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ON STRONGLY h-CONVEX FUNCTIONS

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ABSTRACT. We introduce the notion of strongly h-convex functions (defined on a normed space) and present some properties and representations of such functions. We obtain a characterization of inner product spaces involving the notion of strongly h-convex functions. Finally, a Hermite–Hadamard–type inequality for strongly h-convex functions is given.

1. INTRODUCTION

Let I be an interval in $\mathbb{R}$ and $h : (0,1) \to (0,\infty)$ be a given function. Following Varošanec [17], a function $f : I \to \mathbb{R}$ is said to be *h-convex* if

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in (0,1)$. This notion unifies and generalizes the known classes of convex functions, s - convex functions, Godunova-Levin functions and P-functions, which are obtained by putting in (1.1) $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, respectively. Many properties of them can be found, for instance, in [1, 2, 4, 10, 13, 14, 17].

Recall also that a function $f : I \to \mathbb{R}$ is called *strongly convex with modulus* $c > 0$, if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

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for all $x, y \in I$ and $t \in (0,1)$. Strongly convex functions have been introduced by Polyak [12], and they play an important role in optimization theory and mathematical economics. Various properties and applications of them can be found in the literature (see, for instance, [6, 7, 9, 11, 15, 16] and the references therein).

In this paper we introduce the notion of strongly h-convex functions defined in normed spaces and present some examples and properties of them. In particular we obtain a representation of strongly h-convex functions in inner product spaces and, using the methods of [9], we give a characterization of inner product spaces, among normed spaces, that involves the notion of strongly h-convex function. Finally, a version of Hermite–Hadamard-type inequalities for strongly h-convex functions is presented. This result generalizes the Hermite–Hadamard-type inequalities obtained in [7] for strongly convex functions, and for $c = 0$, coincides with the classical Hermite–Hadamard inequalities, as well as the corresponding Hermite–Hadamard-type inequalities for h-convex functions, s-convex functions, Godunova-Levin functions and P-functions presented in [14, 3, 4], respectively.

2. Some basic properties and representations

In what follows $(X, \|\cdot\|)$ denotes a real normed space, D stands for a convex subset of X, $h : (0,1) \to (0,\infty)$ is a given function and c is a positive constant. We say that a function $f : D \to \mathbb{R}$ is *strongly h-convex with modulus c* if

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) - ct(1-t)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in D$ and $t \in (0,1)$. In particular, if f satisfies (2.1) with $h(t) = t$, $h(t) = t^s$ ($s \in (0,1)$), $h(t) = \frac{1}{t}$, and $h(t) = 1$, then f is said to be strongly convex, strongly s-convex, strongly Godunova-Levin functions and strongly P-function, respectively. The notion of h-convex function corresponds to the case $c = 0$. We start with two lemmas which give some relationships between strongly h-convex functions and h-convex functions in the case where X is a real inner product space (that is, the norm $\|\cdot\|$ is induced by an inner product: $\|x\|^2 :=< x \mid x >$).

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a real inner product space, D be a convex subset of X and $c > 0$. Assume that $h : (0,1) \to (0,\infty)$ satisfies the condition*

$$h(t) \ge t, \qquad t \in (0,1). \tag{2.2}$$

If $g : D \to \mathbb{R}$ is h-convex, then $f : D \to \mathbb{R}$ defined by $f(x) = g(x) + c\|x\|^2, \quad x \in D$ is strongly h-convex with modulus c.

Proof. Assume that g is h-convex. Then

$$\begin{aligned}
& f(tx + (1-t)y) \\
= \; & g(tx + (1-t)y) + c\|(tx + (1-t)y)\|^2 \\
\le \; & h(t)g(x) + h(1-t)g(y) + c\|(tx + (1-t)y)\|^2 \\
= \; & h(t)f(x) + h(1-t)f(y) - ch(t)\|x\|^2 - ch(1-t)\|y\|^2 + c\|(tx + (1-t)y)\|^2 \\
\le \; & h(t)f(x) + h(1-t)f(y) - ct\|x\|^2 - c(1-t)\|y\|^2 \\
+ \; & c(t^2\|x\|^2 + 2t(1-t) < x \mid y > + (1-t)^2\|y\|^2) \\
= \; & h(t)f(x) + h(1-t)f(y) - ct(1-t)\|x-y\|^2,
\end{aligned}$$

which shows that f is strongly h-convex with modulus c.

□

In a similar way we can prove the next lemma

Lemma 2.2. *let $(X, \|\cdot\|)$ be a real inner product space, D be a convex subset of X and $c > 0$. Assume that $h : (0, 1) \to (0, \infty)$ satisfies the condition*

$$h(t) \le t, \qquad t \in (0, 1).$$

If $f : D \to \mathbb{R}$ is strongly h-convex with modulus c, then there exists an h-convex function $g : D \to \mathbb{R}$ such that $f(x) = g(x) + c\|x\|^2$, where $x \in D$.

Remark 2.3. For strongly convex functions (i.e. if f satisfies (2.1) with $h(t) = t$, $t \in (0, 1)$) defined on a convex subset D of an inner product space X the following characterization holds (see [9, 16], cf. also [6, Prop 1.12] for the case $X = \mathbb{R}^n$): A function $f : D \to \mathbb{R}$ is strongly convex with modulus c if and only if $g = f - c\|\cdot\|^2$ is convex. This result follows also from Lemma 1 and Lemma 2 above. However, an analogous characterization is not true for arbitrary h.

Example 2.4. Let $h(t) := 1$, $t \in (0, 1)$. Then $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) := 1$, $x \in [-1, 1]$, is strongly h-convex with modulus $c = 1$. Indeed, for every $x, y \in [-1, 1]$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) = 1 \le 2 - t(1-t)(x-y)^2 = f(x) + f(y) - t(1-t)(x-y)^2.$$

However, $g(x) := f(x) - x^2$ is not h-convex. For instance,

$$g\left(\frac{1}{2}(-1) + \frac{1}{2}1\right) = 1 > 0 = g(-1) + g(1).$$

Now, let $h(t) := t^2$, $t \in (0, 1)$. Then $g : [-1, 1] \to \mathbb{R}$ given by $g(x) := 1$, $x \in [-1, 1]$, is h-convex, but $f(x) := g(x) + x^2$, $x \in [-1, 1]$, is not strongly h-convex with modulus 1. For instance,

$$f\left(\frac{1}{2}(-1) + \frac{1}{2}1\right) = 1 > 0 = \frac{1}{4}f(-1) + \frac{1}{4}f(1) - \frac{1}{4}(1+1)^2.$$

Remark 2.5. Condition (2.2) is satisfied, for instance, for the following functions defined in $(0, 1)$: $h_1(t) = t$, $h_2(t) = t^s$ $(s \in (0, 1))$, $h_3(t) = \frac{1}{t}$, $h_4(t) = 1$. Thus, if a function $g : I \to \mathbb{R}$ is convex, s-convex, a Godunova-Levin function or a P-function, then by Lemma 1, $f : I \to \mathbb{R}$ given by $f(x) = g(x) + cx^2$ is strongly h-convex with $h = h_i$, respectively.

Remark 2.6. We can easily check that if a function $g : D \to [0, \infty)$, defined on a convex subset D of a normed space X, is convex then it is h-convex with any $h : (0, 1) \to (0, \infty)$ satisfying (2.2). Therefore, if X is an inner product space then, by Lemma 1, $f : D \to [0, \infty)$ given by $f(x) = g(x) + c\|x\|^2$ is strongly h-convex.

3. A characterization of inner product spaces via strong h-convexity

The assumption that X is an inner product space in Lemma 1 is essential. Moreover, it appears that the fact that for every h-convex function $g : X \to \mathbb{R}$ the function $f = g + \|\cdot\|^2$ is strongly h-convex characterizes inner product spaces among normed spaces. Similar characterizations of inner product spaces by strongly convex and strongly midconvex functions are presented in [9].

Theorem 3.1. *Let* $(X, \|\cdot\|)$ *be a real normed space. Assume that* $h : (0,1) \to \mathbb{R}$ *satisfies (2.2) and* $h(\frac{1}{2}) = \frac{1}{2}$. *The following conditions are equivalent:*

1. $(X, \|\cdot\|)$ *is an inner product space;*
2. *For every* $c > 0$ *and for every* h*-convex function* $g : D \to \mathbb{R}$ *defined on a convex subset* D *of* X, *the function* $f = g + c\|\cdot\|^2$ *is strongly* h*-convex with modulus* c*;*
3. $\|\cdot\|^2 : X \to \mathbb{R}$ *is strongly* h*-convex with modulus* 1.

Proof. The implication $1 \Rightarrow 2$ follows be Lemma 1.
To see that $2 \Rightarrow 3$ take $g = 0$. Clearly, g is h-convex, whence $f = c\|\cdot\|^2$ is strongly h-convex with modulus c. Consequently, $\|\cdot\|^2$ is strongly h-convex with modulus 1.
To prove $3 \Rightarrow 1$ observe that by the strong h-convexity of $\|\cdot\|^2$ and the assumption $h(\frac{1}{2}) = \frac{1}{2}$, we have

$$\left\|\frac{x+y}{2}\right\|^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}\|x-y\|^2$$

and hence

$$\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \tag{3.1}$$

for all $x, y \in X$. Now, putting $u = x+y$ and $v = x-y$ in (3.1) we get

$$2\|u\|^2 + 2\|v\|^2 \leq \|u+v\|^2 + \|u-v\|^2 \tag{3.2}$$

for all $u, v \in X$

Conditions (3.1) and (3.2) mean that the norm $\|\cdot\|$ satisfies the parallelogram law, which implies, by the classical Jordan-Von Neumann theorem, that $(X, \|\cdot\|)$ is an inner product space.

□

4. Hermite–Hadamard-type Inequalities

It is known that if a function $f : I \to \mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$, $a < b$. These classical Hermite–Hadamard inequalities play an important role in convex analysis and there is an extensive literature dealing with its applications, various generalizations and refinements (see for instance [5, 8], and the references therein). The following result is a counterpart of the Hermite–Hadamard inequalities for strongly h-convex functions.

Theorem 4.1. *let* $h:(0,1)\to(0,\infty)$ *be a given function. If a function* $f:I\to\mathbb{R}$ *is Lebesgue integrable and strongly* h*-convex with modulus* $c>0$*, then*

$$\frac{1}{2h(\frac{1}{2})}\Big[f\big(\frac{a+b}{2}\big)+\frac{c}{12}(b-a)^2\Big]\le\frac{1}{b-a}\int_a^b f(x)dx$$
$$\le(f(a)+f(b))\int_0^1 h(t)dt-\frac{c}{6}(b-a)^2 \qquad (4.1)$$

for all $a,b\in I$*,* $a<b$

Proof. Fix $a,b\in I$, $a<b$, and take $u=ta+(1-t)b$, $v=(1-t)a+tb$. Then, the strong h-convexity of f implies

$$\begin{aligned}
f(\frac{a+b}{2}) &= f(\frac{u+v}{2})\\
&\le h(\frac{1}{2})f(u)+h(\frac{1}{2})f(v)-\frac{c}{4}(u-v)^2\\
&= h(\frac{1}{2})[f(ta+(1-t)b)+f((1-t)a+tb)]\\
&\quad -\frac{c}{4}((2t-1)a+(1-2t)b)^2.
\end{aligned}$$

Integrating the above inequality over the interval $(0,1)$, we obtain

$$\begin{aligned}
& f\Big(\frac{a+b}{2}\Big)\\
\le\ & h\Big(\frac{1}{2}\Big)\Bigg[\int_0^1 f(ta+(1-t)b)dt+\int_0^1 f((1-t)a+tb)dt\Bigg]\\
& -\frac{c}{4}\int_0^1\Big((2t-1)a+(1-2t)b\Big)^2dt\\
=\ & h\Big(\frac{1}{2}\Big)\frac{2}{b-a}\int_a^b f(x)dx-\frac{c}{12}(b-a)^2
\end{aligned}$$

which gives the left-hand side inequality of (4.1).

For the proof of the right-hand side inequality of (4.1) we use inequality (2). Integrating over the interval $(0,1)$, we get

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)dx &= \int_0^1 f((1-t)a+tb)dt\\
&\le f(a)\int_0^1 h(1-t)dt+f(b)\int_0^1 h(t)dt\\
&\quad -c(b-a)^2\int_0^1 t(1-t)dt\\
&= (f(a)+f(b))\int_0^1 h(t)dt-\frac{c}{6}(b-a)^2
\end{aligned}$$

which gives the right-hand side inequality of (4.1). □

Remark 4.2. (1) In the case $c = 0$ the Hermite–Hadamard-type inequalities (4.1) coincide with the Hermite–Hadamard-type inequalities for h-convex functions proved by Sarikaya, Saglam and Yildirim in [14].

(2) If $h(t) = t$, $t \in (0,1)$, then the inequalities (4.1) reduce to

$$f\Big(\frac{a+b}{2}\Big) + \frac{c}{12}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2.$$

These Hermite–Hadamard-type inequalities for strongly convex functions have been proved by Merentes and Nikodem in [7]. For $c = 0$ we get the classical Hermite–Hadamard inequalities.

(3) If $h(t) = t^s$, $t \in (0,1)$, then the inequalities (4.1) give

$$2^{s-1}\left[f\Big(\frac{a+b}{2}\Big) + \frac{c}{12}(b-a)^2\right] \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1} - \frac{c}{6}(b-a)^2.$$

For $c = 0$ it reduces to the Hermite–Hadamard-type inequalities for s-convex functions proved by Dragomir and Fitzpatrik [3].

(4) If $h(t) = \frac{1}{t}$, $t \in (0,1)$, then the inequalities (4.1) give

$$\frac{1}{4}f\Big(\frac{a+b}{2}\Big) + \frac{c}{48}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx \quad (\le +\infty).$$

The case $c = 0$ corresponds to the Hermite–Hadamard-type inequalities for Godunova–Levin functions obtained by Dragomir, Pečarić and Persson [4].

(5) If $h(t) = 1$, $t \in (0,1)$, then the inequalities (4.1) reduce to

$$\frac{1}{2}f\Big(\frac{a+b}{2}\Big) + \frac{c}{24}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx \le f(a) + f(b) - \frac{c}{6}(b-a)^2.$$

In the case $c = 0$ it gives the Hermite–Hadamard-type inequalities for P-convex functions proved by Dragomir, Pečarić and Persson in [4].

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