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MODIFIED α-BERNSTEIN OPERATORS WITH BETTER APPROXIMATION PROPERTIES

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Abstract. In the present note, following a new approach recently described by Khosravian-Arab, Dehghan, and Eslahchi, we construct a new kind of α-Bernstein operator and study a uniform convergence estimate for these operators. We also prove some direct results involving the asymptotic theorems. Finally, we illustrate the convergence of the operators to a certain function with the help of Maple software.

1. Introduction

Let $f : I \to \mathbb{R}$, with $I = [0, 1]$. A Bernstein operator is defined by

$$T_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in I, \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, 1, \ldots, n$, and $p_{n,k}(x) = 0$, if $k < 0$ or $k > n$.

It is known that the Bernstein fundamental polynomials verify that

$$p_{n,k}(x) = (1-x)\, p_{n-1,k}(x) + x p_{n-1,k-1}(x), \quad 0 < k < n. \tag{1.2}$$

Lots of interesting results have been studied by many researchers of Bernstein polynomials (see, e.g., [12], [10], [26], [13], [14], [11], [23], [5], [1], [7]).

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Numerous modifications in Bernstein polynomials—such as Bernstein–Stancu, Bernstein–Schurer, q-Bernstein polynomials, and (p,q)-Bernstein operators—have been made by several researchers, some representative examples of which are [21], [20], [3], [25], [18], [24], [9], [6], [4], and [22].

Chen, Tan, Liu, and Xie [8] considered a generalization of the Bernstein operators (1.1) involving any fixed real parameter $\alpha \in [0,1]$ as follows:

$$T_{n,\alpha}(f;x) = \sum_{k=0}^{n} p_{n,k,\alpha}(x) f\left(\frac{k}{n}\right), \quad x \in I, \tag{1.3}$$

where $p_{n,k,\alpha}(x) = \left(\binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha\, x(1-x)\right)x^{k-1}(1-x)^{n-k-1}$ and $n \geq 2$. It is known that

$$p_{n,k,\alpha}(x) = (1-x)\ p_{n-1,k,\alpha}(x) + x\ p_{n-1,k-1,\alpha}(x), \quad 0 < k < n. \tag{1.4}$$

These operators preserve the constant as well as linear functions. The authors of [8] studied the uniform convergence, Voronovskaya-type asymptotic formula, and shape-preserving properties for these operators. For $\alpha = 1$, (1.3) includes Bernstein operators (1.1). The present authors in [2] defined the bivariate extension of α-Bernstein operators and studied the degree of approximation for the associated GBS (generalized Boolean sum) operators, and in [16] introduced the Durrmeyer variant of the operators (1.3) and established Voronovskaya-type asymptotic formula, local, and global approximation properties. A Kantorovich variant of the α-Bernstein operators (1.3) was considered and studied in [19].

Khosravian-Arab, Dehghan, and Eslahchi in [17] introduced generalized Bernstein operators as follows:

$$\begin{aligned} B_n^{M,1}(f,x) &= \sum_{k=0}^{n} p_{n,k}^{M,1}(x) f\left(\frac{k}{n}\right), \quad x \in I,\\ p_{n,k}^{M,1}(x) &= a(x,n)p_{n-1,k}(x)\\ &\qquad + a(1-x,n)p_{n-1,k-1}(x), \quad 1 \leq k \leq n-1,\\ p_{n,0}^{M,1}(x) &= a(x,n)(1-x)^{n-1},\\ p_{n,n}^{M,1}(x) &= a(1-x,n)x^{n-1}. \end{aligned} \tag{1.5}$$

Here

$$a(x,n) = a_1(n)x + a_0(n), \quad n = 0,1,\ldots, \tag{1.6}$$

where $a_0(n)$ and $a_1(n)$ are two unknown sequences which are determined in an appropriate way. For $a_1(n) = -1$, $a_0(n) = 1$, obviously (1.5) reduces to (1.2).

Gupta, Tachev, and Acu [15] introduced a Kantorovich modification of the operators (1.5) and proved some direct estimates which involve the asymptotic-type theorems.

2. α-Bernstein operators of order I

Inspired by [17], for any $f \in C(I)$, we introduce the modified α-Bernstein operators, given by

$$T_{n,\alpha}^{M,1}(f;x)=\sum_{k=0}^{n} p_{n,k,\alpha}^{M,1}(x)\ f\Big(\frac{k}{n}\Big), \quad x\in I, \tag{2.1}$$

where

$$p_{n,k,\alpha}^{M,1}(x)=a(x,n)p_{n-1,k,\alpha}(x)+a(1-x,n)p_{n-1,k-1,\alpha}(x), \tag{2.2}$$

where $a(x,n)$ is defined as above.

The aim of the present paper is to present these new Bernstein operators (2.1) based on $\alpha \in [0,1]$, studying their uniform convergence and asymptotic behavior. Now, we compute some lemmas which will be useful in the future proofs of the main results. Let $e_i(x)=x^i$, $i\in\mathbb{N}\cup\{0\}$.

Lemma 2.1. *For the operators $T_{n,\alpha}^{M,1}(f;x)$, we have the following:*

(i) $T_{n,\alpha}^{M,1}(e_0;x)=2a_0(n)+a_1(n)$;

(ii) $T_{n,\alpha}^{M,1}(e_1;x)=x(2a_0(n)+a_1(n))+\frac{(1-2x)(a_0(n)+a_1(n))}{n}$;

(iii) $T_{n,\alpha}^{M,1}(e_2;x)=x^2(2a_0(n)+a_1(n))+\frac{x(2a_0(n)(2-3x)+a_1(n)(3-5x))}{n}$
$+\frac{(a_0(n)+a_1(n))-2x(1-x)(2\alpha a_0(n)+(1+\alpha)a_1(n))}{n^2}$;

(iv) $T_{n,\alpha}^{M,1}(e_3;x)=x^3(2a_0(n)+a_1(n))+\frac{3x^2a_0(n)(3-4x)+3x^2a_1(n)(2-3x)}{n}$
$+\frac{(8x-3x^2(5+4\alpha)+2x^3(6+6\alpha))a_0(n)+(7x-6x^2(3+\alpha)+2x^3(7+3\alpha))a_1(n)}{n^2}$
$+\frac{(1+2x(5-9\alpha)+18x^2(3\alpha-2)+6x^3(4-9\alpha))a_0(n)+(1+4x(1-3\alpha)+6x^2(1-2\alpha)(2x-3))a_1(n)}{n^3}$;

(v) $T_{n,\alpha}^{M,1}(e_4;x)=x^4(2a_0(n)+a_1(n))+\frac{4x^3(4-5x)a_0(n)+2x^3(5-7x)a_1(n)}{n}$
$+\frac{2x^2(16-12x(\alpha+3)+x^2(23+12\alpha))a_0(n)+x^2(25-2x(33+6\alpha)+x^2(47+12\alpha))a_1(n)}{n^2}$
$+\frac{4x(4-24x\alpha+22x^2(3\alpha-1)+x^3(17-42\alpha))a_0(n)}{n^3}$
$+\frac{x(15-x(29+60\alpha)+4x^2(42\alpha-1)+2x^3(7-54\alpha))a_1(n)}{n^3}$
$+\frac{(1+16x(3-4\alpha)+32x^2(11\alpha-9)+48x^3(5-6\alpha)(2+x))a_0(n)}{n^4}$
$+\frac{(1+2x(17-25\alpha)+2x^2(133\alpha-101)+24x^3(7-9\alpha)(2+x))a_1(n)}{n^4}$.

Lemma 2.2. *The computation of the central moments up to the second order for modified α-Bernstein type operators (2.1), is given by*

(i) $T_{n,\alpha}^{M,1}(t-x;x)=\frac{(1-2x)(a_0(n)+a_1(n))}{n}$;

(ii) $T_{n,\alpha}^{M,1}((t-x)^2;x)=\frac{x(2a_0(n)(2-3x)+a_1(n)(3-5x))-2x(1-2x)(a_0(n)+a_1(n))}{n}$
$+\frac{(a_0(n)+a_1(n))-2x(1-x)(2\alpha a_0(n)+(1+\alpha)a_1(n))}{n^2}$;

(iii) $T_{n,\alpha}^{M,1}((t-x)^4;x)=\frac{3x^2(1-x)^2(2a_0(n)+a_1(n))}{n^2}$
$+\frac{x(1-x)(12a_0(n)+11a_1(n))-2x^2(1-x)^2(2a_0(n)(7+6\alpha)+(17+6\alpha)a_1(n))}{n^3}$
$+\frac{(a_0(n)+a_1(n))+x(1-x)(16(4\alpha-3)a_0(n)+2(17-25\alpha)a_1(n))}{n^4}$
$+\frac{3x^2(1-x)^2((5-6\alpha)a_0(n)+8(9\alpha-7)a_1(n))}{n^4}$.

Note that throughout the paper we will assume that the sequences $a_i(n), i=0,1$, verify the condition

$$2a_0(n) + a_1(n) = 1. \tag{2.3}$$

This assumption on the sequences $a_i(n), i = 0, 1$, was made in order to study the uniform convergence. We consider the following two cases for unknown sequences $a_0(n)$ and $a_1(n)$:

Case 1. Let

$$a_0(n) \geq 0, \qquad a_0(n) + a_1(n) \geq 0. \tag{2.4}$$

Using condition (2.3), we get $0 \leq a_0(n) \leq 1$ and $-1 \leq a_1(n) \leq 1$; namely, the sequences are bounded. In this case the operator (2.1) is positive.

Case 2. Let

$$a_0(n) < 0, \qquad \text{or} \qquad a_0(n) + a_1(n) < 0. \tag{2.5}$$

If $a_0(n) < 0$, then $a_0(n) + a_1(n) > 1$, and if $a_0(n) + a_1(n) < 0$, then $a_1(n) > 1$. In this case the operator (2.1) is not positive.

Theorem 2.3. *Let $f \in C(I)$. If $a_1(n)$, $a_0(n)$ verify conditions (2.3) and (2.4), then*

$$\lim_{n\to\infty} T^{M,1}_{n,\alpha}(f; x) = f(x),$$

uniformly on I.

Proof. Since the sequences $a_1(n)$, $a_0(n)$ satisfy conditions (2.3) and (2.4), it follows that these sequences are bounded. Using the well-known Korovkin theorem and Lemma 2.1, the uniform convergence of the operators $T^{M,1}_{n,\alpha}$ is proved. □

The above result can be extended for Case 2. In order to prove this result, we recall the extended form of the Korovkin theorem.

Theorem 2.4 ([17, Theorem 10])**.** *Let $0 < h \in C[a, b]$ be a function, and suppose that $(L_n)_{n\geq 1}$ is a sequence of positive linear operators such that $\lim_{n\to\infty} L_n(e_i) = he_i$, $i = 0, 1, 2$, uniformly on $[a, b]$. Then for a given function $f \in C[a, b]$, we have $\lim_{n\to\infty} L_n(f) = hf$ uniformly on $[a, b]$.*

Theorem 2.5. *If $f \in C(I)$, then for all convergence sequences $a_0(n)$ and $a_1(n)$ that verify conditions (2.3) and (2.5), we have*

$$\lim_{n\to\infty} T^{M,1}_{n,\alpha}(f; x) = f(x),$$

uniformly on I.

Proof. Let $a_0(n) < 0$ and $a_1(n) > 1$. Suppose that

$$F^{M,1}_{n,1,\alpha} = \sum_{k=0}^{n} \left[\big(a_1(n)x\big)p_{n-1,k,\alpha}(x) + a_1(n)p_{n-1,k-1,\alpha}(x)\right] f\left(\frac{k}{n}\right)$$

and

$$F^{M,1}_{n,2,\alpha} = \sum_{k=0}^{n} \left[-\big(a_0(n)\big)p_{n-1,k,\alpha}(x) + \big(a_1(n)x - a_0(n)\big)p_{n-1,k-1,\alpha}(x)\right] f\left(\frac{k}{n}\right).$$

It is easy to verify that the operators $F^{M,1}_{n,1,\alpha}$ and $F^{M,1}_{n,2,\alpha}$ are positive; moreover, we can check that

$$T^{M,1}_{n,\alpha} = F^{M,1}_{n,1,\alpha} - F^{M,1}_{n,2,\alpha}.$$

The moments of the operators $F_{n,1,\alpha}^{M,1}$ and $F_{n,2,\alpha}^{M,1}$ are calculated below:

$$F_{n,1,\alpha}^{M,1}(e_0;x) = a_1(n)(1+x),$$
$$F_{n,1,\alpha}^{M,1}(e_1;x) = \frac{x(1+x)(n-1)a_1(n)}{n} + \frac{a_1(n)}{n}, \tag{2.6}$$
$$F_{n,1,\alpha}^{M,1}(e_2;x) = \frac{x^3(n(n-3)+2\alpha)a_1(n)}{n^2} + \frac{x^2(n-1)^2a_1(n)}{n^2} + \frac{x(3n-2\alpha-1)a_1(n)}{n^2} + \frac{a_1(n)}{n^2}, \tag{2.7}$$

and

$$F_{n,2,\alpha}^{M,1}(e_0;x) = a_1(n)x - 2a_0(n), \tag{2.8}$$
$$F_{n,1,\alpha}^{M,1}(e_1;x) = \frac{x(n-1)(a_1(n)x-2a_0(n))}{n} + \frac{(a_1(n)x-a_0(n))}{n}, \tag{2.9}$$
$$F_{n,1,\alpha}^{M,1}(e_2;x) = x^2\big(a_1(n)x-2a_0(n)\big) + \frac{(6x^2a_0(n)+3x^2a_1(n)-3x^3a_(n)-4xa_0(n))}{n} + \frac{(2x^3\alpha a_1(n)-2x^2\alpha a_1(n)+xa_1(n)+4x(1-x)\alpha a_0(n)-a_0(n))}{n^2}. \tag{2.10}$$

Applying Theorem 2.4 and (2.6)–(2.10), we get

$$\lim_{n\to\infty} F_{n,1,\alpha}^{M,1}(f;x) = p_1(1+x)f(x),$$
$$\lim_{n\to\infty} F_{n,2,\alpha}^{M,1}(f;x) = \big(p_1(1+x)-1\big)f(x),$$

where $p_1 = \lim_{n\to\infty} a_1(n)$. Therefore, $T_{n,\alpha}^{M,1}(f;x) = f(x)$. □

Now we present a Voronovskaya-type asymptotic formula for the operators $T_{n,\alpha}^{M,1}$.

Theorem 2.6. *Let $a_i(n)$, $i=0,1$, be convergent sequences that satisfy conditions (2.3) and (2.4), and let $p_i = \lim_{n\to\infty} a_i(n)$, $i=0,1$. If $f'' \in C(I)$, then*

$$\lim_{n\to\infty} n\big(T_{n,\alpha}^{M,1}(f;x) - f(x)\big) = \frac{x(4p_0+3p_1-x(6p_0+5p_1))}{2}f''(x), \tag{2.11}$$

uniformly on I.

Proof. From Taylor's formula, we can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2f''(x) + \zeta(t,x)(t-x)^2, \tag{2.12}$$

where $\zeta(t,x) \to 0$ as $t \to x$ and is a continuous function on I.

Applying modified α-Bernstein operators $T_{n,\alpha}^{M,1}$ to (2.12), we obtain

$$T_{n,\alpha}^{M,1}(f;x) - f(x) = f'(x)T_{n,\alpha}^{M,1}\big((t-x);x\big) + \frac{1}{2}f''(x)T_{n,\alpha}^{M,1}\big((t-x)^2;x\big) + T_{n,\alpha}^{M,1}\big(\zeta(t,x)(t-x)^2;x\big).$$

Applying the Cauchy–Schwarz inequality, we get

$$nT_{n,\alpha}^{M,1}\big(\zeta(t,x)(t-x)^2;x\big) \le \sqrt{T_{n,\alpha}^{M,1}\big(\zeta^2(t,x);x\big)}\sqrt{n^2T_{n,\alpha}^{M,1}\big((t-x)^4;x\big)}.$$

Using Lemma 2.2, we get

$$\lim_{n\to\infty} n^2T_{n,\alpha}^{M,1}\big((t-x)^4;x\big) = 3x^2(1-x)^2.$$

Since $\zeta^2(x,x)=0$ and $\zeta^2(\cdot,x)\in C(I)$, by Theorem 2.3, we get

$$\lim_{n\to\infty} T_{n,\alpha}^{M,1}\big(\zeta(t,x);x\big) = 0,$$

uniformly with respect to $x\in I$. Therefore,

$$\lim_{n\to\infty} nT_{n,\alpha}^{M,1}\big(\zeta(t,x)(t-x)^2;x\big) = 0.$$

Applying the results from Lemma 2.2, the theorem is proved. □

Theorem 2.7. *Let $f\in C(I)$. If $a_0(n)$, $a_1(n)$ satisfy condition (2.3), then*

$$\big|T_{n,\alpha}^{M,1}(f;x)-f(x)\big| \le \big|T_{n,\alpha}(f;x)-f(x)\big| + \frac{3}{2}\Big|\big(1+a_1(n)\big)\Big(\frac{1}{2}-x\Big)\Big|\omega_1\Big(f;\frac{\sqrt{n+2(1-\alpha)}}{n}\Big).$$

Proof. We have

$$\big|T_{n,\alpha}^{M,1}(f;x)-f(x)\big| \le \big|T_{n,\alpha}(f;x)-f(x)\big| + \big|T_{n,\alpha}^{M,1}(f;x)-T_{n,\alpha}(f;x)\big|. \tag{2.13}$$

In the following, we will give an estimate of the quantity $|T_{n,\alpha}^{M,1}(f;x)-T_{n,\alpha}(f;x)|$. So,

$$\begin{aligned}
&T_{n,\alpha}^{M,1}(f;x)-T_{n,\alpha}(f;x)\\
&\quad=\sum_{k=0}^{n}\big\{a(x,n)p_{n-1,k,\alpha}(x)+a(1-x,n)p_{n-1,k-1,\alpha}(x)\big\}f\Big(\frac{k}{n}\Big)\\
&\qquad-\sum_{k=0}^{n}\big\{(1-x)p_{n-1,k,\alpha}(x)+xp_{n-1,k-1,\alpha}(x)\big\}f\Big(\frac{k}{n}\Big)\\
&\quad=\sum_{k=0}^{n-1}\big\{\big(a_1(n)+1\big)x+a_0(n)-1\big\}p_{n-1,k,\alpha}(x)f\Big(\frac{k}{n}\Big)\\
&\qquad+\sum_{k=1}^{n}\big\{\big(-1-a_1(n)\big)x+a_1(n)+a_0(n)\big\}p_{n-1,k-1,\alpha}(x)f\Big(\frac{k}{n}\Big)\\
&\quad=\sum_{k=0}^{n-1}\big\{\big(a_1(n)+1\big)x+a_0(n)-1\big\}p_{n-1,k,\alpha}(x)f\Big(\frac{k}{n}\Big)\\
&\qquad+\sum_{k=0}^{n-1}\big\{(-\big(1+a_1(n)\big)x+a_1(n)+a_0(n)\big\}p_{n-1,k,\alpha}(x)f\Big(\frac{k+1}{n}\Big)\\
&\quad=\sum_{k=0}^{n-1}\Big[f\Big(\frac{k+1}{n}\Big)-f\Big(\frac{k}{n}\Big)\Big]\big\{-\big(1+a_1(n)\big)x+a_0(n)+a_1(n)\big\}p_{n-1,k,\alpha}(x).
\end{aligned}$$

From [8, Theorem 3.4], we get

$$\begin{aligned}&\left|T^{M,1}_{n,\alpha}(f;x)-T_{n,\alpha}(f;x)\right|\\&\quad\le\frac{3}{2}\left|-\big(1+a_1(n)\big)x+a_0(n)+a_1(n)\right|\omega_1\Big(f;\frac{\sqrt{n+2(1-\alpha)}}{n}\Big)\\&\quad=\frac{3}{2}\Big|\big(1+a_1(n)\big)\Big(\frac{1}{2}-x\Big)\Big|\omega_1\Big(f;\frac{\sqrt{n+2(1-\alpha)}}{n}\Big),\end{aligned}$$

and after replacing this estimate in (2.13) the proof is complete. □

Remark 2.8. We note the two following conditions.

(i) For $a_1(n)=-1$, all the estimates for α-Bernstein operator $T_{n,\alpha}$ hold.

(ii) If $a_1(n)$ is bounded, say, $|a_1(n)|\le A_1$, then

$$\left|T^{M,1}_{n,\alpha}(f;x)-f(x)\right|\le\left|T_{n,\alpha}(f;x)-f(x)\right|+\frac{1}{2}(1+A_1)\omega_1\Big(f;\frac{\sqrt{n+2(1-\alpha)}}{n}\Big).$$

Now, we extend the results from Theorem 2.6 when the operator $T_{n,\alpha}$ is non-positive, that is, when the sequences $a_1(n)$ and $a_0(n)$ satisfy (2.3) and (2.5).

Theorem 2.9. *Let $a_1(n)$ and $a_0(n)$ be convergent sequences which satisfy (2.3) and (2.5), and let $p_i=\lim_{n\to\infty}a_i(n)$, $i=0,1$. If f is differentiable in a neighborhood of x and possesses a second derivative $f''(x)$, then*

$$\lim_{n\to\infty}n\big(T^{M,1}_{n,\alpha}(f;x)-f(x)\big)=\frac{x(4p_0+3p_1-x(6p_0+5p_1))}{2}f''(x).\tag{2.14}$$

If $f''\in C(I)$, then the convergence is uniform.

Proof. We follow the same lines as in Theorem 2.6. Therefore, it remains to show that for this case we also have

$$\lim_{n\to\infty}nT^{M,1}_{n,\alpha}\big(\zeta(t,x)(t-x)^2;x\big)=0.$$

For $a=0,1$, we have

$$\begin{aligned}&\Big|\sum_{k=0}^{n-1}p_{n-1,k,\alpha}(x)\Big(\zeta\Big(\frac{k+a}{n}\Big)\Big(\frac{k+a}{n}-x\Big)^2\Big)\Big|\\&\quad=\Big|\Big(\sum_{|\frac{k+a}{n}-x|<\delta}+\sum_{|\frac{k+a}{n}-x|\ge\delta}\Big)p_{n-1,k,\alpha}(x)\Big(\zeta\Big(\frac{k+a}{n}\Big)\Big(\frac{k+a}{n}-x\Big)^2\Big)\Big|\\&\quad\le\epsilon\frac{1}{n}+\frac{M}{2n\delta^2},\end{aligned}$$

where $M=\sup_{0\le t\le 1}\zeta(t,x)(t-x)^2$. Then $\zeta(t,x)(t-x)^2\le\frac{M}{\delta^4}((\frac{k}{n}-x)^2)$. From the boundedness of the sequences $a_i(n), i=0,1$, we have

$$\begin{aligned}&\left|T^{M,1}_{n,\alpha}\big(\zeta(t,x)(t-x)^2;x\big)\right|\\&\quad\le\left|a_1(n)x+a_0(n)\right|\sum_{k=0}^{n-1}p_{n-1,k,\alpha}(x)\Big|\zeta\Big(\frac{k}{n}\Big)\Big(\frac{k+1}{n}-x\Big)^2\Big|\end{aligned}$$

$$+\left|a_1(n)(1-x)+a_0(n)\right|\sum_{k=0}^{n-1}p_{n-1,k,\alpha}(x)\left|\zeta\left(\frac{k}{n}\right)\left(\frac{k+1}{n}-x\right)^2\right|\leq\epsilon.$$

The proof of Theorem 2.9 is completed. □

3. α-Bernstein operators of order II

Now, we are ready to extend the previous results to obtain the second-order approximation formula

$$T_{n,\alpha}^{M,2}(f;x)=\sum_{k=0}^{n}p_{n,k,\alpha}^{M,2}(x)\ f\left(\frac{k}{n}\right),\quad x\in I,\tag{3.1}$$

where

$$\begin{aligned}p_{n,k,\alpha}^{M,2}(x)&=b(x,n)p_{n-2,k,\alpha}(x)\\&\quad+d(x,n)p_{n-2,k-1,\alpha}(x)+b(1-x,n)p_{n-2,k-2,\alpha}(x)\end{aligned}\tag{3.2}$$

and

$$b(x,n)=b_2(n)x^2+b_1(n)x+b_0(n),\qquad d(x,n)=d_0(n)x(1-x),\tag{3.3}$$

and where $b_i(n), i=0,1,2$ and $d_0(n)$ are unknown sequences to be determined later. For $b_2(n)=b_0(n)=1$, $b_1(n)=-2$, $d_0(n)=2$ operators (3.4) with (3.2) and (3.3) reduces to the α-Bernstein operators

$$T_{n,\alpha}^{M,2}(e_0;x)=\sum_{k=0}^{n}p_{n,k,\alpha}^{M,2}(x)=b(x,n)+d(x,n)+b(1-x,n).\tag{3.4}$$

We set $T_{n,\alpha}^{M,2}(e_0;x)=1$. Then this yields

$$d_0(n)=2b_2(n),\qquad b_2(n)+b_1(n)+2b_0(n)=1,\quad\text{as } n\to\infty.\tag{3.5}$$

Applying (3.5), we obtain

$$T_{n,\alpha}^{M,2}(e_1;x)=\frac{x(4b_0(n)+n-4)}{n}+\frac{2(1-b_0(n))}{n}.\tag{3.6}$$

If we set $b_0(n)=1$, then we get

$$\begin{aligned}T_{n,\alpha}^{M,2}(e_1;x)&=x,\qquad b_1(n)=-b_2(n)-1,\\d_0(n)&=-2\big(1+b_1(n)\big).\end{aligned}\tag{3.7}$$

$$\begin{aligned}T_{n,\alpha}^{M,2}(e_2;x)&=x^2+\frac{2x((2-\alpha)+x(\alpha-2)-b_2(n)+xb_2(n))}{n^2}\\&\quad+\frac{x(1-x)}{n}.\end{aligned}\tag{3.8}$$

In order to have a second-order approximation formula, we set $b_2(n)=\frac{n}{2}$. Then (3.8) reduces to

$$T_{n,\alpha}^{M,2}(e_2;x)=x^2+\frac{2x(1-x)(2-\alpha)}{n^2}.\tag{3.9}$$

We follow our analysis for the second case, that is, for the following case:

$$b_2(n)=\frac{n}{2},\qquad b_1(n)=-1-\frac{n}{2},\qquad b_0(n)=1,\qquad d_0(n)=n,\ (3.10)$$

$$T_{n,\alpha}^{M,2}(f;x)=\sum_{k=0}^{n}p_{n,k,\alpha}^{M,2}(x)f\Big(\frac{k}{n}\Big),\quad x\in I,\tag{3.11}$$

where

$$p_{n,k,\alpha}^{M,2}(x)=\Big(\frac{n}{2}x^2-\Big(1+\frac{n}{2}\Big)x+1\Big)p_{n-2,k,\alpha}(x)+nx(1-x)p_{n-2,k-1,\alpha}(x)$$
$$+\Big(\frac{n}{2}x^2-\frac{n}{2}x+x\Big)p_{n-2,k-2,\alpha}(x).\tag{3.12}$$

We note that the operator (3.11) with (3.12) is not positive and therefore does not fulfill the conjecture of Theorem 2.3. Next, we will state the first important result concerning the uniform convergence of the new operator (3.11) together with (3.12).

Lemma 3.1. *The moments of the operators (3.11) are given by the following:*

(i) $T_{n,\alpha}^{M,2}(e_0;x)=1$;

(ii) $T_{n,\alpha}^{M,2}(e_1;x)=x$;

(iii) $T_{n,\alpha}^{M,2}(e_2;x)=x^2+\frac{2x(1-x)(2-\alpha)}{n^2}$;

(iv) $T_{n,\alpha}^{M,2}(e_3;x)=x^3+\frac{6x(1-x)(2x-1)(\alpha-2)}{n^3}+\frac{2x(1-x)(-1+x(8-3\alpha))}{n^2}$;

(v) $T_{n,\alpha}^{M,2}(e_4;x)=x^4+\frac{2x(1-x)(7(\alpha-2)+12x(1-x)(-5+2\alpha))}{n^4}$
$+\frac{6x(1-x)(-1+x(15-4\alpha+x(8\alpha-23)))}{n^3}+\frac{x^2(1-x)(-11+x(12\alpha-43))}{n^2}$;

(vi) $T_{n,\alpha}^{M,2}(e_5;x)=x^5+\frac{30x(1-x)(2x-1)(-2+8x(1-x)+\alpha)}{n^5}$
$+\frac{2x(1-x)(-7+x(158-15\alpha-2x(261-30\alpha+8x(5\alpha-28))))}{n^4}$
$+\frac{30x^2(1-x)(-2+x(14-17x-2\alpha+4x\alpha))}{n^3}+\frac{x^3(1-x)(-7+x(19-4\alpha))}{n^2}$;

(vii) $T_{n,\alpha}^{M,2}(e_6;x)=x^6+\frac{2x(1-x)(31(\alpha-2)+60x(1-x)(5+(4-30x(1-x)\alpha)))}{n^6}$
$+\frac{30x(1-x)(-1-x(-27-8\alpha+x(127+8x(17x-28)+76\alpha+74(x-2)x\alpha)))}{n^5}$
$+\frac{x^2(1-x)(-239-x(-2371+5944x-4361x^2+30(-4+x(6+5x))\alpha))}{n^4}$
$+\frac{30x^3(1-x)(-10+x(49-4\alpha+x(8\alpha-50)))}{n^3}+\frac{5x^4(1-x)(-17+x(37-6\alpha))}{n^2}$.

Lemma 3.2. *The central moments of the operators (3.11) are given by the following:*

(i) $T_{n,\alpha}^{M,2}((t-x)^2;x)=\frac{2x(1-x)(2-\alpha)}{n^2}$;

(ii) $T_{n,\alpha}^{M,2}((t-x)^3;x)=\frac{2x(1-x)(2x-1)}{n^2}+\frac{6x(1-x)(2x-1)(\alpha-2)}{n^3}$;

(iii) $T_{n,\alpha}^{M,2}((t-x)^4;x)=-\frac{3x^2(1-x)^2}{n^2}+\frac{6x(1-x)(-1+7x(1-x))}{n^3}$
$+\frac{2x(1-x)(7(2-\alpha)+12x(1-x)(2\alpha-5))}{n^4}$;

(iv) $T_{n,\alpha}^{M,2}((t-x)^5;x)=\frac{30x^2(1-x)^2(2x-1)}{n^3}+\frac{2x(1-x)(2x-1)(7-x(1-x)(37+10\alpha))}{n^4}$
$+\frac{2x(1-x)(2x-1)(15\alpha-30(2x-1)^2)}{n^5}$;

(v) $T_{n,\alpha}^{M,2}((t-x)^6;x)=-\frac{x^3(1-x)^3}{n^3}+\frac{5x^2(1-x)^2(-31+2x(1-x)(9\alpha+74))}{n^4}$
$+\frac{30x(1-x)(-1+x(1-x)(15+14\alpha-2x(1-x)(20+27\alpha)))}{n^5}$
$+\frac{2x(1-x)(31(2-\alpha)-60x(1-x)(5+(4-30x(1-x))\alpha))}{n^6}$.

Theorem 3.3 (Voronovskaya's theorem)**.** *Let f be bounded in I, and let x be a point of I at which $f'''(x)$ exists. Then we have*

$$\lim_{n\to\infty} n^2\big(T_{n,\alpha}^{M,2}(f;x) - f(x)\big) = x(1-x)\Big((2-\alpha)f''(x) + \frac{(2x-1)}{3}f'''(x)\Big).$$

If $f'''(x) \in C(I)$, then the convergence is uniform.

Proof. Using Lemma 3.2, the proof of this theorem closely follows the ideas developed in ([17], Theorem 15), and hence we omit the proof. □

4. Numerical examples

Example 4.1. The convergence of the modified α-Bernstein operators is illustrated in Figure 1, where $f(x) = (x-1)^2\cos(2\pi x)$, $n = 5$, $\alpha = 0.9$, $a_0(n) = \frac{n}{2n+1}$, and $a_1(n) = \frac{1}{2n+1}$. We observe that the approximation by modified α-Bernstein operators $T_{n,\alpha}^{M,1}(f;x)$ and $T_{n,\alpha}^{M,2}(f;x)$ is better than that of α-Bernstein operator $T_{n,\alpha}(f;x)$.

Example 4.2. The comparison of the convergence of α-Bernstein operator $T_{n,\alpha}(f;x)$ and the modified α-Bernstein operators $T_{n,\alpha}^{M,1}(f;x)$, $T_{n,\alpha}^{M,2}(f;x)$ for $f(x) = (x-1)^2\sin(2\pi x)$, $\alpha = 0.9$, $n = 5$, $a_0(n) = \frac{n}{2n+1}$, and $a_1(n) = \frac{1}{2n+1}$ is illustrated in Figure 2.

Example 4.3. The comparison of the convergence of α-Bernstein operator $T_{n,\alpha}(f;x)$ and the modified α-Bernstein operators $T_{n,\alpha}^{M,1}(f;x)$, $T_{n,\alpha}^{M,2}(f;x)$ for $f(x) = (x-1)^4\cos(2\pi x)$, $\alpha = 0.9$, $n = 10$, $a_0(n) = 0$, and $a_1(n) = 1$ is illustrated in Figure 3.

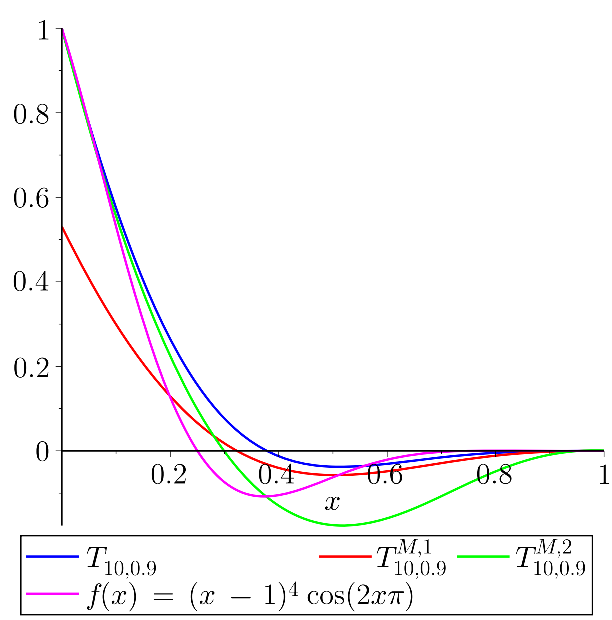

FIGURE 1. Approximation process.

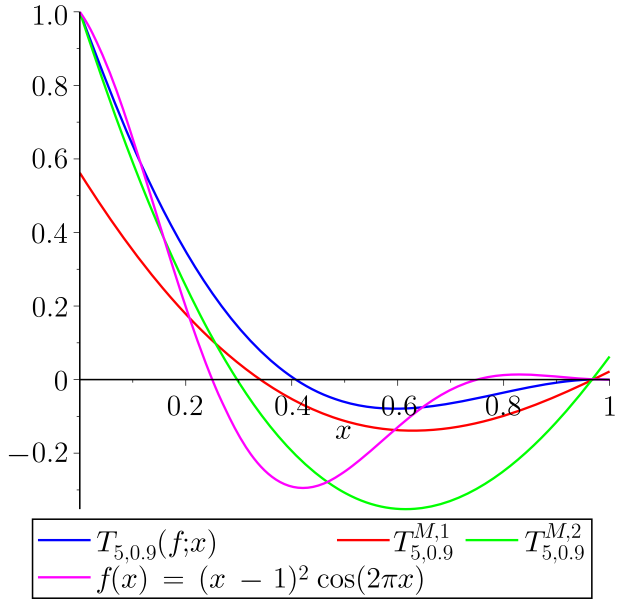

FIGURE 2. Approximation process.

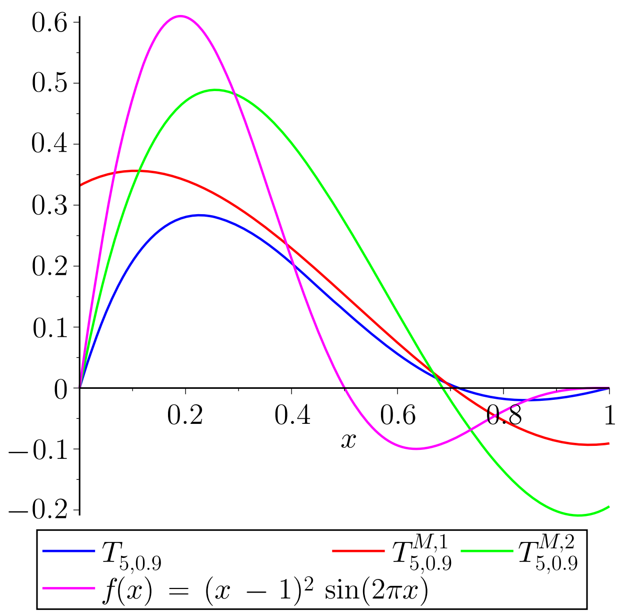

FIGURE 3. Approximation process.

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