Ann. Funct. Anal. 4 (2013), no. 1, 138–148
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL:www.emis.de/journals/AFA/

COUPLED COINCIDENCE POINT THEOREMS FOR NONLINEAR CONTRACTIONS UNDER C-DISTANCE IN CONE METRIC SPACES

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Communicated by M. A. Japón Pineda

ABSTRACT. In this paper, among others, we prove the following results: (1) Let (X, d) be a complete cone metric space partially ordered by $\sqsubseteq$ and q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let F satisfy mixed g-monotone property and $q(F(x,y), F(u,v)) \preceq \frac{k}{2}(q(gx,gu) + q(gy,gv))$ for some $k \in [0,1)$ and all $x,y,u,v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*). (2) If, in (1), we replace completeness of (X, d) by completeness of $(g(X), d)$ and commutativity, continuity of mappings F and g by the condition: (i) for any nondecreasing sequence $\{x_n\}$ in X converging to x we have $x_n \sqsubseteq x$ for all n. (ii) for any nonincreasing sequence $\{y_n\}$ in Y converging to y we have $y \sqsubseteq y_n$ for all n, then F and g have a coupled coincidence point (x^*, y^*).

1. Introduction and preliminaries

Since Banach's fixed point theorem in 1922, because of its simplicity and usefulness, it has become a very important tool in solving the existence problems in many branches of nonlinear analysis. Ran and Reurings [15] extended the Banach contraction principle to metric spaces endowed with a partial ordering, and they gave application of their results to matrix equations. In [14] Nieto and López extended the result of Ran and Reurings [15] for nondecreasing mappings and

Date: Received: 15 July 2012; Accepted 30 October 2012.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 46B40, 54H25, 55M20.
Key words and phrases. Fixed point, coincidence point, cone metric space, c-distance.

applied their results to get a unique solution for a first order differential equation.

Bhaskar and Lakshmikantham [4] introduced the notion of a coupled fixed point of a mapping $F : X \times X \to X$. They established some coupled fixed point results and applied their results to the study of existence and uniqueness of solution for a periodic boundary value problem. Lakshmikantham and Ćirić [12] introduced the concept of coupled coincidence points and proved coupled coincidence and coupled common fixed point results for mappings $F : X \times X \to X$ and $g : X \to X$ satisfying nonlinear contractive condition in ordered metric space.

The concept of cone metric spaces is a generalization of metric spaces, where each pair of points is assigned to a member of a real Banach space with a cone. This cone naturally induces a partial order in the Banach spaces. The concept of cone metric space was introduced in the work of Huang and Zhang [7] where they also established the Banach contraction mapping principle in this space. Then, several authors have studied fixed point problems in cone metric spaces. For more study on fixed point theory on cone metric spaces see [1, 2, 6, 7] and for many recent results on fixed point theory on other spaces see [3, 9, 11, 16, 17]. The studies of asymmetric structures and their application in mathematics are important (see, e.g. [8, 13, 19, 20, 22, 23, 24, 25, 26, 27]).

Recently Cho et al. [6] introduced a new concept of c-distance in cone metric spaces which is a cone version of w-distance of Kada et al. [8]. In [5] Cho et al. established coupled fixed point theorems under weak contraction mappings by using the concept of mixed monotone property and c-distance in partially ordered cone metric spaces. In this paper we extend the results of Cho et al. [5] and establish the existence of coupled coincidence point for mappings $F : X \times X \to X$ and $g : X \to X$ satisfying nonlinear contractive condition and mixed g-monotone property under c-distance in cone metric spaces.

Throughout this paper, $(X, \sqsubseteq)$ denotes a partially ordered set with partial order $\sqsubseteq$.

Definition 1.1 ([4])**.** A mapping $F : X \times X \to X$ is said to have mixed monotone property if for any $x, y \in X$

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \sqsubseteq x_2 \quad &\Longrightarrow \quad F(x_1, y) \sqsubseteq F(x_2, y), \\ y_1, y_2 \in X, \quad y_1 \sqsubseteq y_2 \quad &\Longrightarrow \quad F(x, y_1) \sqsupseteq F(x, y_2). \end{aligned}$$

Definition 1.2 ([12])**.** A mapping $F : X \times X \to X$ is said to have mixed g-monotone property if for any $x, y \in X$

$$\begin{aligned} x_1, x_2 \in X, \quad gx_1 \sqsubseteq gx_2 \quad &\Longrightarrow \quad F(x_1, y) \sqsubseteq F(x_2, y), \\ y_1, y_2 \in X, \quad gy_1 \sqsubseteq gy_2 \quad &\Longrightarrow \quad F(x, y_1) \sqsupseteq F(x, y_2). \end{aligned}$$

Definition 1.3 ([4])**.** An element $(x, y) \in X \times X$ is called a coupled fixed point of the mappings $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.4 ([12])**.** An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.5 ([12])**.** Let $F : X \times X \to X$ and $g : X \to X$. The mappings F and g are said to commute if $gF(x,y) = F(gx, gy)$ for all $x, y \in X$.

In [7], cone metric space was introduced in the following manner:

Let $(E, \|\cdot\|)$ be a real Banach space and θ denote the zero element in E. Assume that P is a subset of E. Then P is called a cone if and only if:

(1) P is non empty, closed and $P \neq \{\theta\}$.
(2) If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$.
(3) $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subseteq E$ and $x, y \in E$, the partial ordering $\preceq$ on E with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in intP$. It can be easily shown that $\lambda \cdot intP \subseteq intP$ for all $\lambda > 0$ and $intP + intP \subseteq intP$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number K satisfying above is called the normal constant of P.

In the following we always suppose E is a real Banach space, P is a cone in E with $intP \neq \phi$ and $\preceq$ is partial ordering with respect to P.

Definition 1.6 ([7])**.** Let X be a non empty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(i) $\theta \prec d(x,y)$ for all $x, y \in X$ with $x \neq y$ and $d(x,y) = \theta \Leftrightarrow x = y$.
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$.
(iii) $d(x,z) \preceq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.7 ([7])**.** Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$ then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.
(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$ then $\{x_n\}$ is called a Cauchy sequence in X.
(3) A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 1.8 ([7])**.** *Let (X, d) be a cone metric space, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then,*

(1) *the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ (or equivalently $\|d(x_n, x)\| \to 0$),*
(2) *the sequence $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ (or equivalently $\|d(x_n, x_m)\| \to 0$).*
(3) *the sequence $\{x_n\}$ (respectively, $\{y_n\}$) converges to x (respectively, y) then $d(x_n, y_n) \to d(x, y)$.*

Lemma 1.9 ([21]). *Every cone metric space $(X,\ d)$ is a topological space. For $c \gg 0, c \in E$, $x \in X$ let $B(x,c) = \{y \in X : d(y,x) \ll c\}$ and $\beta = \{B(x,c) : x \in X, c \gg 0\}$. Then $\tau_c = \{U \subseteq X : \forall\, x \in U, \exists\, B \in \beta, x \in B \subseteq U\}$ is a topology on X.*

Definition 1.10 ([21]). Let (X,d) be a cone metric space. A map $T:(X,d)\to(X,d)$ is called sequentially continuous if $x_n \in X$, $x_n \to x$ implies $Tx_n \to Tx$.

Lemma 1.11 ([21]). *Let (X,d) be a cone metric space, and $T:(X,d)\to(X,d)$ be any map. Then, T is continuous if and only if T is sequentially continuous.*

Let (X,d) be a cone metric space and $X^2 = X \times X$. Define a function $\rho : X^2 \times X^2 \to E$ by $\rho((x_1,y_1),(x_2,y_2)) = d(x_1,x_2) + d(y_1,y_2)$ for all (x_1,y_1) and $(x_2,y_2) \in X^2$. Then (X^2,ρ) is a cone metric space [10].

Lemma 1.12 ([10]). *Let $z_n = (x_n,y_n) \in X^2$ be a sequence and $z = (x,y) \in X^2$. Then $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.*

Next we give the notation of c-distance on a cone metric space which is generalization of w-distance of Kada et al. [8] with some properties.

Definition 1.13 ([6]). Let (X,d) be a cone metric space. A function $q:X\times X\to E$ is called a c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x,y)$ for all $x,y \in X$,
(q2) $q(x,z) \preceq q(x,y) + q(y,z)$ for all $x,y,z \in X$,
(q3) For each $x \in X$ and $n \in \mathbb{N}$, if $q(x,y_n) \preceq u$ for some $u = u_x \in P$, then $q(x,y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,
(q4) For all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Remark 1.14. The c-distance q is a w-distance on X if we let (X,d) be a metric space, $E = \mathbb{R}$, $P = [0,\infty)$, and (q3) is replaced by the following condition: for any $x \in X$, $q(x,\cdot) : X \to \mathbb{R}$ is lower semicontinuous. Moreover, (q3) holds whenever $q(x,\cdot)$ is lower semi-continuous. Thus, if (X,d) is a metric space, $E = \mathbb{R}$, and $P = [0,\infty)$, then every w-distance is a c-distance. But the converse is not true in the general case. Therefore, the c-distance is a generalization of the w-distance.

Example 1.15 ([18]). Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0,\infty)$ and define a mapping $d : X \times X \to E$ by $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = y$ for all $x,y \in X$. Then q is a c-distance on X.

Example 1.16 ([18]). Let X be a non empty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the normal cone P. Let $d : X \times X \to E$ be the corresponding cone metric. Define a mapping $q : X \times X \to E$ by $q(x,y) = d(x,y)$ for all $x,y \in X$. Then, q is c-distance.

Example 1.17 ([18]). Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\|_1 = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0, t \in [0,1]\}$. Let $X = [0,+\infty)$ (with usual order), and $d(x,y)(t) = |x-y|\varphi(t)$ where $\varphi : [0,1] \to \mathbb{R}$ is given by $\varphi(t) = e^t$ for all $t \in [0,1]$. Then (X,d) is an ordered cone metric space (see [6, Example 2.9]). This cone is not normal.

Define a mapping $q : X \times X \to E$ by $q(x, y) = (x + y)\varphi$ for all $x, y \in X$. Then q is a c-distance.

Example 1.18 ([18])**.** Let X be a non empty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the normal cone P. Let $d : X \times X \to E$ be the corresponding cone metric. Define a mapping $q : X \times X \to E$ by $q(x, y) = d(u, y)$ for all $x, y \in X$, where u is a fixed point in X. Then, q is a c-distance.

Lemma 1.19 ([6])**.** *Let (X, d) be a cone metric space and q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $y, z \in X$. Suppose that u_n is a sequence in P converging to θ. Then the following hold:*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.*
(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then y_n converges to z.*
(3) *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*
(4) *If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Remark 1.20 ([6]). (1) $q(x, y) = q(y, x)$ may not be true for all $x, y \in X$.
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

2. Main Results

Theorem 2.1. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let F satisfy mixed g-monotone property and*

$$q(F(x, y), F(u, v)) \preceq \frac{k}{2}(q(gx, gu) + q(gy, gv))$$

for some $k \in [0, 1)$ and all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Choose $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$. Since $F(X \times X) \subseteq g(X)$, one can find $x_1, y_1 \in X$ in a way that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Repeating the same argument one can find $x_2, y_2 \in X$ in a way that $gx_2 = F(x_1, y_1)$ and $F(y_1, x_1) = gy_2$. Continuing this process one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X that satisfy $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for all $n \geq 0$. It is asserted that $\{gx_n\}$ is a nondecreasing and $\{gy_n\}$ is a nonincreasing sequence. That is

$$gx_n \sqsubseteq gx_{n+1} \quad \text{and} \quad gy_n \sqsupseteq gy_{n+1} \tag{2.1}$$

for all $n \geq 0$. For $n = 0$, (2.1) follows by the choice of x_0 and y_0. Let us assume that (2.1) holds good for $n = k$, $k \geq 0$. So we have $gx_k \sqsubseteq gx_{k+1}$ and $gy_k \sqsupseteq gy_{k+1}$. Mixed g-monotonicity of F now implies that $gx_{k+1} = F(x_k, y_k) \sqsubseteq F(x_{k+1}, y_k) \sqsubseteq F(x_{k+1}, y_{k+1}) = gx_{k+2}$. Similarly we have $gy_{k+1} \sqsupseteq gy_{k+2}$. Thus (2.1) follows for

$k+1$. Hence, by induction, our assertion follows. Now for all $n \in \mathbb{N}$

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq \frac{k}{2}(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)) \end{aligned}$$

and

$$\begin{aligned} q(gy_n, gy_{n+1}) &= q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\preceq \frac{k}{2}(q(gy_{n-1}, gy_n) + q(gx_{n-1}, gx_n)). \end{aligned}$$

Put $q_n = q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1})$. Then, we have

$$\begin{aligned} q_n &= q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \\ &\preceq k\, q_{n-1} \\ &\vdots \\ &\preceq k^n\, q_0. \end{aligned}$$

Let $m > n \geq 1$. It follows that

$$q(gx_n, gx_m) \preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \ldots + q(gx_{m-1}, gx_m)$$

and

$$q(gy_n, gy_m) \preceq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \ldots + q(gy_{m-1}, gy_m).$$

Then we have

$$\begin{aligned} q(gx_n, gx_m) + q(gy_n, gy_m) &\preceq q_n + q_{n+1} + \ldots + q_{m-1} \\ &\preceq k^n\, q_0 + k^{n+1}\, q_0 + \ldots + k^{m-1}\, q_0 \\ &\preceq \frac{k^n}{1-k}\, q_0. \end{aligned} \tag{2.2}$$

From (2.2) we have

$$q(gx_n, gx_m) \preceq \frac{k^n}{1-k}\, q_0 \tag{2.3}$$

and also

$$q(gy_n, gy_m) \preceq \frac{k^n}{1-k}\, q_0. \tag{2.4}$$

Thus, Lemma 1.19(3) shows that gx_n and gy_n are Cauchy sequences in X. Since X is complete, there exists there exists $x^*, y^* \in X$ such that $gx_n \to x^*$ and $gy_n \to y^*$ as $n \to \infty$. By continuity of g we get $\lim_{n\to\infty} ggx_n = gx^*$ and $\lim_{n\to\infty} ggy_n = gy^*$ Commutativity of F and g now implies that

$$ggx_n = g(F(x_{n-1}, y_{n-1})) = F(gx_{n-1}, gy_{n-1})$$

for all $n \in \mathbb{N}$ and

$$ggy_n = gF(y_{n-1}, x_{n-1}) = F(gy_{n-1}, gx_{n-1})$$

for all $n \in \mathbb{N}$. Since F is continuous, therefore,

$$\begin{aligned} gx^* &= \lim_{n\to\infty} ggx_n \\ &= \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) \\ &= F\Big(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}\Big) \\ &= F(x^*, y^*) \end{aligned}$$

and

$$\begin{aligned} gy^* &= \lim_{n\to\infty} ggy_n \\ &= \lim_{n\to\infty} F(gy_{n-1}, gx_{n-1}) \\ &= F\Big(\lim_{n\to\infty} gy_{n-1}, \lim_{n\to\infty} gx_{n-1}\Big) \\ &= F(y^*, x^*). \end{aligned}$$

Thus (x^*, y^*) is a coupled coincidence point of F and g. □

Corollary 2.2 ([5]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ is a continuous functions satisfying mixed monotone property and*

$$q(F(x, y), F(u, v)) \preceq \frac{k}{2}(q(x, u) + q(y, v))$$

for some $k \in [0, 1)$ and all $x, y, u, v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$. If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Theorem 2.3. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two functions such that $F(X \times X) \subseteq g(X)$ and $(g(X), d)$ is complete subspace of X. Let F satisfy mixed g-monotone property and*

$$q(F(x, y), F(u, v)) \preceq \frac{k}{2}(q(gx, gu) + q(gy, gv))$$

for some $k \in [0, 1)$ and all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. Suppose X has the following property:

(i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.*
(ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*).

Proof. Consider Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ as in the proof of Theorem 2.1. Since $(g(X), d)$ is complete, there exists $x^*, y^* \in X$ such that $gx_n \to gx^*$ and

$gy_n \to gy^*$. By (q3), (2.3) and (2.4) we have

$$q(gx_n, gx^*) \preceq \frac{k^n}{1-k}\, q_0 \tag{2.5}$$

for all $n \geq 0$ and

$$q(gy_n, gy^*) \preceq \frac{k^n}{1-k}\, q_0 \tag{2.6}$$

for all $n \geq 0$. Sequence $\{gx_n\}$ is nondecreasing and converges to gx^*. By given condition (i) we have, therefore, $gx_n \sqsubseteq gx^*$ for all $n \geq 0$ and similarly $gy_n \sqsupseteq gy^*$ for all $n \geq 0$. Thus for all $n \in \mathbb{N}$

$$\begin{aligned} q(gx_n, F(x^*, y^*)) &= q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \\ &\preceq \frac{k}{2}[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)] \\ &\preceq \frac{k}{2}\left[\frac{k^{n-1}}{1-k}\, q_0 + \frac{k^{n-1}}{1-k}\, q_0\right] \\ &= \frac{k^n}{1-k}\, q_0 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} q(gy_n, F(y^*, x^*)) &= q(F(y_{n-1}, x_{n-1}), F(y^*, x^*)) \\ &\preceq \frac{k}{2}[q(gy_{n-1}, gy^*) + q(gx_{n-1}, gx^*)] \\ &\preceq \frac{k}{2}\left[\frac{k^{n-1}}{1-k}\, q_0 + \frac{k^{n-1}}{1-k}\, q_0\right] \\ &= \frac{k^n}{1-k}\, q_0. \end{aligned} \tag{2.8}$$

By Lemma 1.19(1), (2.5) and (2.7), we have $F(x^*, y^*) = gx^*$. Similarly, by Lemma 1.19(1), (2.6) and (2.8) we have $F(y^*, x^*) = gy^*$. Thus (x^*, y^*) is a coupled coincidence point of F and g. $\square$

Corollary 2.4. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ is a functions satisfying mixed monotone property and*

$$q(F(x, y), F(u, v)) \preceq \frac{k}{2}(q(x, u) + q(y, v))$$

for some $k \in [0, 1)$ and all $x, y, u, v \in X$ with $(x \sqsubseteq u)$ and $(y \sqsupseteq v)$ or $(x \sqsupseteq u)$ and $(y \sqsubseteq v)$. Suppose X has the following property:

(i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.*
(ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ satisfying $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*).

Theorem 2.5. *Under the hypothesis of either Theorem 2.1 or Theorem 2.3 we have* $q(gx^*, gx^*) = \theta$ *and* $q(gy^*, gy^*) = \theta$.

Proof. We have

$$\begin{aligned} q(gx^*, gx^*) &= q(F(x^*, y^*), F(x^*, y^*) \\ &\preceq \frac{k}{2}\big(q(gx^*, gx^*) + q(gy^*, gy^*)\big) \end{aligned}$$

and also

$$\begin{aligned} q(gy^*, gy^*) &= q(F(y^*, x^*), F(y^*, x^*)) \\ &\preceq \frac{k}{2}\big(q(gy^*, gy^*) + q(gx^*, gx^*)\big). \end{aligned}$$

This implies that $q(gx^*, gx^*) + q(gy^*, gy^*) \preceq k(q(gx^*, gx^*) + q(gy^*, gy^*))$.
Since $0 \leq k < 1$, we have $q(gx^*, gx^*) + q(gy^*, gy^*) = \theta$.
But $q(gx^*, gx^*) \geq \theta$ and $q(gy^*, gy^*) \geq \theta$, hence $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$. ☐

Corollary 2.6. *Under the hypothesis of either Corollary 2.2 or Corollary 2.4 we have* $q(x^*, x^*) = \theta$ *and* $q(y^*, y^*) = \theta$.

Theorem 2.7. *In addition to hypothesis of either Theorem 2.1 or Theorem 2.3, suppose that any two elements of* $g(X)$ *are comparable and* g *is one-one. Then there exists a coupled coincidence point of* F *and* g *which is of the form* (x^*, x^*) *for some* $x^* \in X$.

Proof. Consider coupled coincidence point (x^*, y^*) of F and g. Then we have

$$\begin{aligned} q(gx^*, gy^*) &= q(F(x^*, y^*), F(y^*, x^*) \\ &\preceq \frac{k}{2}\big(q(gx^*, gy^*) + q(gy^*, gx^*)\big) \end{aligned}$$

and also

$$\begin{aligned} q(gy^*, gx^*) &= q(F(y^*, x^*), F(x^*, y^*)) \\ &\preceq \frac{k}{2}\big(q(gy^*, gx^*) + q(gx^*, gy^*)\big). \end{aligned}$$

This implies that $q(gx^*, gy^*) + q(gy^*, gx^*) \preceq k(q(gy^*, gx^*) + q(gx^*, gy^*))$.
Since $0 \leq k < 1$, we have $q(gx^*, gy^*) + q(gy^*, gx^*) = \theta$.
But $q(gx^*, gy^*) \geq \theta$ and $q(gy^*, gx^*) \geq \theta$, hence $q(gx^*, gy^*) = \theta$ and $q(gy^*, gx^*) = \theta$.
Let $u_n = \theta$, $x_n = gx^*$ for all $n \geq 0$, then we have $q(x_n, gx^*) \preceq u_n$ for all $n \geq 0$ and $q(x_n, gy^*) \preceq u_n$ for all $n \geq 0$. By Lemma 1.19(1) we have $gx^* = gy^*$. Since g is one-one, therefore, $x^* = y^*$. Thus there exists a coupled coincidence point of the form (x^*, x^*) for some $x^* \in X$. This completes the proof. ☐

Corollary 2.8. *In addition to hypothesis of either Corollary 2.2 or Corollary 2.4, suppose that any two elements of* X *are comparable. Then there exists a coupled fixed point of* F *which is of the form* (x^*, x^*) *for some* $x^* \in X$.

Example 2.9. Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\|_1 = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0, t \in [0,1]\}$. Let $X = [0,+\infty)$ (with usual order) and $d(x,y)(t) = |x-y|\, e^t$. Then (X,d) is an ordered cone metric space (see [6, Example 2.9]). Further, let $q : X \times X \to E$ be defined by $q(x,y)(t) = y\, e^t$. It is easy to check that q is a c-distance on X. Consider now the function defined by

$$F(x,y) = \begin{cases} \frac{1}{8}(x-y) & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

and $gx = \frac{1}{2}x$ for all $x \in X$. Then $F(X \times X) \subseteq g(X) = X$ and F satisfy mixed g-monotone property. Also it can be seen easily that $q(F(x,y), F(u,v)) \preceq \frac{1}{3}(q(gx,gu) + q(gy,gv))$ for all $x,y,u,v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. Further F and g are continuous, commuting, $g(0) \sqsubseteq F(0,1)$ and $g(1) \sqsupseteq F(1,0)$. Thus, by Theorem 2.1, F and g have a coincidence point. Here F and g have a coincidence point at $(0,0)$.

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