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# GENERALIZED FRAMES FOR CONTROLLED OPERATORS IN HILBERT SPACES

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ABSTRACT. Controlled frames and g-frames were considered recently as generalizations of frames in Hilbert spaces. In this paper we generalize some of the known results in frame theory to controlled g-frames. We obtain some new properties of controlled g-frames and obtain new controlled g-frames by considering controlled g-frames for its components. And we also find some new resolutions of the identity. Furthermore, we study the stabilities of controlled g-frames under small perturbations.

## 1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [9] in order to study problems in nonharmonic Fourier series, and they were widely studied after the great 1986 work by Daubechies, Grossmann, and Meyer [8]. Today, frame theory has broad applications in pure mathematics, such as for the Kadison–Singer problem and statistics (see, e.g., [5], [10]), as well as in applied mathematics (see, e.g., [3]), computer science (see, e.g., [11], [14]), and emerging applications (see, e.g., [16], [20]). We refer the reader to [7] for an introduction to frame theory and its applications.

In 2006, Sun [21] introduced the concept of g-frames, generalized frames which include ordinary frames, bounded invertible linear operators, fusion frames, as well as many recent generalizations of frames (for more details see [12], [15],


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[17]). G-frames and g-Riesz bases in Hilbert spaces have some properties similar to those of frames, but not all the properties are similar (see [21]). Controlled frames for spherical wavelets were introduced in [4] and have been used recently to improve the numerical efficiency of iterative algorithms (see [2]). The role of controller operators is like the role played by precondition matrices or operators in linear algebra. So we give some new properties of controlled g-frames.

Controlled g-frames were introduced in [19]. In the present article, we give some new properties of controlled g-frames and construct new controlled g-frames from a given controlled g-frame, and we generalize some known results of g-frames to controlled g-frames in Section 2. In Section 3 we obtain some new resolutions of the identity with controlled g-frames, and in Section 4 we study the stability of controlled g-frames under small perturbations.

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are two separable Hilbert spaces and $\{\mathcal{H}_i : i \in I\}$ is a sequence of subspaces of $\mathcal{K}$, where $I$ is a subset of $\mathbb{Z}$. We denote by $L(\mathcal{H}, \mathcal{H}_i)$ the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_i$, and $\mathrm{GL}(\mathcal{H})$ denotes the set of all bounded linear operators which have bounded inverse. It is easy to see that if $T, U \in \mathrm{GL}(\mathcal{H})$, then $T^*$, $T$, and $TU$ are also in $\mathrm{GL}(\mathcal{H})$. Let $\mathrm{GL}^+(\mathcal{H})$ be the set of all positive operators in $\mathrm{GL}(\mathcal{H})$. Also $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$.

Note that for any sequence $\{\mathcal{H}_i : i \in I\}$ of Hilbert spaces, we can always find a large Hilbert space $\mathcal{K}$ such that for all $i \in I$, $\mathcal{H}_i \subset \mathcal{K}$ (e.g., $\mathcal{K} = \bigoplus_{i\in I} \mathcal{H}_i$).

*Definition* 1. A sequence $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a *generalized frame*, or simply *g-frame*, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The numbers $A$ and $B$ are called *g-frame bounds.*

We call $\Lambda$ a *tight g-frame* if $A = B$ and a *Parseval g-frame* if $A = B = 1$. If the second inequality in (1.1) holds, the sequence is called a *g-Bessel sequence.*

$\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a *g-frame sequence* if it is a g-frame for $\overline{\mathrm{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in I}$. For each sequence $\{\mathcal{H}_i\}_{i\in I}$, we define the space $(\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell_2}$ by

$$\Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{\ell_2} = \Big\{\{f_i\}_{i\in I} : f_i \in \mathcal{H}_i, i \in I \text{ and } \sum_{i\in I} \|f_i\|^2 < +\infty\Big\}$$

with the inner product defined by

$$\big\langle \{f_i\}, \{g_i\}\big\rangle = \sum_{i\in I} \langle f_i, g_i\rangle.$$

*Definition* 2. Let $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$. Then the *synthesis operator* for $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is the operator

$$\Theta_\Lambda : \Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{\ell_2} \longrightarrow \mathcal{H}$$

defined by

$$\Theta_\Lambda\big(\{f_i\}_{i\in I}\big) = \sum_{i\in I} \Lambda_i^*(f_i).$$

The adjoint $\Theta_\Lambda^*$ of the synthesis operator is called the *analysis operator* which is given by

$$\Theta_\Lambda^* : \mathcal{H} \longrightarrow \Big(\sum_{i\in I} \oplus \mathcal{H}_i\Big)_{\ell_2}, \qquad \Theta^*(f) = \{\Lambda_i f\}_{i\in I}.$$

By composing $\Theta_\Lambda$ and $\Theta_\Lambda^*$, we obtain the g-frame operator

$$S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H}, \qquad S_\Lambda f = \Theta_\Lambda \Theta_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f.$$

It is easy to see that the g-frame operator is a bounded, positive, and invertible operator.

## 2. Controlled g-frames and constructing new controlled g-frames

Controlled g-frames with two controller operators were studied in [18], [19]. Next, we give the definition of controlled g-frames.

*Definition* 3. Let $T, U \in \mathrm{GL}^+(\mathcal{H})$. The family $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ will be called a $(T, U)$-*controlled g-frame* for $\mathcal{H}$, if $\Lambda$ is a g-Bessel sequence and there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \langle \Lambda_i T f, \Lambda_i U f\rangle \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

$A$ and $B$ are called the *lower* and *upper controlled frame bounds*, respectively.

If $U = I_{\mathcal{H}}$, then we call $\Lambda = \{\Lambda_i\}$ a $T$-controlled g-frame for $\mathcal{H}$ with bounds $A$ and $B$. If the second part of the above inequality holds, then it is called a $(T, U)$-controlled g-Bessel sequence with bound $B$. Let $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $(T, U)$-controlled g-frame for $\mathcal{H}$. Then the $(T, U)$-controlled g-frame operator is defined by

$$S_{T\Lambda U} : \mathcal{H} \longrightarrow \mathcal{H}, \qquad S_{T\Lambda U} f = \sum_{i\in I} U^* \Lambda_i^* \Lambda_i T f, \quad \forall f \in \mathcal{H}.$$

It follows from the definition that for a g-frame, this operator is positive and invertible and

$$A I_{\mathcal{H}} \leq S_{T\Lambda U} \leq B I_{\mathcal{H}}.$$

Also $S_{T\Lambda U} = U^* S_\Lambda T$. For the reader's convenience, we state the following lemma.

**Lemma 1** ([2, Proposition 2.4])**.** *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator. Then the following conditions are equivalent.*

(i) *There exist $m > 0$ and $M < \infty$ such that $m I_{\mathcal{H}} \leq T \leq M I_{\mathcal{H}}$.*

(ii) *$T$ is positive and there exist $m > 0$ and $M < \infty$ such that*

$$m\|f\|^2 \leq \|T^{1/2} f\|^2 \leq M\|f\|^2.$$

(iii) $T \in \mathrm{GL}^+(\mathcal{H})$.

**Proposition 1.** *Let $T, U \in \mathrm{GL}^+(\mathcal{H})$, and let $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of operators. Then the following statements hold.*

(i) *If $\{\Lambda_i : i \in I\}$ is a $(T,U)$-controlled g-frame for $\mathcal{H}$, then $\{\Lambda_i : i \in I\}$ is a g-frame for $\mathcal{H}$.*

(ii) *If $\{\Lambda_i : i \in I\}$ is a frame for $\mathcal{H}$ and $T, U \in \mathrm{GL}^+(\mathcal{H})$, which commute with each other and commute with $S_\Lambda$, then $\{\Lambda_i : i \in I\}$ is a $(T,U)$-controlled g-frame for $\mathcal{H}$.*

*Proof.* The proof consists of two parts.

(i). For $f \in \mathcal{H}$, since the operator

$$S_\Lambda(f) = (U^*)^{-1} S_{T\Lambda U} T^{-1}(f) = \sum_{i\in I} \Lambda_i^* \Lambda_i f$$

is well defined, we can show that it is a bounded and invertible operator and also that it is a positive linear operator on $\mathcal{H}$ because

$$\langle S_\Lambda f, f\rangle = \sum_{i\in I} \|\Lambda_i f\|^2.$$

Also, we have

$$\|S_\Lambda^{-1}\| = \|T S_{T\Lambda U}^{-1} U^*\| \le \|T\| \|S_{T\Lambda U}^{-1}\| \|U^*\| \le \frac{1}{A}\|T\|\|U^*\|.$$

So $S \in \mathrm{GL}^+(\mathcal{H})$. Therefore, by Lemma 1, we have $CI_{\mathcal{H}} \le S_\Lambda \le DI_{\mathcal{H}}$ for some $0 < C \le D < \infty$. So the result follows.

(ii). Let $\{\Lambda_i : i \in I\}$ be a g-frame with bounds $C$, $D$, and let $m, m' > 0$, $M, M' < \infty$ be such that

$$mI_{\mathcal{H}} \le T \le MI_{\mathcal{H}}, \qquad m'I_{\mathcal{H}} \le U^* \le M'I_{\mathcal{H}}.$$

By Lemma 1, we then have

$$mAI_{\mathcal{H}} \le S_\Lambda T \le MBI_{\mathcal{H}}$$

because $T$ commutes with $S_\Lambda$. Again $U^*$ commutes with $S_\Lambda T$ and then

$$mm'AI_{\mathcal{H}} \le S_{T\Lambda U} \le MM'BI_{\mathcal{H}}.$$

So we have the result. □

Theorem 2.8 in [1] leads us to the following result.

**Proposition 2.** *Let $T, U \in \mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i : i \in I\}$ be a $(T,U)$-controlled g-frame for $\mathcal{H}$ with lower and upper bounds $A$ and $B$, respectively. Let $\{\Gamma_i : i \in I\}$ be a g-complete family of bounded operators. If there exists a number $0 < R < A$ such that*

$$0 \le \sum_{i\in I} \big\langle U^*(\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i) Tf, f\big\rangle \le R\|f\|^2, \quad \forall f \in \mathcal{H},$$

*then $\{\Gamma_i : i \in I\}$ is also a $(T,U)$-controlled g-frame for $\mathcal{H}$.*

*Proof.* Let $f$ be an arbitrary element of $\mathcal{H}$. Since $\{\Lambda_i : i \in I\}$ is a $(T,U)$-controlled g-frame for $\mathcal{H}$, we have

$$C\|f\|^2 \leq \sum_{i\in I}\langle U^*\Lambda_i^*\Lambda_i Tf, f\rangle \leq B\|f\|^2.$$

Hence,

$$\begin{aligned}\sum_{i\in I}\langle U^*\Gamma_i^*\Gamma_i Tf, f\rangle &= \sum_{i\in I}\big\langle U^*(\Gamma_i^*\Gamma_i - \Lambda_i^*\Lambda_i)Tf, f\big\rangle + \sum_{i\in I}\langle U^*\Lambda_i^*\Lambda_i Tf, f\rangle \\ &\leq R\|f\|^2 + B\|f\|^2 = (R+B)\|f\|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i\in I}\langle U^*\Gamma_i^*\Gamma_i Tf, f\rangle &= \sum_{i\in I}\langle U^*\Lambda_i^*\Lambda_i Tf, f\rangle + \sum_{i\in I}\big\langle U^*(\Gamma_i^*\Gamma_i - \Lambda_i^*\Lambda_i)Tf, f\big\rangle \\ &\geq \sum_{i\in I}\langle U^*\Lambda_i^*\Lambda_i Tf, f\rangle - \sum_{i\in I}\big\langle U^*(\Gamma_i^*\Gamma_i - \Lambda_i^*\Lambda_i)Tf, f\big\rangle \\ &\geq A\|f\| - R\|f\|^2 = (A-R)\|f\|^2 > 0.\end{aligned}$$

So we have the result. □

**Proposition 3.** *Let $T, U \in \mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i : i \in I\}$ be a $(T,U)$-controlled g-frame for $\mathcal{H}$. Let $\{\Gamma_i : i \in I\}$ be a g-complete family of bounded operators. Suppose that $\Phi : \mathcal{H} \longrightarrow \mathcal{H}$ defined by*

$$\Phi(f) = \sum_{i\in I} U^*(\Gamma_i^*\Gamma_i - \Lambda_i^*\Lambda_i)Tf, \quad \forall f \in \mathcal{H},$$

*is a positive and compact operator. Then $\{\Gamma_i : i \in I\}$ is a $(T,U)$-controlled g-frame for $\mathcal{H}$.*

*Proof.* Let $\{\Lambda_i : i \in I\}$ be a $(T,U)$-controlled g-frame for $\mathcal{H}$. Then by Proposition 1 it is a g-frame for $\mathcal{H}$. On the other hand, since $\Phi$ is a positive compact operator, $U^{-1}\Phi T^{-1}$ is also a positive compact operator. Hence,

$$(U^*)^{-1}\Phi T^{-1} f = \sum_{i\in I}\Gamma_i^*\Gamma_i^* f - \Lambda_i^*\Lambda_i f, \quad \forall f \in \mathcal{H}.$$

Let $\Psi = (U^*)^{-1}\Phi T^{-1}$, and let $P : \mathcal{H} \longrightarrow \mathcal{H}$ be an operator defined by

$$P = S_\Lambda + \Psi.$$

A simple computation shows that $\Psi$ is bounded and self-adjoint and that $P$ is bounded, linear, and self-adjoint. Let $f$ be an arbitrary element of $\mathcal{H}$. We have

$$\|Pf\| = \|S_\Lambda f + \Psi f\| \leq \|S_\Lambda f\| + \|\Psi f\| \leq \big(B + \|\Psi\|\big)\|f\|.$$

Therefore,

$$\sum_{i\in I}\|\Gamma_i f\|^2\langle Pf, f\rangle \leq \big(B + \|\Psi\|\big)\|f\|^2.$$

Since $\Psi$ is a compact operator, $\Psi S_\Lambda^{-1}$ is also a compact operator on $\mathcal{H}$. By Theorem 2.8 in [1], $P$ has closed range. Now we show that $P$ is injective. Let $g$ be an element of $\mathcal{H}$ such that $Pf = 0$. Then

$$\sum_{i\in I} \|\Gamma_i g\|^2 = \langle Pg, g\rangle = 0.$$

Hence, $\Gamma_i g = 0$ for each $i \in I$. Since $\{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g-complete, we have $g = 0$. Furthermore, we have

$$\text{Range}(P) = \big(N(P^*)\big)^{\perp} = N(P)^{\perp} = \mathcal{H}.$$

Hence $P$ is onto and therefore invertible on $\mathcal{H}$. Similar to the proof of Theorem 2.8 of [1], we have

$$\sum_{i\in I} \|\Gamma_i g\|^2 \geq \big(B + \|\Psi\|\big)^{-1} \|P^{-1}\|^{-2} \|f\|^2.$$

Then $\{\Gamma_i : i \in I\}$ is a g-frame for $\mathcal{H}$. Since $\Phi = U^* S_\Gamma T - U^* S_\Lambda T$, $U^* S_\Gamma T = \Phi + U^* S_\Lambda T$. It is easy to see that $U^* S_\Gamma T$ is a bounded positive operator. By Lemma 1, we have that $\{\Gamma_i : i \in I\}$ is a $(T, U)$-controlled g-frame for $\mathcal{H}$. □

The next result is a generalization of Theorem 3.3 of [6].

**Theorem 1.** *Let $T, U \in \mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of bounded operators. Let $\{\Gamma_{ij} \in L(\mathcal{H}_i, \mathcal{H}_{ij}) : j \in J_i\}$ be a $(C_i, D_i)$-$(T, U)$-controlled g-frame for each $\mathcal{H}_i$, and suppose that they are $(C, D)$-bounded. Then the following conditions are equivalent.*

(i) *$\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a $(T, U)$-controlled g-frame for $\mathcal{H}$.*
(ii) *$\{\Gamma_{ij}\Lambda_i \in L(\mathcal{H}_i, \mathcal{H}_{ij}) : i \in I,\ j \in J_i\}$ is a $(T, U)$-controlled g-frame for $\mathcal{H}$.*

*Proof.* The proofs consists of two parts.

(i) ⇒ (ii). Let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $(T, U)$-controlled g-frame with bounds $(A,\ B)$ for $\mathcal{H}$. Then for all $f \in \mathcal{H}$ we have

$$\begin{aligned}
&\sum_{i\in I}\sum_{j\in J_i} \langle \Gamma_{ij}\Lambda_i T f, \Gamma_{ij}\Lambda_i U f\rangle \\
&\quad = \sum_{i\in I}\sum_{j\in J_i} \langle \Gamma_{ij}^*\Gamma_{ij}\Lambda_i T f, \Lambda_i U f\rangle \\
&\quad \leq \sum_{i\in I} D_i \langle \Lambda_i T f, \Lambda_i U f\rangle \\
&\quad \leq DB\|f\|^2.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\sum_{i\in I}\sum_{j\in J_i} \langle \Gamma_{ij}\Lambda_i T f, \Gamma_{ij}\Lambda_i U f\rangle \\
&\quad = \sum_{i\in I}\sum_{j\in J_i} \langle \Gamma_{ij}^*\Gamma_{ij}\Lambda_i T f, \Lambda_i U f\rangle \\
&\quad \geq \sum_{i\in I} C_i \langle \Lambda_i T f, \Lambda_i U f\rangle \\
&\quad \geq CA\|f\|^2.
\end{aligned}$$

(ii) $\Rightarrow$ (i). Let $\{\Gamma_{ij}\Lambda_i \in L(\mathcal{H}_i, \mathcal{H}_{ij}) : i \in I,\ j \in J_i\}$ be a $(T, U)$-controlled g-frame with bounds $A$, $B$ for $\mathcal{H}$. Since $\Lambda_i f \in \mathcal{H}_i$, we have

$$\sum_{i\in I}\langle \Lambda_i Tf, \Lambda_i Uf\rangle \le \sum_{i\in I}\frac{1}{C_i}\sum_{j\in J_i}\langle \Gamma_{ij}\Lambda_i Tf, \Gamma_{ij}\Lambda_i Uf\rangle \le \frac{B}{C}\|f\|^2.$$

Also,

$$\sum_{i\in I}\langle \Lambda_i Tf, \Lambda_i Uf\rangle \ge \sum_{i\in I}\frac{1}{D_i}\sum_{j\in J_i}\langle \Gamma_{ij}\Lambda_i Tf, \Gamma_{ij}\Lambda_i Uf\rangle \ge \frac{A}{D}\|f\|^2.$$

□

Our next result is a characterization theorem for $(T, U)$-controlled g-frames.

**Theorem 2.** *Let $T, U \in \mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of bounded operators. Suppose that $\{e_{ij} : j \in J_i\}$ is an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$. Then $\{\Lambda_i : i \in I\}$ is a $(T, U)$-controlled g-frame for $\mathcal{H}$ if and only if $\{T^* u_{ij} : i \in I, j \in J_i\}$ is a $U^*(T^*)^{-1}$-controlled frame for $\mathcal{H}$, where $u_{ij} = \Lambda_i^* e_{ij}$.*

*Proof.* Let $\{e_{ij} : j \in J_i\}$ be an orthonormal basis for $\mathcal{H}_i$ for each $i \in I$. For any $f \in \mathcal{H}$, since $\Lambda_i f \in \mathcal{H}_i$, we have

$$\Lambda_i(Tf) = \sum_{j\in J_i}\big\langle \Lambda_i(Tf), e_{ij}\big\rangle e_{ij} = \sum_{j\in J_i}\langle f, T^*\Lambda_i^* e_{ij}\rangle e_{ij}.$$

Also,

$$\Lambda_i(Uf) = \sum_{j\in J_i}\big\langle \Lambda_i(Uf), e_{ij}\big\rangle e_{ij} = \sum_{j\in J_i}\langle f, U^*\Lambda_i^* e_{ij}\rangle e_{ij}.$$

Hence,

$$\langle \Lambda_i Tf, \Lambda_i Uf\rangle = \sum_{j\in J_i}\langle f, T^*\Lambda_i^* e_{ij}\rangle\langle U^*\Lambda^* e_{ij}, f\rangle.$$

Now, if we take $u_{ij} = \Lambda_i^* e_{ij}$, $f_{ij} = T^* u_{ij}$, and $\Omega = U^*(T^*)^{-1}$, then

$$A\|f\|^2 \le \sum_{i\in I}\langle \Lambda_i Tf, \Lambda_i Uf\rangle \le B\|f\|^2$$

is equivalent to

$$A\|f\| \le \sum_{i\in I}\sum_{j\in J_i}\langle f, f_{ij}\rangle\langle \Omega f_{ij}, f\rangle \le B\|f\|^2.$$

So we have the result. □

Note that $\{u_{ij} : i \in I, j \in J_i\}$ is the sequence induced by $\{\Lambda_i : i \in I\}$ with respect to $\{e_{ij} : j \in J_i\}$.

By the above result, finding suitable operators $T$ and $U$ such that $\{\Lambda_i : i \in I\}$ forms a $(T, U)$-controlled fusion frame for $\mathcal{H}$ with optimal bounds, is equivalent to finding suitable operators $T$ and $U$ such that $\{T^* u_{ij} : i \in I, j \in J_i\}$ is a $U^*(T^*)^{-1}$-controlled frame for $\mathcal{H}$ with optimal frame bounds.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. We recall that $\mathcal{H} \oplus \mathcal{K} = \{(f, g) : f \in \mathcal{H}, g \in \mathcal{K}\}$ is a Hilbert space with pointwise operations and inner product

$$\big\langle (f, g), (f', g')\big\rangle := \langle f, f'\rangle_{\mathcal{H}} + \langle g, g'\rangle_{\mathcal{K}}, \quad \forall f, f' \in \mathcal{H}, g, g' \in \mathcal{K}.$$

Also, if $\Lambda \in L(\mathcal{H}, V)$ and $\Gamma \in L(\mathcal{K}, W)$, then for all $f \in \mathcal{H}$, $g \in \mathcal{K}$ we define

$$\Lambda \oplus \Gamma \in L(\mathcal{H} \oplus \mathcal{K}, V \oplus W) \quad \text{by } (\Lambda \oplus \Gamma)(Tf, Ug) := (\Lambda Tf, \Gamma Ug),$$

where $V$, $W$ are Hilbert spaces and $T \in \mathrm{GL}(\mathcal{H})$, $U \in \mathrm{GL}(\mathcal{K})$.

**Theorem 3.** *Let $T \in \mathrm{GL}(\mathcal{H})$, $U \in \mathrm{GL}(\mathcal{K})$. Let $\{\Lambda_i \in L(\mathcal{H}, V_i) : i \in I\}$ and $\{\Gamma_i \in L(\mathcal{K}, W_i) : i \in I\}$ be a $(T,T)$-controlled g-frame with bounds $(A,B)$ and a $(U,U)$-controlled g-frame with bounds $(C,D)$, respectively. Then $\{\Lambda_i \oplus \Gamma_i \in L(\mathcal{H} \oplus \mathcal{K}, V_i \oplus W_i) : i \in I\}$ is a $(T,U)$-controlled g-frame with bounds $(\min\{A,C\}, \max\{B,D\})$.*

*Proof.* Let $(f,g)$ be an arbitrary element of $\mathcal{H} \oplus \mathcal{K}$. Then we have

$$\begin{aligned}
\sum_{i\in I} \big\|(\Lambda_i \oplus \Gamma_i)(Tf,Ug)\big\|^2 &= \sum_{i\in I} \big\langle(\Lambda_i \oplus \Gamma_i)(Tf,Ug), (\Lambda_i \oplus \Gamma_i)(Tf,Ug)\big\rangle \\
&= \sum_{i\in I} \big\langle(\Lambda_i Tf, \Gamma_i Ug), (\Lambda_i Tf, \Gamma_i Ug)\big\rangle \\
&= \sum_{i\in I} \langle \Lambda_i Tf, \Lambda_i f\rangle + \langle \Gamma_i Ug, \Gamma_i Ug\rangle \\
&= \sum_{i\in I} \|\Lambda_i Tf\|^2 + \sum_{i\in I} \|\Gamma_i Uf\|^2 \\
&\le B\|f\|^2 + D\|g\|^2 \\
&\le \max\{B,D\}\big(\|f\|^2 + \|g\|^2\big) \\
&= \max\{B,D\}\big\|(f,g)\big\|^2.
\end{aligned}$$

Similarly, we have

$$\min\{A,C\}\big(\|f\|^2 + \|g\|^2\big) \le \sum_{i\in I} \big\|(\Lambda_i \oplus \Gamma_i)(Tf,Ug)\big\|^2.$$

So we have the result. □

Our next result is a generalization of Proposition 3.9 in [18].

**Proposition 4.** *Let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with frame operator $S_\Lambda$ and bounds $A$, $B$, and let $\varepsilon > 0$ be a real number. Let $T \in \mathrm{GL}(\mathcal{H})$ be an operator such that $\|T - S_\Lambda^{-1}\| \le \varepsilon\|T\|$. If $\|T\| < \frac{1}{B\sqrt{\varepsilon^2+2\varepsilon}}$, then $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a $(T,T)$-controlled g-frame for $\mathcal{H}$ with bounds*

$$\frac{1}{B} - B(\varepsilon^2 + 2\varepsilon)\|T\|^2 \qquad \text{and} \qquad B\Big(\varepsilon\|T\| + \frac{1}{A}\Big)^2.$$

*Proof.* Let $f \in \mathcal{H}$ be an arbitrary element, and let $\Theta_\Lambda$ be the synthesis operator of $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$. Then we have

$$\begin{aligned}
\|\Theta^*_{\Lambda T} f\|^2 &= \|\Theta^*_{\Lambda(T-S_\Lambda^{-1})} f\|^2 + \langle \Theta^*_{\Lambda(T-S_\Lambda^{-1})} f, \Theta^*_{\Lambda S_\Lambda^{-1}} f\rangle \\
&\quad + \langle \Theta^*_{\Lambda S_\Lambda^{-1}} f, \Theta^*_{\Lambda(T-S_\Lambda^{-1})} f\rangle + \|\Theta^*_{\Lambda S_\Lambda^{-1}} f\|^2.
\end{aligned}$$

Now by the hypothesis and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|\Theta^*_{\Lambda T}f\|^2 &\leq B\big(\|T - S_\Lambda^{-1}\|^2 + 2\|T - S_\Lambda^{-1}\|\|S_\Lambda^{-1}\| + \|S_\Lambda^{-1}\|^2\big)\|f\|^2 \\ &\leq B\Big(\varepsilon^2\|T\|^2 + 2\varepsilon\|t\|\frac{1}{A} + \frac{1}{A^2}\Big)\|f\|^2 \\ &= B\Big(\varepsilon\|T\| + \frac{1}{A}\Big)^2\|f\|^2.\end{aligned}$$

On the other hand, since $\{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ is also a g-frame with lower frame bound $\frac{1}{B}$, we have

$$\begin{aligned}\frac{1}{B}\|f\|^2 &\leq \|\Theta^*_{\Lambda S_\Lambda^{-1}}f\|^2 \\ &= \|\Theta^*_{\Lambda(S_\Lambda^{-1}-T)}f\|^2 + \langle\Theta^*_{\Lambda(S_\Lambda^{-1}-T)}f, \Theta^*_{\Lambda T}f\rangle \\ &\quad + \langle\Theta^*_{\Lambda T}f, \Theta^*_{\Lambda(S_\Lambda^{-1}-T)}f\rangle + \|\Theta^*_{\Lambda T}f\|^2 \\ &= B\big(\|S_\Lambda^{-1} - T\|^2 + 2\|S_\Lambda^{-1} - T\|\|T\|\big)\|f\|^2 + \|\Theta^*_{\Lambda T}f\|^2.\end{aligned}$$

Therefore, we have

$$\Big(\frac{1}{B} - B(\varepsilon^2 + 2\varepsilon)\|T\|^2\Big)\|f\|^2 \leq \|\Theta^*_{\Lambda T}f\|^2.$$

Now the result holds. □

We end this section by giving the following results concerning the constructions of new controlled g-frames.

**Theorem 4.** *Let $T \in \mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $(T,T)$-controlled g-frame with bounds $(A,B)$. Let $\{\Gamma_i\}_{i\in I}$ be a g-sequence with synthesis operator $\Theta_\Gamma$. For any two positively confined sequences $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$, if $\|\Theta_\Gamma\|^2 < \frac{A\inf_{i\in I} a_i^2}{2\|T\|^2\sup_{i\in I} b_i^2}$, then $\{a_i\Lambda_i + b_i\Gamma_i\}_{i\in I}$ is a $(T,T)$-controlled g-frame for $\mathcal{H}$.*

*Proof.* For any $f \in \mathcal{H}$, we have

$$\begin{aligned}&\sum_{i\in I}\big\|(a_i\Lambda_i + b_i\Gamma_i)Tf\big\|^2 \\ &\quad = \sum_{i\in I}\|a_i\Lambda_i Tf\|^2 + \sum_{i\in I}\|b_i\Gamma_i Tf\|^2 \\ &\qquad + 2\,\mathrm{Re}\sum_{i\in I}\langle a_i\Lambda_i Tf, b_i\Gamma_i Tf\rangle \\ &\quad \leq 2\Big(\sum_{i\in I}\|a_i\Lambda_i Tf\|^2 + \sum_{i\in I}\|b_i\Gamma_i Tf\|^2\Big) \\ &\quad \leq 2\Big(\big(\sup_{i\in I} a_i^2\big)\sum_{i\in I}\|\Lambda_i Tf\|^2 + \big(\sup_{i\in I} b_i^2\big)\sum_{i\in I}\|\Gamma_i Tf\|^2\Big) \\ &\quad \leq 2\big(\big(\sup_{i\in I} a_i^2\big)B\|f\|^2 + \big(\sup_{i\in I} b_i^2\big)\|\Theta_\Gamma^* Tf\|^2\big) \\ &\quad \leq 2\big(\big(\sup_{i\in I} a_i^2\big)B + \big(\sup_{i\in I} b_i^2\big)\|T\|^2\|\Theta_\Gamma\|^2\big)\|f\|^2.\end{aligned}$$

Since

$$\sum_{i\in I}\|a_i\Lambda_i Tf\|^2=\sum_{i\in I}\big\|(a_i\Lambda_i+b_i\Gamma_i)Tf-b_i\Gamma_i Tf\big\|^2 \le 2\Big(\sum_{i\in I}\big\|(a_i\Lambda_i+b_i\Gamma_i)Tf\big\|^2+\sum_{i\in I}\|b_i\Gamma_i Tf\|^2\Big),$$

we have

$$\begin{aligned}2\sum_{i\in I}\big\|(a_i\Lambda_i+b_i\Gamma_i)Tf\big\|^2 &\ge \sum_{i\in I}\|a_i\Lambda_i Tf\|^2-2\sum_{i\in I}\|b_i\Gamma_i Tf\|^2\\ &\ge \big(\inf_{i\in I}a_i^2\big)\sum_{i\in I}\|\Lambda_i Tf\|^2-2\big(\sup_{i\in I}b_i^2\big)\|\Theta_\Gamma^* Tf\|^2\\ &\ge \Big(\big(\inf_{i\in I}a_i^2\big)A-2\big(\sup_{i\in I}b_i^2\big)\|T\|^2\|\Theta_\Gamma\|^2\Big)\|f\|^2.\end{aligned}$$

From $\|\Theta_\Gamma\|^2<\frac{A\inf_{i\in I}a_i^2}{2\|T\|^2\sup_{i\in I}b_i^2}$, we obtain that $\{a_i\Lambda_i+b_i\Gamma_i\}_{i\in I}$ is a $(T,T)$-controlled g-frame for $\mathcal{H}$. □

## 3. Resolutions of the identity

In this section, we will find new resolutions of the identity. In fact, let $T,U\in\mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i\in L(\mathcal{H},\mathcal{H}_i):i\in I\}$ be a $(T,U)$-controlled g-frame. Then we have

$$f=\sum_{i\in I}S_{T\Lambda U}^{-1}U^*\Lambda_i^*\Lambda_i Tf=\sum_{i\in I}U^*\Lambda_i^*\Lambda_i TS_{T\Lambda U}^{-1}f,\quad \forall f\in\mathcal{H}.$$

By choosing suitable control operators we may obtain more suitable approximations. Now we will give a new resolution of the identity by using two controlled operators.

*Definition* 4. Let $T,U\in\mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i\in L(\mathcal{H},\mathcal{H}_i):i\in I\}$ and $\{\Gamma_i\in L(\mathcal{H},\mathcal{H}_i):i\in I\}$ be $(T,T)$-controlled and $(U,U)$-controlled g-Bessel sequences, respectively. We define a $(T,U)$-controlled g-frame operator for this pair of controlled g-Bessel sequences as follows:

$$S_{T\Gamma\Lambda U}(f)=\sum_{i\in I}U^*\Gamma_i^*\Lambda_i T(f),\quad \forall f\in\mathcal{H}.$$

As mentioned before, $\{\Lambda_i\in L(\mathcal{H},\mathcal{H}_i):i\in I\}$ and $\{\Gamma_i\in L(\mathcal{H},\mathcal{H}_i):i\in I\}$ are also two Bessel g-sequences. So by [13], the g-frame operator $S_{\Gamma\Lambda}(f)=\sum_{i\in I}\Gamma_i^*\Lambda_i(f)$ for this pair of g-Bessel sequences is well defined and bounded. Since $S_{T\Gamma\Lambda U}=U^*S_{\Gamma\Lambda}T$, $S_{T\Gamma\Lambda U}$ is a well-defined and bounded operator.

**Lemma 2.** *Let $T,U\in\mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i:i\in I\}$ and $\{\Gamma_i:i\in I\}$ be $(T,T)$-controlled and $(U,U)$-controlled g-Bessel sequences, respectively. If $S_{T\Gamma\Lambda U}$ is bounded below, then $\{\Lambda_i:i\in I\}$ and $\{\Gamma_i:i\in I\}$ are $(T,T)$-controlled and $(U,U)$-controlled g-frames, respectively.*

*Proof.* Suppose that there exists a number $\lambda > 0$ such that for all $f \in \mathcal{H}$,

$$\lambda\|f\| \leq \|S_{T\Gamma\Lambda U}\|.$$

Then we have

$$\begin{aligned}\lambda\|f\| \leq \|S_{T\Gamma\Lambda U}\| &= \sup_{g\in\mathcal{H},\|g\|=1}\Big|\Big\langle\sum_{i\in I} U^*\Gamma_i^*\Lambda_i Tf, g\Big\rangle\Big| \\ &= \sup_{\|g\|=1}\Big|\Big\langle\sum_{i\in I}\Lambda_i Tf, \Gamma_i Ug\Big\rangle\Big| \\ &\leq \sup_{\|g\|=1}\Big(\sum_{i\in I}\|\Lambda_i Tf\|^2\Big)^{1/2}\Big(\sum_{i\in I}\|\Gamma_i Ug\|^2\Big)^{1/2} \\ &\leq \sqrt{B}\Big(\sum_{i\in I}\|\Lambda_i Tf\|^2\Big)^{1/2}.\end{aligned}$$

Hence,

$$\frac{\lambda^2}{D}\|f\|^2 \leq \sum_{i\in I}\|\Lambda_i Tf\|^2.$$

On the other hand, since

$$S_{T\Gamma\Lambda U}^* = (U^*S_{\Gamma\Lambda}T)^* = T^*S_{\Gamma\Lambda}^*U = T^*S_{\Lambda\Gamma}U = S_{U\Lambda\Gamma T},$$

we can say that $S_{U\Lambda\Gamma T}$ is also bounded below. So by the above result, $\{\Gamma_i : i \in I\}$ is a $(U, U)$-controlled g-frame. □

**Theorem 5.** *Let $T \in \mathrm{GL}(\mathcal{H})$, and let $\Lambda = \{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $(T, T)$-controlled g-Bessel sequence. Then the following conditions are equivalent.*

(i) *$\Lambda$ is a $(T, T)$-controlled g-frame for $\mathcal{H}$.*

(ii) *There exists an operator $U \in \mathrm{GL}(\mathcal{H})$ and a $(U, U)$-controlled g-Bessel sequence $\Gamma = \{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ such that $S_{U\Gamma\Lambda T} \geq mI_{\mathcal{H}}$ on $\mathcal{H}$, for some $m > 0$.*

*Proof.* The proofs consists of two parts.

(i) ⇒ (ii). Let $\Lambda$ be a $(T, T)$-controlled g-frame with lower and upper g-frame bounds $A_T$ and $B_T$, respectively. Then we take $U = T$, $\Gamma_i = \Lambda_i$, for all $i \in I$. Hence, we have

$$\langle S_{T\Lambda\Lambda T}f, f\rangle = \Big\langle\sum_{i\in I} T^*\Lambda_i^*\Lambda_i Tf, f\Big\rangle = \sum_{i\in I}\langle\Lambda_i Tf, \Lambda_i Tf\rangle \geq A_T\|f\|^2$$

for all $f \in \mathcal{H}$. Moreover,

$$C_T\|f\|^2 \leq \|S_{T\Lambda\Lambda T}^{1/2}\|^2 \leq D_T\|f\|^2.$$

By Lemma 1, $S_{T\Lambda\Lambda T} \in \mathrm{GL}^+(\mathcal{H})$.

(ii) ⇒ (i). Suppose that there exist an operator $U \in \mathrm{GL}(\mathcal{H})$ and a $(U, U)$-controlled g-Bessel sequence $\Gamma = \{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ with Bessel bound $B_U$. Also, let $m > 0$ be a constant such that

$$\langle S_{U\Gamma\Lambda T}f, f\rangle \geq m\|f\|^2$$

for all $f \in \mathcal{H}$. Then we have

$$\begin{aligned} m\|f\|^2 &\le \langle S_{U\Gamma\Lambda T} f, f\rangle \\ &= \sum_{i\in I} \langle \Lambda_i T f, \Gamma_i U f\rangle \\ &\le \Big(\sum_{i\in I} \|\Lambda_i T f\|^2\Big)^{1/2} \Big(\sum_{i\in I} \|\Gamma_i U f\|^2\Big)^{1/2} \\ &\le \sqrt{B_U}\|f\| \Big(\sum_{i\in I} \|\Lambda_i T f\|^2\Big)^{1/2}, \end{aligned}$$

by the Cauchy–Schwarz inequality. Hence,

$$\frac{m^2}{B_U}\|f\|^2 \le \sum_{i\in I} \|\Lambda_i T f\|^2 \le B_T\|f\|^2.$$

So $\Lambda$ is a $(T,T)$-controlled g-frame for $\mathcal{H}$. □

**Theorem 6.** *Let $T, U \in \mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $(T,T)$-controlled g-frame with bounds $(A,B)$ for $\mathcal{H}$. Let the family $\{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $(U,U)$-controlled g-Bessel sequence. Suppose that there exists a number $0 < \lambda \le A$ such that*

$$\big\|(S_{T\Gamma\Lambda U} - S_{T\Lambda T})f\big\| \le \lambda\|f\|, \quad \forall f \in \mathcal{H}.$$

*Then $S_{T\Gamma\Lambda U}$ is invertible and also $\{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a $(U,U)$-controlled g-frame for $\mathcal{H}$.*

*Proof.* Let $f \in \mathcal{H}$ be an arbitrary element of $\mathcal{H}$. Then we have

$$\begin{aligned} \|S_{T\Gamma\Lambda U} f\| &= \|S_{T\Gamma\Lambda U} f - S_{T\Lambda T} f + S_{T\Lambda T} f\| \\ &\ge \|S_{T\Lambda T} f\| - \|S_{T\Gamma\Lambda U} f - S_{T\Lambda T} f\| \\ &\ge (A - \lambda)\|f\|. \end{aligned}$$

So $S_{T\Gamma\Lambda U}$ is bounded below and therefore one-to-one with closed range. On the other hand, since

$$\|S_{U\Gamma\Lambda T} - S_{T\Lambda T}\| = \big\|(S_{T\Gamma\Lambda U} - S_{T\Lambda T})^*\big\| \le \lambda,$$

by the above result $S_{U\Gamma\Lambda T}$ is also bounded below $(A-\lambda)$ and therefore one-to-one with closed range. Hence, both $S_{T\Gamma\Lambda U}$ and $S_{U\Gamma\Lambda T}$ are invertible. And

$$\begin{aligned} (A-\lambda)\|f\| \le \|S_{U\Gamma\Lambda T}\| &= \sup_{g\in\mathcal{H}, \|g\|=1} \Big|\Big\langle \sum_{i\in I} T^*\Lambda_i^*\Gamma_i U f, g\Big\rangle\Big| \\ &= \sup_{\|g\|=1} \Big|\Big\langle \sum_{i\in I} \Gamma_i U f, \Lambda_i T g\Big\rangle\Big| \\ &\le \sup_{\|g\|=1} \Big(\sum_i \|\Gamma_i U f\|^2\Big)^{1/2} \Big(\sum_i \|\Lambda_i T g\|^2\Big)^{1/2} \\ &\le \sqrt{B}\Big(\sum_i \|\Gamma_i U f\|^2\Big)^{1/2}. \end{aligned}$$

Hence,

$$\frac{(A-\lambda)^2}{B}\|f\|^2 \leq \sum_{i\in I}\|\Gamma_i U f\|^2. \qquad \square$$

Another version of these cases is as follows.

**Proposition 5.** *Let $\Lambda$ and $\Gamma$ be controlled g-Bessel sequences as mentioned in Definition 3. Suppose that there exists $0<\varepsilon<1$ such that*

$$\|f - S_{T\Gamma\Lambda U}f\| \leq \varepsilon\|f\|, \quad \forall f \in \mathcal{H}.$$

*Then $\Lambda$ and $\Gamma$ are $(T,T)$-controlled and $(U,U)$-controlled g-frames, respectively. Furthermore, $S_{T\Gamma\Lambda U}$ is invertible.*

*Proof.* First,

$$\|I_{\mathcal{H}} - S_{T\Gamma\Lambda U}\| \leq \varepsilon < 1;$$

therefore, $S_{T\Gamma\Lambda U}$ is invertible. Second, let $f$ be an arbitrary element of $\mathcal{H}$ of $\mathcal{H}$. Then we have

$$\|S_{T\Gamma\Lambda U}f\| \geq \|f\| - \|f - S_{T\Gamma\Lambda U}f\| \geq (1-\varepsilon)\|f\|.$$

Hence, $S_{T\Gamma\Lambda U}$ is bounded below. By Lemma 2, we know that $\Lambda$ is a $(T,T)$-controlled g-frame.

On the other hand, we have

$$\|I_{\mathcal{H}} - S_{U\Lambda\Gamma T}\| = \left\|(I_{\mathcal{H}} - S_{T\Gamma\Lambda U})^*\right\| \leq \varepsilon.$$

Hence, we can similarly say that $\Gamma$ is a $(U,U)$-controlled g-frame. $\square$

With the hypotheses, both $S_{T\Gamma\Lambda U}$ and $S_{U\Gamma\Lambda T}$ are invertible. Then the family

$$\{S^{-1}_{T\Gamma\Lambda U}U^*\Gamma_i^*\Lambda_i T\}_{i\in I}$$

is a resolution of the identity. Also, we have the new reconstruction formulas

$$f = \sum_{i\in I} S^{-1}_{T\Gamma\Lambda U}U^*\Gamma_i^*\Lambda_i T f = \sum_{i\in I}\Gamma_i^*\Lambda_i T S^{-1}_{T\Gamma\Lambda U} f$$

and

$$f = \sum_{i\in I} S^{-1}_{U\Lambda\Gamma T}T^*\Lambda_i^*\Gamma_i U f = \sum_{i\in I} T^*\Lambda_i^*\Gamma_i U S^{-1}_{U\Lambda\Gamma T} f.$$

Suppose that $\|I_{\mathcal{H}} - S_{T\Gamma\Lambda U}\| < 1$. Then as we mentioned in Proposition 5, $S_{T\Gamma\Lambda U}$ is invertible and we have

$$S^{-1}_{T\Gamma\Lambda U} = \sum_{n=0}^{\infty}(I_{\mathcal{H}} - S_{T\Gamma\Lambda U})^n.$$

Then we have

$$f = \sum_{i\in I}\sum_{n=0}^{\infty}(I_{\mathcal{H}} - S_{T\Gamma\Lambda U})^n U^*\Gamma_i^*\Lambda_i T f = \sum_{i\in I}\sum_{n=0}^{\infty}U^*\Gamma_i^*\Lambda_i T(I_{\mathcal{H}} - S_{T\Gamma\Lambda U})^n f.$$

Furthermore,

$$\|S^{-1}_{T\Gamma\Lambda U}\| \leq \left(1 - \|I_{\mathcal{H}} - S_{T\Gamma\Lambda U}\|\right)^{-1}.$$

Therefore,

$$\left\{(I_{\mathcal{H}} - S_{T\Gamma\Lambda U})^n U^*\Gamma_i^*\Lambda_i T\right\}_{i\in I, n\in\mathbb{Z}^+}$$

is a new resolution of the identity.

## 4. Perturbation of controlled g-frames

The perturbation of frames is important for constructing new frames from a given one. In this section we give new definitions of perturbations of g-frames with respect to the operators $T$, $U$.

*Definition* 5. Let $T, U \in \mathrm{GL}(\mathcal{H})$, and let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two g-complete families of bounded operators. Let $0 \le \lambda_1, \lambda_2 < 1$ be real numbers, and let $\mathcal{C} = \{c_i\}_{i\in I}$ be an arbitrary sequence of positive numbers such that $\|\mathcal{C}\|_2 < \infty$. We say that the family $\{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a $(\lambda_1, \lambda_2, \mathcal{C}, T, U)$-*perturbation* of $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ if we have

$$\|\Lambda_i Tf - \Gamma_i Uf\| \le \lambda_1\|\Lambda_i Tf\| + \lambda_2\|\Gamma_i Uf\| + c_i\|f\|, \quad \forall f \in \mathcal{H}.$$

We have the following important result.

**Proposition 6.** *Let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with frame bounds $A$, $B$. Suppose that $T, U \in \mathrm{GL}(\mathcal{H})$. Let $\{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $(\lambda_1, \lambda_2, \mathcal{C}, T, U)$-perturbation of $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, in which*

$$(1-\lambda_1)\sqrt{A}\|T^{-1}\|^{-1} > \|\mathcal{C}\|_2.$$

*Then $\{\Gamma_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ with g-frame bounds*

$$\Big(\frac{(1-\lambda_1)\sqrt{A}\|T^{-1}\|^{-1} - \|\mathcal{C}\|_2}{1+\lambda_2}\|U\|^{-1}\Big)^2, \qquad \Big(\frac{(1+\lambda_1)\sqrt{B}\|T\| + \|\mathcal{C}\|_2}{1-\lambda_2}\|U\|^{-1}\Big)^2$$

*Proof.* Since $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-frame for $\mathcal{H}$ with frame bounds $A$, $B$, for all $f \in \mathcal{H}$, we have

$$\frac{\sqrt{A}}{\|T^{-1}\|}\|f\| \le \sum_{i\in I}\big(\|\Lambda_i Tf\|^2\big)^{\frac{1}{2}} \le \sqrt{B}\|T\|f\|.$$

Then by triangular inequality we have

$$\begin{aligned}\Big(\sum_{i\in I}\|\Gamma_i Uf\|^2\Big)^{\frac{1}{2}} &\le \Big(\sum_{i\in I}\big(\|\Lambda_i Tf\| + \|\Lambda_i Tf - \Gamma_i Uf\|\big)^2\Big)^{\frac{1}{2}}\\ &\le \Big(\sum_{i\in I}\big(\|\Lambda_i Tf\| + \lambda_1\|\Lambda_i Tf\| + \lambda_2\|\Gamma_i Uf\| + c_i\|f\|\big)^2\Big)^{\frac{1}{2}}\\ &\le (1+\lambda_1)\sum_{i\in I}\big(\|\Lambda_i Tf\|^2\big)^{\frac{1}{2}} + \lambda_2\sum_{i\in I}\big(\|\Gamma_i Uf\|^2\big)^{\frac{1}{2}}\\ &\quad + \|\mathcal{C}\|_2\|f\|.\end{aligned}$$

Hence,

$$(1-\lambda_2)\sum_{i\in I}\big(\|\Gamma_i Uf\|^2\big)^{\frac{1}{2}} \le (1+\lambda_1)\sqrt{B}\|T\|\frac{\|Uf\|}{\|U\|^{-1}} + \|\mathcal{C}\|_2\frac{\|Uf\|}{\|U\|^{-1}}.$$

Since $Uf \in \mathcal{H}$, finally we have

$$\sum_{i\in I} \|\Gamma_i f\|^2 \leq \Big(\frac{(1+\lambda_1)\sqrt{B}\|T\| + \|\mathcal{C}\|_2)}{1-\lambda_2}\|U\|^{-1}\Big)^2 \|f\|^2.$$

Now for the lower bound we have

$$\begin{aligned}\Big(\sum_{i\in I} \|\Gamma_i U f\|^2\Big)^{\frac{1}{2}} &\geq \Big(\sum_{i\in I}\big(\|\Lambda_i T f\| - \|\Lambda_i T f - \Gamma_i U f\|\big)^2\Big)^{\frac{1}{2}} \\ &\geq \Big(\sum_{i\in I}\big(\|\Lambda_i T f\| - \lambda_1\|\Lambda_i T f\| - \lambda_2\|\Gamma_i U f\| - c_i\|f\|\big)^2\Big)^{\frac{1}{2}} \\ &\geq (1-\lambda_1)\sum_{i\in I}\big(\|\Lambda_i T f\|^2\big)^{\frac{1}{2}} - \lambda_2 \sum_{i\in I}\big(\|\Gamma_i U f\|^2\big)^{\frac{1}{2}} \\ &\quad - \|\mathcal{C}\|_2\|f\|.\end{aligned}$$

Hence,

$$(1+\lambda_2)\sum_{i\in I}\big(\|\Gamma_i U f\|^2\big)^{\frac{1}{2}} \geq (1-\lambda_1)\sqrt{A}\|T^{-1}\|^{-1}\frac{\|Uf\|}{\|U\|^{-1}} - \|\mathcal{C}\|_2\frac{\|Uf\|}{\|U\|^{-1}},$$

which yields

$$\sum_{i\in I} \|\Gamma_i f\|^2 \geq \Big(\frac{(1-\lambda_1)\sqrt{A}\|T^{-1}\|^{-1} - \|\mathcal{C}\|_2}{1+\lambda_2}\|U\|^{-1}\Big)^2 \|f\|^2. \qquad \square$$

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## References

1. M. S. Asgari and A. Khosravi, *Frames and bases of subspaces in Hilbert spaces*, J. Math. Anal. Appl. **308** (2005), no. 2, 541–553. Zbl 1091.46006. MR2150106. DOI 10.1016/j.jmaa.2004.11.036. 540, 541, 542
2. P. Balazs, J.-P. Antoine, and A. Gryboś, *Weighted and controlled frames: Mutual relationship and first numerical properties*, Int. J. Wavelets, Multiresolut. Inf. Process. **8** (2010), no. 1, 109–132. Zbl 1192.42016. MR2654396. DOI 10.1142/S0219691310003377. 538, 539
3. B. G. Bodmann and V. I. Paulsen, *Frames, graphs and erasures*, Linear Algebra Appl. **404** (2005), no. 1, 118–146. Zbl 1088.46009. MR2149656. DOI 10.1016/j.laa.2005.02.016. 537
4. I. Bogdanova, P. Vandergheynst, J.-P. Antoine, L. Jacques, and M. Morvidone, *Stereographic wavelet frames on the sphere*, Appl. Comput. Harmon. Anal. **19** (2005), no. 2, 223–252. Zbl 1082.42026. MR2163079. DOI 10.1016/j.acha.2005.05.001. 538
5. P. G. Casazza, O. Christensen, A. M. Lindner, and R. Vershynin, *Frames and the Feichtinger conjecture*, Proc. Amer. Math. Soc. **133** (2005), no. 4, 1025–1033. Zbl 1082.46018. MR2117203. DOI 10.1090/S0002-9939-04-07594-X. 537
6. P. G Casazza, G. Kutyniok, and S. Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal. **25** (2008), no. 1, 114–132. Zbl 1258.42029. MR2419707. DOI 10.1016/j.acha.2007.10.001. 542
7. O. Christensen, *An Introduction to Frames and Riesz Bases*, 2nd ed., Birkhäuser/Springer, Cham, 2016. Zbl 1348.42033. MR3495345. DOI 10.1007/978-3-319-25613-9. 537

8. I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), no. 5, 1271–1283. Zbl 0608.46014. MR0836025. DOI 10.1063/1.527388. 537
9. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), no. 2, 341–366. Zbl 0049.32401. MR0047179. DOI 10.2307/1990760. 537
10. M. Ehler, *Random tight frames*, J. Fourier Anal. Appl. **18** (2012), no. 1, 1–20. Zbl 1247.42026. MR2885555. DOI 10.1007/s00041-011-9182-5. 537
11. V. K. Goyal, J. Kovačević, and J. A. Kelner, *Quantized frame expansions with erasures*, Appl. Comput. Harmon. Anal. **10** (2001), no. 3, 203–233. Zbl 0992.94009. MR1829801. DOI 10.1006/acha.2000.0340. 537
12. X. Guo, *Characterizations of disjointness of g-frames and constructions of g-frames in Hilbert spaces*, Complex Anal. Oper. Theory **8** (2014), no. 7, 1547–1563. Zbl 1304.42072. MR3261712. DOI 10.1007/s11785-014-0364-4. 537
13. A. Khosravi and K. Musazadeh, *Fusion frames and g-frames*, J. Math. Anal. Appl. **342** (2008), no. 2, 1068–1083. Zbl 1143.42033. MR2445260. DOI 10.1016/j.jmaa.2008.01.002. 546
14. J. Leng, D. Han, and T. Huang, *Optimal dual frames for communication coding with probabilistic erasures*, IEEE Trans. Signal Process. **59** (2011), no. 11, 5380–5389. Zbl 1393.94747. MR2883607. DOI 10.1109/TSP.2011.2162955. 537
15. D. Li, J. Leng, and T. Huang, *Generalized frames for operators associated with atomic systems*, Banach J. Math. Anal. **12** (2018), no. 1, 206–221. Zbl 1382.42019. MR3745581. DOI 10.1215/17358787-2017-0050. 537
16. D. Li, J. Leng, T. Huang, and Q. Gao, *Frame expansions with probabilistic erasures*, Digit. Signal Process. **72** (2018), 75–82. MR3726074. DOI 10.1016/j.dsp.2017.10.002. 537
17. D. Li, J. Leng, T. Huang, and G. Sun, *On sum and stability of g-frames in Hilbert spaces*, Linear Multilinear Algebra **66** (2018), no. 8, 1578–1592. Zbl 1392.42029. MR3806240. DOI 10.1080/03081087.2017.1364338. 538
18. K. Musazadeh and H. Khandani, *Some results on controlled frames in Hilbert spaces*, Acta Math. Sci. Ser. B (Engl. Ed.) **36** (2016), no. 3, 655–665. Zbl 1363.42059. MR3479244. DOI 10.1016/S0252-9602(16)30029-7. 539, 544
19. A. Rahimi and A. Fereydooni, *Controlled g-frames and their g-multipliers in Hilbert spaces*, An. Ştiinţ Univ. "Ovidius" Constanţa Ser. Mat. **21** (2013), no. 2, 223–236. Zbl 1313.42094. MR3096895. DOI 10.2478/auom-2013-0035. 538, 539
20. T. Strohmer and R. W. Heath, Jr., *Grassmannian frames with applications to coding and communication*, Appl. Comput. Harmon. Anal. **14** (2003), no. 3, 257–275. Zbl 1028.42020. MR1984549. DOI 10.1016/S1063-5203(03)00023-X. 537
21. W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), no. 1, 437–452. Zbl 1129.42017. MR2239250. DOI 10.1016/j.jmaa.2005.09.039. 537, 538

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