

RANK EQUALITIES FOR MOORE-PENROSE INVERSE AND DRAZIN INVERSE OVER QUATERNION

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ABSTRACT. In this paper, we consider the ranks of four real matrices $G_i (i = 0, 1, 2, 3)$ in M^\dagger , where $M = M_0 + M_1i + M_2j + M_3k$ is an arbitrary quaternion matrix, and $M^\dagger = G_0 + G_1i + G_2j + G_3k$ is the Moore-Penrose inverse of M . Similarly, the ranks of four real matrices in Drazin inverse of a quaternion matrix are also presented. As applications, the necessary and sufficient conditions for M^\dagger is pure real or pure imaginary Moore-Penrose inverse and N^D is pure real or pure imaginary Drazin inverse are presented, respectively.

1. INTRODUCTION

Throughout this paper, we denote the real number field by \mathbb{R} , the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m \times n}$, the identity matrix with the appropriate size by I , the conjugate transpose of a matrix A by A^* , the column right space, the row left space of a matrix A over \mathbb{H} by $\mathcal{R}(A), \mathcal{N}(A)$, respectively. The Moore-penrose inverse of $A \in \mathbb{H}^{m \times n}$, denoted by A^\dagger , is defined to be the unique solution X to the four matrix equations

$$(i) AXA = A, (ii) XAX = X, (iii) (AX)^* = AX, (iv) (XA)^* = XA.$$

Let $A \in \mathbb{H}^{m \times m}$ be given with $IndA = k$, the smallest positive integer such that $r(A^{k+1}) = r(A^k)$. The Drazin inverse of matrix A , denoted by A^D , is defined to be the unique solution X of the following three matrix equations

$$(i) A^k XA = A^k, (ii) XAX = X, (iii) XA = AX.$$

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Suppose

$$M = M_0 + M_1i + M_2j + M_3k, N = N_0 + N_1i + N_2j + N_3k \quad (1.1)$$

be a quaternion matrix, where $M_i \in \mathbb{R}^{m \times n}$, $N_i \in \mathbb{R}^{m \times m}$, $i = 0, 1, 2, 3$, and let

$$\overline{M} = \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix}, \overline{N} = \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ N_1 & N_0 & N_3 & -N_2 \\ N_2 & -N_3 & N_0 & N_1 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}, \quad (1.2)$$

and the Moore-Penrose inverse of M , the Drazin inverse of N are denoted by

$$M^\dagger = G_0 + G_1i + G_2j + G_3k, N^D = D_0 + D_1i + D_2j + D_3k, \quad (1.3)$$

respectively, where $G_i \in \mathbb{R}^{n \times m}$, $D_i \in \mathbb{R}^{m \times m}$, $i = 0, 1, 2, 3$.

Moore-Penrose inverse of matrix is an attractive topic in matrix theory and have extensively been investigated by many authors (see, e.g., [1]-[11]). Drazin inverse is also one of the important types of generalized inverses of matrices, and have well been examined in the literatures, (see, e.g., [1]-[2], [13]-[16]). For example, Campbell and Meyer gave a basic identity on Drazin inverse of a matrix in [1]

$$A^D = A^k (A^{2k+1})^\dagger A^k. \quad (1.4)$$

L. Zhang presented a characterization of the Drazin inverse of any $n \times n$ singular matrix and proposed a method for solving the Drazin inverse and an algorithm with detailed steps to compute the Drazin inverse in [13].

As well known, the expressions of G_i, D_i ($i = 0, 1, 2, 3$) in M^\dagger, N^D are quite complicated if there are no restrictions (see, e.g., [3], [5]). In that case, it is difficult to find properties of G_i, D_i ($i = 0, 1, 2, 3$) in M^\dagger, N^D . In this paper, we derived the ranks of G_i, D_i ($i = 0, 1, 2, 3$) in M^\dagger, N^D through a simpler method, and then give some interesting consequences.

As a continuation of the above works, we in this paper investigate the ranks of real matrices G_i, D_i ($i = 0, 1, 2, 3$) in M^\dagger and N^D . In Section 2, we derive the formulas of rank equalities of four real matrices G_0, G_1, G_2 and G_3 in $M^\dagger = G_0 + G_1i + G_2j + G_3k$. Moreover, we established the necessary and sufficient conditions for M^\dagger is pure real or pure imaginary Moore-Penrose inverse. In Section 3, the formulas of rank equalities of four real matrices D_0, D_1, D_2 and D_3 in $N^D = D_0 + D_1i + D_2j + D_3k$ are established, and the necessary and sufficient conditions for N^D is pure real or pure imaginary Drazin inverse are presented. Some further research topics related to this paper are also given.

2. RANK EQUALITY FOR G_i ($i = 0, 1, 2, 3$) IN M^\dagger

We begin with the following lemmas which can be generalized to \mathbb{H} .

Lemma 2.1. (see [6]) *Let $A_1, A_2, \dots, A_k \in \mathbb{H}^{m \times n}$. Then the Moore-Penrose inverse of their sum satisfies*

$$(A_1 + A_2 + \cdots + A_k)^\dagger = \frac{1}{k} [I_n, I_n, \cdots, I_n] \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}.$$

Lemma 2.2. (see [6]) Let $A_1, A_2, \dots, A_k \in \mathbb{H}^{m \times n}$. Then the Drazin inverse of their sum satisfies

$$(A_1 + A_2 + \cdots + A_k)^D = \frac{1}{k} [I_n, I_n, \cdots, I_n] \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^D \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}.$$

Lemma 2.3. (see [7]) Let $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{m \times k}$, $C \in \mathbb{H}^{l \times n}$ and $D \in \mathbb{H}^{l \times k}$ be given. Then the rank of the Schur complement $S = D - CA^\dagger B$ satisfies the equality

$$r(D - CA^\dagger B) = r \begin{bmatrix} A^* A A^* & A^* B \\ C A^* & D \end{bmatrix} - r(A). \quad (2.1)$$

Lemma 2.4. (see [8]) Let $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{m \times k}$ and $C \in \mathbb{H}^{l \times n}$ be given, and suppose that

$$\mathcal{R}(AQ) = \mathcal{R}(A), \mathcal{R}[(PA)^*] = \mathcal{R}(A^*).$$

Then

$$r[AQ, B] = r[A, B], r \begin{bmatrix} PA \\ C \end{bmatrix} = r \begin{bmatrix} A \\ C \end{bmatrix}.$$

where P and Q are arbitrary matrices over \mathbb{H} .

Now we establish the main result about Moore-Penrose inverse.

Theorem 2.5. Let M, \overline{M} and M^+ be given by (1.1), (1.2) and (1.3). Then the ranks of G_i ($i = 0, 1, 2, 3$) in (1.3) can be determined by the following formulas

$$r(G_0) = r \begin{bmatrix} \widehat{M}_0 & \widetilde{M}_0 \\ \widetilde{M} & 0 \end{bmatrix} - r(\overline{M}), \quad r(G_1) = r \begin{bmatrix} \widehat{M}_1 & \widetilde{M}_1 \\ \widetilde{M} & 0 \end{bmatrix} - r(\overline{M}), \quad (2.2)$$

$$r(G_2) = r \begin{bmatrix} \widehat{M}_2 & \widetilde{M}_2 \\ \widetilde{M} & 0 \end{bmatrix} - r(\overline{M}), \quad r(G_3) = r \begin{bmatrix} \widehat{M}_3 & \widetilde{M}_3 \\ \widetilde{M} & 0 \end{bmatrix} - r(\overline{M}), \quad (2.3)$$

where

$$\begin{aligned} & \widehat{M}_0 \\ & = \begin{bmatrix} -M_1 & -M_2 & -M_3 \\ M_0 & M_3 & -M_2 \\ -M_3 & M_0 & M_1 \\ M_2 & -M_1 & M_0 \end{bmatrix} \begin{bmatrix} M_0^* & -M_3^* & M_2^* \\ M_3^* & M_0^* & -M_1^* \\ -M_2^* & M_1^* & M_0^* \end{bmatrix}^* \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix}, \end{aligned}$$

$$\widehat{M}_1 = \begin{bmatrix} M_0 & -M_2 & -M_3 \\ M_1 & M_3 & -M_2 \\ M_2 & M_0 & M_1 \\ M_3 & -M_1 & M_0 \end{bmatrix} \begin{bmatrix} M_1^* & M_2^* & M_3^* \\ M_3^* & M_0^* & -M_1^* \\ -M_2^* & M_1^* & M_0^* \end{bmatrix}^* \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix},$$

$$\widehat{M}_2 = \begin{bmatrix} M_0 & -M_1 & -M_3 \\ M_1 & M_0 & -M_2 \\ M_2 & -M_3 & M_1 \\ M_3 & M_2 & M_0 \end{bmatrix} \begin{bmatrix} M_1^* & M_2^* & M_3^* \\ M_0^* & -M_3^* & M_2^* \\ -M_2^* & M_1^* & M_0^* \end{bmatrix}^* \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix},$$

$$\widehat{M}_3 = \begin{bmatrix} M_0 & -M_1 & -M_2 \\ M_1 & M_0 & M_3 \\ M_2 & -M_3 & M_0 \\ M_3 & M_2 & -M_1 \end{bmatrix} \begin{bmatrix} M_1^* & M_2^* & M_3^* \\ M_0^* & -M_3^* & M_2^* \\ M_3^* & M_0^* & -M_1^* \end{bmatrix}^* \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix},$$

$$\widetilde{M}_0 = \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \widetilde{M}_1 = \begin{bmatrix} -M_1 \\ M_0 \\ -M_3 \\ M_2 \end{bmatrix}, \widetilde{M}_2 = \begin{bmatrix} -M_2 \\ M_3 \\ M_0 \\ -M_1 \end{bmatrix}, \widetilde{M}_3 = \begin{bmatrix} -M_3 \\ -M_2 \\ M_1 \\ M_0 \end{bmatrix},$$

and

$$\widetilde{M} = [M_0, -M_1, -M_2, -M_3].$$

Proof. According to Lemma 1, we have

$$\begin{aligned} & (M_0 + M_1i + M_2j + M_3k)^\dagger \\ &= \frac{1}{4} [I_n, I_n, I_n, I_n] \begin{bmatrix} M_0 & M_1i & M_2j & M_3k \\ M_1i & M_0 & M_3k & M_2j \\ M_2j & M_3k & M_0 & M_1i \\ M_3k & M_2j & M_1i & M_0 \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ I_m \\ I_m \\ I_m \end{bmatrix} \\ &= \frac{1}{4} [I_m, iI_m, jI_m, kI_m] \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix} \\ &= \frac{1}{4} [I_m, iI_m, jI_m, kI_m] \begin{bmatrix} G_0 & -G_1 & -G_2 & -G_3 \\ G_1 & G_0 & G_3 & -G_2 \\ G_2 & -G_3 & G_0 & G_1 \\ G_3 & G_2 & -G_1 & G_0 \end{bmatrix} \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix}. \end{aligned}$$

Obviously, G_0 can be written as

$$G_0 = [I_n, 0, 0, 0] \overline{M}^\dagger \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0 \end{bmatrix} = P \overline{M}^\dagger Q, \quad (2.4)$$

where

$$P = [I_m, 0, 0, 0], Q = \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then it follows by Lemma 2, Lemma 3, (1.4) and (2.4) we get

$$\begin{aligned} r(G_0) &= \begin{bmatrix} \overline{M^*MM^*} & \overline{M^*Q} \\ P\overline{M^*} & 0 \end{bmatrix} - r(\overline{M}) \\ &= \begin{bmatrix} \overline{MM^*M} & \overline{MP^*} \\ Q^*\overline{M} & 0 \end{bmatrix} - r(\overline{M}) \\ &= \begin{bmatrix} \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} & \overline{M^*} \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} \\ \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \end{bmatrix} & \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ 0 \end{bmatrix} \end{bmatrix} \\ &\quad - r(\overline{M}) \\ &= \begin{bmatrix} \begin{bmatrix} 0 & -M_1 & -M_2 & -M_3 \\ 0 & M_0 & M_3 & -M_2 \\ 0 & -M_3 & M_0 & M_1 \\ 0 & M_2 & -M_1 & M_0 \end{bmatrix} & \overline{M^*} \begin{bmatrix} 0 & 0 & 0 & 0 \\ M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} \\ \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \end{bmatrix} & \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ 0 \end{bmatrix} \end{bmatrix} \\ &\quad - r(\overline{M}) \\ &= \begin{bmatrix} \begin{bmatrix} -M_1 & -M_2 & -M_3 \\ M_0 & M_3 & -M_2 \\ -M_3 & M_0 & M_1 \\ M_2 & -M_1 & M_0 \end{bmatrix} & \overline{M^*} \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} \\ \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3 \end{bmatrix} & \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ 0 \end{bmatrix} \end{bmatrix} - r(\overline{M}) \end{aligned}$$

which is the first equality in (2.2). The other equalities (2.2) and (2.3) can also be derived by the similar approach. \square

If $M_0 = 0$, then the result in (2.2) and (2.3) can be simplified to the following.

Corollary 2.6. *Let $M = M_1i + M_2j + M_3k$, and denote the Moore-Penrose inverse of M as $M^\dagger = G_0 + G_1i + G_2j + G_3k$,*

$$\widetilde{\overline{M}} = \begin{bmatrix} 0 & -M_1 & -M_2 & -M_3 \\ M_1 & 0 & M_3 & -M_2 \\ M_2 & -M_3 & 0 & M_1 \\ M_3 & M_2 & -M_1 & 0 \end{bmatrix},$$

Then

$$\begin{aligned} r(G_0) &= r \begin{bmatrix} A \\ C \end{bmatrix} + r[A, B] - r(\widetilde{M}), \\ r(G_1) &= r(C), \quad r(G_2) = r(B), \\ r(G_3) &= r \begin{bmatrix} AA^*A & AA^*B & B \\ CA^*A & CA^*B & 0 \\ C & 0 & 0 \end{bmatrix} - r(\widetilde{M}), \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & -M_1 & -M_2 \\ M_1 & 0 & M_3 \\ M_2 & -M_3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -M_3 \\ -M_2 \\ M_1 \end{bmatrix}, \quad C = [M_3, M_2, -M_1].$$

Let $M_2 = M_3 = 0$, we get a complex matrix $\widehat{M} = M_0 + M_1i$. As a special case of Theorem 2.1, we have the following corollary.

Corollary 2.7. *Suppose that $\widehat{M} = M_0 + M_1i$ and $\widehat{M}^\dagger = G_0 + G_1i$. Then the ranks of G_0, G_1 can be determined by the following formulas*

$$\begin{aligned} r(G_0) &= r \begin{bmatrix} \widehat{V}_0 & V_0 \\ W & 0 \end{bmatrix} - r \begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix}, \\ r(G_1) &= r \begin{bmatrix} \widehat{V}_1 & V_1 \\ W & 0 \end{bmatrix} - r \begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix}, \end{aligned}$$

where

$$V_0 = \begin{bmatrix} -M_1 \\ M_0 \end{bmatrix}, \quad \widehat{V}_0 = \begin{bmatrix} -M_1 \\ M_0 \end{bmatrix} M_0^* [M_1, M_0],$$

$$V_1 = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix}, \quad \widehat{V}_1 = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix} M_1^* [M_1, M_0], \quad W = [M_0, -M_1].$$

Now we give a group of rank inequalities derived from (2.2) and (2.3).

Corollary 2.8. *Let M, \overline{M} and M^\dagger be given by (1.1), (1.2) and (1.3). Then the ranks of G_0 in M^\dagger satisfies the rank inequalities*

$$\begin{aligned} r(G_0) \leq & r \begin{bmatrix} M_0 & -M_3 & M_2 \\ M_3 & M_0 & -M_1 \\ -M_2 & M_1 & M_0 \end{bmatrix} + r[M_0, -M_1, -M_2, -M_3] \\ & + r \begin{bmatrix} -M_3 \\ -M_2 \\ M_1 \\ M_0 \end{bmatrix} - r(\overline{M}), \end{aligned} \quad (2.5)$$

$$r(G_0) \geq r[M_0, -M_1, -M_2, -M_3] + r \begin{bmatrix} -M_3 \\ -M_2 \\ M_1 \\ M_0 \end{bmatrix} - r(\overline{M}), \quad (2.6)$$

$$\begin{aligned} r(G_0) \geq & r \begin{bmatrix} M_0 & -M_3 & M_2 \\ M_3 & M_0 & -M_1 \\ -M_2 & M_1 & M_0 \end{bmatrix} - r \begin{bmatrix} M_1 & M_0 & M_3 & -M_2 \\ M_2 & -M_3 & M_0 & M_1 \\ M_3 & M_2 & -M_1 & M_0 \end{bmatrix} \\ & - r \begin{bmatrix} -M_1 & -M_2 & -M_3 \\ M_0 & M_3 & -M_2 \\ -M_3 & M_0 & M_1 \\ M_2 & -M_1 & M_0 \end{bmatrix} + r(\overline{M}). \end{aligned} \quad (2.7)$$

Proof. It is clearly that

$$r(\widetilde{M}_0) + r(\widetilde{M}) \leq r \begin{bmatrix} \widetilde{M}_0 & \widetilde{M}_0 \\ \widetilde{M} & 0 \end{bmatrix} \leq r \begin{bmatrix} M_0^* & -M_3^* & M_2^* \\ M_3^* & M_0^* & -M_1^* \\ -M_2^* & M_1^* & M_0^* \end{bmatrix}^* + r(\widetilde{M}) + r(\widetilde{M}_0),$$

where $\widetilde{M}_0, \widetilde{M}$ and \widetilde{M} are defined as same as Theorem 2.1.

Putting them in the first rank equality in (2.2), we obtain (2.5) and (2.6). To show (2.7), we need the following rank equality

$$r(CA^\dagger B) \geq r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r \begin{bmatrix} A \\ C \end{bmatrix} - r[A, B] + r(A),$$

Now applying above inequality to $P\overline{M}^\dagger Q$ in (2.4), we have

$$r(G_0) = r(P\overline{M}^\dagger Q) \geq r \begin{bmatrix} \overline{M} & Q \\ P & 0 \end{bmatrix} - r \begin{bmatrix} \overline{M} \\ P \end{bmatrix} - r[\overline{M}, Q] + r(\overline{M}),$$

which is (2.7). \square

Rank inequalities for the G_1, G_2 and G_3 in M^\dagger can also be derived in the similar way shown above. We omit them here for simplicity.

Using the result of Theorem 2.1 and Corollary 2.2, we give a necessary and sufficient condition for an arbitrary quaternion matrix M to have a pure real or pure imaginary Moore-Penrose inverse. As a special case, a necessary and sufficient condition for an arbitrary complex matrix to have a pure real or pure imaginary Moore-Penrose inverse is also presented.

Theorem 2.9. Let M, \overline{M} and M^\dagger be given by (1.1), (1.2) and (1.3). Then
(a) the Moore-Penrose inverse of M is a pure real matrix if and only if

$$r(\overline{M}) = r \begin{bmatrix} \widehat{M}_1 & M_1 \\ M & 0 \end{bmatrix} = r \begin{bmatrix} \widehat{M}_2 & M_2 \\ M & 0 \end{bmatrix} = r \begin{bmatrix} \widehat{M}_3 & M_3 \\ M & 0 \end{bmatrix},$$

(b) the Moore-Penrose inverse of M is a pure imaginary matrix if and only if

$$r \begin{bmatrix} \widehat{M}_0 & M_0 \\ M & 0 \end{bmatrix} = r(\overline{M})$$

where M, \widehat{M}_i and M_i ($i = 0, 1, 2, 3$) are defined as Theorem 2.1.

Proof. From Theorem 2.1, the Moore-Penrose inverse of M is a pure real matrix if and only if

$$r(G_1) = r(G_2) = r(G_3) = 0.$$

That is

$$r \begin{bmatrix} \widehat{M}_1 & M_1 \\ M & 0 \end{bmatrix} - r(\overline{M}) = 0, r \begin{bmatrix} \widehat{M}_2 & M_2 \\ M & 0 \end{bmatrix} - r(\overline{M}) = 0, r \begin{bmatrix} \widehat{M}_3 & M_3 \\ M & 0 \end{bmatrix} - r(\overline{M}) = 0.$$

Thus we have part (a). By the same manner, we can get part (b). \square

Corollary 2.10. Suppose that $\widehat{M} = M_0 + M_1i$ and $\widehat{M}^\dagger = G_0 + G_1i$. Then
(a) the Moore-Penrose inverse of \widehat{M} is a pure real matrix if and only if

$$r \begin{bmatrix} \widehat{V}_0 & V_0 \\ W & 0 \end{bmatrix} = r \begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix}$$

(b) the Moore-Penrose inverse of \widehat{M} is a pure imaginary matrix if and only if

$$r \begin{bmatrix} \widehat{V}_1 & V_1 \\ W & 0 \end{bmatrix} = r \begin{bmatrix} M_0 & -M_1 \\ M_1 & M_0 \end{bmatrix},$$

where

$$V_0 = \begin{bmatrix} -M_1 \\ M_0 \end{bmatrix}, \widehat{V}_0 = \begin{bmatrix} -M_1 \\ M_0 \end{bmatrix} M_0^* [M_1, M_0],$$

and

$$V_1 = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix}, \widehat{V}_1 = \begin{bmatrix} M_0 \\ M_1 \end{bmatrix} M_1^* [M_1, M_0], W = [M_0, -M_1].$$

3. RANK EQUALITY FOR D_i ($i = 0, 1, 2, 3$) IN N^D

In this section, we derive the formulas of rank equalities of four real matrices D_0, D_1, D_2 and D_3 in $N^D = D_0 + D_1i + D_2j + D_3k$. Moreover, we established the necessary and sufficient conditions for N have a pure real or pure imaginary Drazin inverse.

Theorem 3.1. Let N, \overline{N} and N^D be given by (1.1), (1.2) and (1.3) with $\text{Ind}M \geq 1$. Then the ranks of in (1.3) can be determined by the following formulas

$$r(D_0) = r \begin{bmatrix} \overline{N}^k \widehat{N}_0 \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widetilde{N}_0 \overline{N}^k & 0 \end{bmatrix} - r(\overline{N}^k), \quad (3.1)$$

$$r(D_1) = r \begin{bmatrix} \overline{N}^k \widehat{N}_1 \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widetilde{N}_1 \overline{N}^k & 0 \end{bmatrix} - r(\overline{N}^k), \quad (3.2)$$

$$r(D_2) = r \begin{bmatrix} \overline{N}^k \widehat{N}_2 \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widetilde{N}_2 \overline{N}^k & 0 \end{bmatrix} - r(\overline{N}^k), \quad (3.3)$$

$$r(D_3) = r \begin{bmatrix} \overline{N}^k \widehat{N}_3 \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widetilde{N}_3 \overline{N}^k & 0 \end{bmatrix} - r(\overline{N}^k), \quad (3.4)$$

where

$$\widetilde{N} = \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \widehat{N}_0 = \begin{bmatrix} N_0 & 0 & 0 & 0 \\ 0 & N_0 & N_3 & -N_2 \\ 0 & -N_3 & N_0 & N_1 \\ 0 & N_2 & -N_1 & N_0 \end{bmatrix}, \widetilde{N}_0 = [N_0, -N_1, -N_2, -N_3],$$

$$\widehat{N}_1 = \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ 0 & N_0 & 0 & 0 \\ N_2 & -N_3 & N_0 & N_1 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}, \widetilde{N}_1 = [N_1 \quad N_0 \quad N_3 \quad -N_2],$$

$$\widehat{N}_2 = \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ N_1 & N_0 & N_3 & -N_2 \\ 0 & -N_3 & 0 & 0 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}, \widetilde{N}_2 = [N_2 \quad -N_3 \quad N_0 \quad N_1],$$

and

$$\widehat{N}_3 = \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ N_1 & N_0 & N_3 & -N_2 \\ 0 & -N_3 & 0 & 0 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}, \widetilde{N}_3 = [N_3 \quad N_2 \quad -N_1 \quad N_0].$$

Proof. According to Lemma 1, we have

$$\begin{aligned}
& (N_0 + N_1i + N_2j + N_3k)^D \\
&= \frac{1}{4} [I_m, I_m, I_m, I_m] \begin{bmatrix} N_0 & N_1i & N_2j & N_3k \\ N_1i & N_0 & N_3k & N_2j \\ N_2j & N_3k & N_0 & N_1i \\ N_3k & N_2j & N_1i & N_0 \end{bmatrix}^D \begin{bmatrix} I_m \\ I_m \\ I_m \\ I_m \end{bmatrix} \\
&= \frac{1}{4} [I_m, iI_m, jI_m, kI_m] \begin{bmatrix} N_0 & -N_1 & -N_2 & -N_3 \\ N_1 & N_0 & N_3 & -N_2 \\ N_2 & -N_3 & N_0 & N_1 \\ N_3 & N_2 & -N_1 & N_0 \end{bmatrix}^D \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix} \\
&= \frac{1}{4} [I_m, iI_m, jI_m, kI_m] \begin{bmatrix} D_0 & -D_1 & -D_2 & -D_3 \\ D_1 & D_0 & D_3 & -D_2 \\ D_2 & -D_3 & D_0 & D_1 \\ D_3 & D_2 & -D_1 & D_0 \end{bmatrix} \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix}.
\end{aligned}$$

Obviously, G_0 can be written as

$$G_0 = [I_m, 0, 0, 0] \bar{N}^D \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0 \end{bmatrix} = P\bar{N}^D Q = P\bar{N}^k (\bar{N}^{2k+1})^D \bar{N}^k Q,$$

where $P = [I_m, 0, 0, 0]$ and $Q = \begin{bmatrix} I_m \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Then it follows by Lemma 2, Lemma 3, (1.4) and (2.4) we get

$$\begin{aligned}
r(D_0) &= \begin{bmatrix} (\bar{N}^{2k+1})^* \bar{N}^{2k+1} (\bar{N}^{2k+1})^* & (\bar{N}^{2k+1})^* \bar{N}^k Q \\ P\bar{N}^k (\bar{N}^{2k+1})^* & 0 \end{bmatrix} - r(\bar{N}^{2k+1}) \\
&= \begin{bmatrix} \bar{N}^{2k+1} & \bar{N}^k Q \\ P\bar{N}^k & 0 \end{bmatrix} - r(\bar{N}^{2k+1}) \\
&= \begin{bmatrix} \bar{N}^{2k+1} - \bar{N}^k Q P \bar{N} \bar{N}^k - \bar{N}^k \bar{N} Q P \bar{N}^k & \bar{N}^k Q \\ P\bar{N}^k & 0 \end{bmatrix} - r(\bar{N}^k) \\
&= \begin{bmatrix} \bar{N}^k (\bar{N} - Q P \bar{N} - \bar{N} Q P) \bar{N}^k & \bar{N}^k Q \\ P\bar{N}^k & 0 \end{bmatrix} - r(\bar{N}^k) \\
&= \begin{bmatrix} \bar{N}^k \begin{bmatrix} N_0 & 0 & 0 & 0 \\ 0 & N_0 & N_3 & -N_2 \\ 0 & -N_3 & N_0 & N_1 \\ 0 & N_2 & -N_1 & N_0 \end{bmatrix} \bar{N}^k & \bar{N}^k \bar{N}^{k-1} \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} \\ [N_0, -N_1, -N_2, -N_3] \bar{N}^k & 0 \end{bmatrix} - r(\bar{N}^k),
\end{aligned}$$

which is the equality in (3.1). The equalities (3.2-3.4) can also be derived by the similar approach. \square

Let $N_2 = N_3 = 0$, we get a complex matrix $\widehat{N} = N_0 + N_1i$. As a special case of Theorem 3.1, we have the following corollary.

Corollary 3.2. *Suppose that $\widehat{N} = N_0 + N_1i$ and $\widehat{N}^+ = D_0 + D_1i$. Then the ranks of D_0, D_1 can be determined by the following formulas*

$$r(D_0) = r \begin{bmatrix} \widetilde{W}^k \widehat{V} \widetilde{W}^k & \widetilde{W}^{k-1} V_1 \\ \widehat{V} \widetilde{W}^k & 0 \end{bmatrix} - r(\widetilde{W}^k),$$

$$r(D_1) = r \begin{bmatrix} \widetilde{W}^k \widetilde{V} \widetilde{W}^k & \widetilde{W}^{k-1} V_1 \\ \widetilde{V} \widetilde{W}^k & 0 \end{bmatrix} - r(\widetilde{W}^k),$$

where

$$V_1 = \begin{bmatrix} N_0 \\ N_1 \end{bmatrix}, \widehat{V} = \begin{bmatrix} N_0 & 0 \\ 0 & N_0 \end{bmatrix}, \widetilde{V} = \begin{bmatrix} 0 & -N_1 \\ N_1 & 0 \end{bmatrix},$$

$$\widetilde{W} = \begin{bmatrix} N_0 & -N_1 \\ N_1 & N_0 \end{bmatrix}, W_1 = [N_0, -N_1], W_2 = [N_1, N_0].$$

Using the result of Theorem 3.1 and Corollary 3.2, we give a necessary and sufficient condition for an arbitrary quaternion matrix N to have a pure real or pure imaginary Drazin inverse. As a special case, a necessary and sufficient condition for an arbitrary square complex matrix to have a pure real or pure imaginary Drazin inverse is also presented.

Theorem 3.3. *Let N, \overline{N} and N^D be given by (1.1), (1.2) and (1.3) with $\text{Ind}M \geq 1$. Then*

(a) *the Drazin inverse of N is a pure real matrix if and only if*

$$r(\overline{N}^k) = r \begin{bmatrix} \overline{N}^k \widehat{N}_1 \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widehat{N}_1 \overline{N}^k & 0 \end{bmatrix} = r \begin{bmatrix} \overline{N}^k \widehat{N}_2 \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widehat{N}_2 \overline{N}^k & 0 \end{bmatrix}$$

$$= r \begin{bmatrix} \overline{N}^k \widehat{N}_3 \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widehat{N}_3 \overline{N}^k & 0 \end{bmatrix},$$

(b) *the Drazin inverse of N is a pure imaginary matrix if and only if*

$$r \begin{bmatrix} \overline{N}^k \widehat{N}_0 \overline{N}^k & \overline{N}^{k-1} \widetilde{N} \\ \widehat{N}_0 \overline{N}^k & 0 \end{bmatrix} = r(\overline{N}^k),$$

where $\widetilde{N}, \widehat{N}_i$ and \widetilde{N}_i ($i = 0, 1, 2, 3$) are defined as Theorem 3.1.

Corollary 3.4. *Suppose that $\widehat{N} = N_0 + N_1i$ and $\widehat{N}^D = D_0 + D_1i$. Then*

(a) *the Drazin inverse of \widehat{N} is a pure real matrix if and only if*

$$r \begin{bmatrix} \widetilde{W}^k \widetilde{V} \widetilde{W}^k & \widetilde{W}^{k-1} V_1 \\ \widetilde{V} \widetilde{W}^k & 0 \end{bmatrix} = r(\widetilde{W}^k),$$

(b) the Drazin inverse of \widehat{N} is a pure imaginary matrix if and only if

$$r \begin{bmatrix} \widetilde{W}^k \widehat{V} \widetilde{W}^k & \widetilde{W}^{k-1} V_1 \\ \widehat{V} \widetilde{W}^k & 0 \end{bmatrix} = r(\widetilde{W}^k),$$

where

$$V_1 = \begin{bmatrix} N_0 \\ N_1 \end{bmatrix}, \widehat{V} = \begin{bmatrix} N_0 & 0 \\ 0 & N_0 \end{bmatrix}, \widetilde{V} = \begin{bmatrix} 0 & -N_1 \\ N_1 & 0 \end{bmatrix},$$

$$\widetilde{W} = \begin{bmatrix} N_0 & -N_1 \\ N_1 & N_0 \end{bmatrix}, W_1 = [N_0, -N_1], W_2 = [N_1, N_0].$$

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