admirable work of Professor Tsuyoshi Ando on positive semi-definiteness of operator-matrices within a very general framework subjected to the consideration of strict contractions on a Hilbert space; see [1].

Moreover, in [9], as the Aveiro discretization method, a new discretization principle was introduced with many concrete examples and further, with typical and historical real inversion formulas for the Laplace transform. The above-mentioned method gives some ultimate discretization method, ultimate sampling theory and ultimate realizations of reproducing kernel Hilbert spaces.

Here, as the analytical problems, we shall give analytical characterizations of the associated images and inversion formulas based on [20].

Without loss of generality, we assume that $a > 0$ and $c = 0$. We shall look for good representations of the associated reproducing kernels to the integral transform. Using the great reference [13] and MATLAB®, we found the following formulas:

$$\begin{aligned} K_{\epsilon,\xi^2}(z, \bar{u}) &= \int_{\mathbb{R}} \exp(-iz(a\xi^2 + b\xi))\overline{\exp(-iu(a\xi^2 + b\xi))}\xi^2 \exp(-\epsilon\xi^2) d\xi \\ &= \frac{1}{4} \exp\left(\frac{(z - \bar{u})^2(-b^2)}{4\{a(z - \bar{u})i + \epsilon\}} \right) \times \\ &\quad \frac{\sqrt{\pi}}{\{a(z - \bar{u})i + \epsilon\}^{\frac{5}{2}}} [-b^2(z - \bar{u})^2 + 2\{a(z - \bar{u})i + \epsilon\}], \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} K_{\epsilon}(z, \bar{u}) &= \int_{\mathbb{R}} \exp(-iz(a\xi^2 + b\xi))\overline{\exp(-iu(a\xi^2 + b\xi))} \exp(-\epsilon\xi^2) d\xi \\ &= \exp\left(\frac{(-b^2)(z - \bar{u})^2}{4\{a(z - \bar{u})i + \epsilon\}} \right) \sqrt{\frac{\pi}{a(z - \bar{u})i + \epsilon}}. \end{aligned} \tag{1.5}$$

Both identities are valid on the lower half-plane:

$$L_{a,\epsilon} = \left\{ y < \frac{\epsilon}{2a} \right\}, \quad z = x + iy.$$

In particular, we obtain the explicit and useful formulas:

$$K_{0,\xi^2}(z, \bar{u}) = \frac{1}{4} \exp\left(\frac{(z-\bar{u})(-b^2)}{4ai} \right) \frac{\sqrt{\pi}}{\{a(z-\bar{u})i\}^{\frac{5}{2}}} [-b^2(z-\bar{u})^2 + 2\{a(z-\bar{u})i\}],$$

$$K_0(z, \bar{u}) = \exp\left(\frac{(-b^2)(z-\bar{u})}{4ia} \right) \times \sqrt{\frac{\pi}{ia(z-\bar{u})}}.$$

2. Bergman-Selberg spaces

In order to see the image spaces of the integral transforms in various situations, we recall the related reproducing kernel Hilbert spaces.

For $q > \frac{1}{2}$,

$$K_q(z, \bar{u}) = \frac{\Gamma(2q)}{(z+\bar{u})^{2q}}$$

is the Bergman-Selberg reproducing kernel. On the half-plane

$$\mathbb{R}^2_+ = \{z \;:\; \Re z := \mathrm{Re} z = x > 0\},$$

let H_{K_q} denote the RKHS consisting of all analytic functions f on $\mathbb{R}^2_+$ with finite norms

$$\|f\|^2_{H_{K_q}} = \frac{1}{\pi\,\Gamma(2q-1)} \int\int_{\mathbb{R}^2_+} |f(z)|^2 [2\Re z]^{2q-2} dxdy.$$

For $q = \frac{1}{2}$, $K_{1/2}(z, \bar{u})$ is the Szegö reproducing kernel. Let $H_{K_{\frac{1}{2}}}$ denote the RKHS consisting of all analytic functions f on $\mathbb{R}^2_+$ with finite norms

$$\|f\|^2_{H_{K_{\frac{1}{2}}}} = \frac{1}{2\pi} \sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 dy.$$

Then, a member f of $H_{K_{\frac{1}{2}}}$ has nontangential boundary values on the imaginary axis belonging to L_2, and

$$\|f\|^2_{H_{K_{\frac{1}{2}}}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(iy)|^2 dy.$$

We now recall the representations of the norms in Bergman-Selberg spaces on strips and the complex half-plane.

For the Bergman-Selberg spaces $H_{K_q}(S_r)$ $(q > 0)$ on the strip

$$S_r = \{z \;:\; |\Im z| := |\mathrm{Im} z| = |y| < r\},$$

we have the following result.

Proposition 2.1. *Let $q > \frac{1}{2}$. For $f \in H_{K_q}(S_r)$ we have the identity*

$$\|f\|^2_{H_{K_q(S_r)}} = \frac{1}{\Gamma(2q-1)\,\pi^q} \int\int_{S_r} |f(z)|^2 K_{S_r}(z, \bar{z})^{1-q} dxdy,$$

where $K_{S_r}(z, \bar{z})$ is the usual Bergman kernel on S_r.

Proposition 2.2. *For the right half-plane $\mathbb{R}^2_+$ and $q > \frac{1}{2}$ we have the identity*

$$\|f\|^2_{H_{K_q(\mathbb{R}^2_+)}} = \frac{1}{\Gamma(2q-1)\pi} \int\int_{\mathbb{R}^2_+} |f(z)|^2 (2x)^{2q-2} dxdy$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^{\infty} |\partial_x^n (xf'(x))|^2 x^{2n+2q-1} dx. \tag{2.1}$$

Conversely, any C^∞-function f defined on the positive real line with convergent summation can be extended analytically onto the right half-plane $\mathbb{R}^2_+$. The analytic extension, denoted by f also, satisfying

$$\lim_{\substack{x\in\mathbb{R}^+ \\ x\to\infty}} f(x) = 0$$

belongs to $H_{K_q}(\mathbb{R}^2_+)$ and the identity (2.1) is valid for this f.

For any $q > 0$, let $K_q(z, \bar{u})$ denote the usual Bergman kernel. In Propositions 2.1 and 2.2, the Bergman-Selberg space H_{K_q} can be defined as the RKHS admitting the reproducing kernel

$$K_q(z, \bar{u}) = \Gamma(2q)\pi^q K_q(z, \bar{u})^q.$$

For $q > \frac{1}{2}$ the Bergman-Selberg space admits the norms as in Proposition 2.1 and Proposition 2.2. For $q = \frac{1}{2}$ we have the classical Szegö space. For $0 < q < \frac{1}{2}$ we do not have the representations of the norms in the Bergman-Selberg spaces. We can, however, use the isometric mapping between $H_{K_{1/2}}$ and H_{K_q} for some special cases to express the norms in H_{K_q}.

3. Some Properties of Reproducing Kernel Hilbert Spaces

We shall further recall the needed fundamental properties of the reproducing kernel Hilbert spaces. Let us define, for a general abstract reproducing kernel $K(p, q)$ on the set $E \times E$,

$$K_s(p, q) = s(p)\overline{s(q)}K(p, q)$$

for $p, q \in E$. Then, we have

Proposition 3.1.

$$H_{K_s} = \{F \in \mathcal{F}(E) : F = f \cdot s \quad \textit{for some} \quad f \in H_K\}.$$

Furthermore, we have

$$\langle f \cdot s, g \cdot s\rangle_{H_{K_s}} = \langle f, g\rangle_{H_K}$$

for all $f, g \in H_K$.

Proposition 3.2. *Let $K_0, K : E \times E \longrightarrow \mathbb{C}$ be positive definite quadratic form functions. Then, the following statements are equivalent:*

(i) *The Hilbert space H_{K_0} is a subset of H_K.*
(ii) *There exists $\gamma > 0$ such that $K_0 \ll \gamma^2 K$.*

If each one of those items holds, then the following embedding is continuous:

$$H_{K_0} \hookrightarrow H_K,$$

where the embedding norm M is given by

$$M = \inf\left\{\gamma : K_0 \ll \gamma^2 K\right\}.$$

Here, the notation $K_0 \ll \gamma^2 K$ means that $\gamma^2 K - K_0$ is positive semi-definite.

4. Quadratic Fourier Transform Analysis

In view of our proposed analysis, we will consider, step-by-step, the following cases:

- First case:
$$\int_{\mathbb{R}} \exp(-ix(a\xi^2))F(\xi)\xi^2 d\xi = f(x).$$
- Second case:
$$\int_{\mathbb{R}} \exp(-ix(a\xi^2 + b\xi))F(\xi)d\xi = f(x).$$
- Third case:
$$\int_{-\infty}^{\infty} \exp\left(-ix(a\xi^2)\right)|\xi|^{2n+1}e^{-\varepsilon\xi^2}F(\xi)d\xi.$$
- Fourth case:
$$\int_{0}^{\infty} \exp\left(-ix(a\xi^2)\right)\xi e^{-\varepsilon\xi^2}F(\xi)d\xi.$$
- Fifth case: with the weight $\exp(-\varepsilon\xi^2)$.
- Sixth case: the Fourier transform
$$\int_{-\infty}^{\infty} \exp\left(-iz(b\xi)\right)e^{-\varepsilon\xi^2}F(\xi)d\xi.$$

It is worth saying that for any integral transform, in general, an inversion formula is very important (see [2, 3, 6, 7, 10, 12, 14, 15, 16, 17, 22, 23]). In the following subsections, we deal with the isometrical identities and the inversion formulas.

4.1. **First case.** In this case ($b = 0$), thanks to (1.4) the associated reproducing kernel is given by

$$K_{0,\xi^2}(z, \bar{u}) = \frac{1}{2}\frac{\sqrt{\pi}}{a^{3/2}\{(z-\bar{u})i\}^{\frac{3}{2}}}. \tag{4.1}$$

It is worth saying that

$$\frac{1}{(z-\bar{u})i} \tag{4.2}$$

is the Szegö kernel on the lower half-plane, and the Bergman-Selberg spaces are conformally invariant. By comparing Propositions 2.1 and 2.2, we see that the reproducing kernel Hilbert space admitting the reproducing kernel (4.1) is composed of analytic functions on the lower half-plane L with finite norms:

$$\|f\|^2_{H_K(L)} = \frac{2a^{3/2}}{\pi^2} \int\int_L |f(z)|^2 (-2y)^{-1/2} dxdy.$$

In Propositions 2.1 and 2.2, $q = \frac{3}{4}$. Hence, from the general theory of integral transforms, we directly derive the following result.

Theorem 4.1. *For the integral transform $F \longrightarrow f$ defined by*

$$\int_{\mathbb{R}} \exp(-iz(a\xi^2))F(\xi)\xi^2 d\xi = f(z)$$

in which the functions F satisfy

$$\int_{\mathbb{R}} |F(\xi)|^2 \xi^2 d\xi < \infty, \tag{4.3}$$

the isometric identity yields

$$\int_{\mathbb{R}} |F(\xi)|^2 \xi^2 d\xi = \frac{2a^{3/2}}{\pi^2} \int\int_L |f(z)|^2 (-2y)^{-\frac{1}{2}} dxdy. \tag{4.4}$$

Furthermore, the complex inversion formula holds true:

$$F(\xi) = \operatorname*{l.i.m}_{n\to\infty} \frac{2a^{3/2}}{\pi^2} \int\int_{L_n} f(z) \exp(\{i\overline{z}(a\xi^2)\})(-2y)^{\frac{-1}{2}} dxdy.$$

Here, $\{L_n\}$ is a compact exhaustion of the lower half-plane L and the notation l. i. m *is considered in the sense of the norm of the space satisfying* (4.3)*.*

Proposition 2.2 shows that, in Theorem 4.1, the complex integral (4.4) is representable on the half-line of $\{iy\}$ $(0 > y > -\infty)$ in terms of infinite order Sobolev spaces and on the half-line, we can obtain the inversion formula as in the real inversion formula of the Laplace transform; see [20].

4.2. **Second case.** Here, we shall use the identity (1.5). Recall that (4.2) is the Szegö kernel on the lower half-plane. By Proposition 3.1, the images f of the integral transform

$$\int_{\mathbb{R}} \exp(-iz(a\xi^2 + b\xi))F(\xi)d\xi = f(z)$$

for functions F satisfying

$$\int_{\mathbb{R}} |F(\xi)|^2 d\xi < \infty,$$

are representable in the form

$$f(z) = f_1(z)\sqrt{\frac{\pi}{a}} \exp\left(\frac{ib^2 z}{4a}\right),$$

where $f_1(z)$ belongs to the reproducing kernel Hilbert space admitting the kernel

$$\sqrt{\frac{1}{i(z-\overline{u})}}. \tag{4.5}$$

However, the structure of this reproducing kernel Hilbert space is involved. We recall two interesting properties:

(1) $f_1(z)^2$ belongs to the Szegö space on the half-plane, in general, and we obtain the norm inequality in the form

$$\|f_1^2\| \leq M\|f_1\|^2,$$

where M is determined precisely and $\|f_1\|$ is the norm in the reproducing kernel Hilbert space admitting the kernel (4.5) (see [20], Appendix 2). Meanwhile, $\|f_1^2\|$ is the norm in the Szegö space. By using this norm, we can obtain the isometric identity in the integral transform. This fact means that for the integral transform we can obtain the norm inequality in terms of the Szegö norm for the square of the image of the integral transform.

(2) In general, for the Bergman-Selberg spaces, when we consider its derivative f' for a member, we have the isometric identity in such a way: in the notation in Proposition 2.2,

$$\|f'\|^2_{H_{K_{q+1}}(\mathbb{R}^2_+)} = \|f\|^2_{H_{K_q}(\mathbb{R}^2_+)}$$

(cf. [4]; [20], Appendix 2). Therefore, by taking the derivative, we shall consider the image identification of the integral transform. By considering Proposition 3.1 and Proposition 2.2 (for the case of the lower half-plane version, and $q = \frac{3}{4}$), we have the norm realization of f as follows:

$$\|f\|^2 = \frac{a^{\frac{1}{2}}}{\pi^2} \int\int_L \left| \left\{ f(z) \exp\left(\frac{-ib^2 z}{4a} \right) \right\}' \right|^2 (-2y)^{-\frac{1}{2}} dxdy.$$

We thus obtain the following theorem.

Theorem 4.2. *For the integral transform*

$$\int_{\mathbb{R}} \exp(-iz(a\xi^2 + b\xi))F(\xi)d\xi = f(z)$$

in which F satisfies

$$\int_{\mathbb{R}} |F(\xi)|^2 d\xi < \infty,$$

we have the isometric identity

$$\int_{\mathbb{R}} |F(\xi)|^2 d\xi = \frac{a^{\frac{1}{2}}}{\pi^2} \int\int_L \left| \left\{ f(z) \exp\left(\frac{-ib^2 z}{4a} \right) \right\}' \right|^2 (-2y)^{-\frac{1}{2}} dxdy.$$

In Theorem 4.2, the isometric identity is already involved, and so any analytical inversion formula seems to be very complicated. For the general approach for the inversions, see [20, p. 24–27].

4.3. **Third case.** We find a specially interesting identity

$$\int_{-\infty}^{\infty} e^{-i(z-\overline{u})(a\xi^2)}|\xi|^{2n+1}e^{-\varepsilon\xi^2}d\xi = \frac{n!}{[a(z-\overline{u})i+\varepsilon]^{n+1}}.$$

Put $\varepsilon = 0$, then we have

$$\int_{-\infty}^{\infty} e^{-i(z-\overline{u})(a\xi^2)}|\xi|^{2n+1}d\xi = \frac{n!}{[a(z-\overline{u})i]^{n+1}}.$$

If $n = 0$, then we obtain the simple identity

$$\int_{-\infty}^{\infty} e^{-i(z-\overline{u})(a\xi^2)}|\xi|e^{-\varepsilon\xi^2}d\xi = \frac{1}{[a(z-\overline{u})i+\varepsilon]}.$$

Put $\varepsilon = 0$, then we have

$$\int_{-\infty}^{\infty} e^{-i(z-\overline{u})(a\xi^2)}|\xi|d\xi = \frac{1}{[a(z-\overline{u})i]}.$$

By using those identities, we obtain the previously considered corresponding results.

4.4. **Fourth case.** We find a specially interesting identity

$$\int_{0}^{\infty} \exp\left(-i(z-\overline{u})(a\xi^2)\right)\xi e^{-\varepsilon\xi^2}d\xi = \frac{1}{2\{a(z-\overline{u})i+\varepsilon\}}.$$

Since

$$\frac{1}{2\{a(z-\overline{u})i+\varepsilon\}}$$

is the Szegö kernel on the half-plane $L_{\varepsilon,a} = \{y < \frac{\varepsilon}{2a}\}$ we immediately obtain the following theorem.

Theorem 4.3. *In the integral transform*

$$\int_{0}^{\infty} \exp\left(-iz(a\xi^2)\right)\xi e^{-\varepsilon\xi^2}F(\xi)d\xi = f(z)$$

in which F fulfills the condition

$$\int_{0}^{\infty} |F(\xi)|^2\xi e^{-\varepsilon\xi^2}d\xi < \infty,$$

we have the isometric identity

$$\int_{0}^{\infty} |F(\xi)|^2\xi e^{-\varepsilon\xi^2}d\xi = \frac{a}{\pi}\int_{-\infty}^{\infty}\left|f\left(x+\frac{\varepsilon}{2a}i\right)\right|^2 dx.$$

Moreover, the inversion formula is given by

$$F(\xi) = \underset{n\to\infty}{\mathrm{l.i.m}}\ \frac{a}{\pi}\int_{I_n} f\left(x+\frac{\varepsilon}{2a}i\right)\exp\left((ix+\frac{\varepsilon}{2a})a\xi^2\right)dx,$$

where $\{I_n\}$ is a compact exhaustion of the line $\left(x+\frac{\varepsilon}{2a}i\right)$ composing of finite intervals.

4.5. **Fifth case.** The realizations of the reproducing kernel Hilbert spaces admitting the kernels (1.4) and (1.5) are complicated. The kernel (1.4) is composed of the product and sum of the concretely known reproducing kernels whose structures are known. Therefore, the reproducing kernel Hilbert space admitting the kernel (1.4) is given by the methods of the restriction of some tensor product and sum of reproducing kernels, and so its structure is very complicated (see [20]). The kernel (1.5) is a much more complicated one as it contains the exponential of a known reproducing kernel (cf. [20], Appendix 2). In order to derive general properties by Proposition 3.2, having considered both cases, we have monotonicity in ε for $\varepsilon_1 > \varepsilon_2$:

$$K_{\varepsilon_1,\xi^2}(z,\overline{u}) \ll K_{\varepsilon_2,\xi^2}(z,\overline{u}), \qquad K_{\varepsilon_1}(z,\overline{u}) \ll K_{\varepsilon_2}(z,\overline{u}).$$

For $\varepsilon = 0$ the corresponding reproducing kernel Hilbert spaces are concretely realized. Due to Proposition 3.2 we have the inclusion relations as functions and norm inequalities. This means that in the general integral transforms with the weight $e^{-\varepsilon\xi^2}$, we obtain the norm inequalities.

4.6. **Sixth case.** Following the identity (1.5), we shall examine the Fourier transform for $a = 0$. The reproducing kernel is as follows:

$$\sqrt{\frac{\pi}{\varepsilon}} \exp\left(\frac{-b^2 z^2}{4\varepsilon}\right) \cdot \exp\left(\frac{-b^2 \overline{u}^2}{4\varepsilon}\right) \cdot \exp\left(\frac{b^2 z \overline{u}}{2\varepsilon}\right). \tag{4.6}$$

Note that

$$\exp\left(\frac{b^2 z\overline{u}}{2\varepsilon}\right)$$

is the reproducing kernel comprising of entire functions with finite norm squares:

$$\frac{A}{\pi}\iint_{\mathbf{C}} |f(z)|^2 \exp\left(-A|z|^2\right) dxdy \qquad \text{for} \qquad A = \frac{b^2}{2\varepsilon}$$

(cf. [20, p. 61]). Hence, the reproducing kernel Hilbert space admitting kernel (4.6) can be realized concretely. Namely, we can obtain the following theorem.

Theorem 4.4. *In the Fourier transform*

$$f(z) = \int_{-\infty}^{\infty} \exp\left(-iz(b\xi)\right) F(\xi) e^{-\varepsilon\xi^2} d\xi$$

in which the function F satisfies

$$\int_{-\infty}^{\infty} |F(\xi)|^2 e^{-\varepsilon\xi^2} d\xi < \infty,$$

f is an entire function. Moreover, the isometric identity holds

$$\frac{b^2}{2\pi^{3/2}\varepsilon^{1/2}} \iint_{\mathbb{C}} |f(z)|^2 \exp\left(-\frac{b^2}{\varepsilon}y^2\right) dxdy = \int_{-\infty}^{\infty} |F(\xi)|^2 e^{-\varepsilon\xi^2} d\xi.$$

Further, the complex inversion formula is given by

$$F(\xi) = \operatorname*{l.i.m}_{n\to\infty} \iint_{K_n} f(z) \exp(i\,\overline{z}\,b\,\xi) \exp\left(-\frac{b^2}{\varepsilon}y^2\right) dxdy$$

where K_n is a compact exhaustion of the whole complex plane.

From the identity (1.4), we see that when the weight $\xi^2 e^{-\varepsilon\xi^2}$, the Fourier transform structure is very complicated.

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