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LACK OF MONOTONICITY IN FERROMAGNETIC ISING MODEL PHASE DIAGRAMS

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We study patterns of the phase diagram of ferromagnetic Ising models on graphs under an external magnetic field. We provide an example of a tree with only two types of vertices on which for a range of values of the external field there is a unique Gibbs distribution at low enough and at high enough temperatures, while at intermediate temperatures there is phase coexistence (in other words, a reentrance transition takes place).

1. Introduction. The investigation reported in this paper started with the following thoughts. The ferromagnetic Ising model can be seen as the most basic model in mathematical statistical mechanics and its natural mathematical setting is a graph, with the vertices being the sites, which we can think of as the locations of spins $+1$ or -1, and the edges indicating the pairs of spins which do interact. Two parameters appear in this model: an external field $h \in \mathbb{R}$ and a temperature $T \in \mathbb{R}_+$. A fundamental problem is then to locate on the phase diagram $h \times T$ the region where there is more than one Gibbs distribution (the so called "phase-coexistence" region). It is natural to ask what the general features of the phase-coexistence region are. Regarding its intersection with the T-axis (i.e., the reduced problem in which h is held fixed equal to 0), there is a well-known argument—to be reviewed later, when we have enough notation available—based on one of Griffiths' inequalities, which shows that this intersection is always an interval (possibly degenerate), which has one of its end points at $(h, T) = (0, 0)$. This simple feature of the phase-coexistence region can, of course, be seen as a monotonicity property: if there is a unique Gibbs measure at the point $(0, T_1)$ of the phase diagram, then the same is also true at all the points $(0, T)$, with $T > T_1$. The main point of this paper is that if $h \neq 0$, then the same sort of monotonicity in T is not true, even if we restrict ourselves to very simple graphs (our example will be an almost homogeneous tree, with only two types of vertices).

Before proceeding, we will have to introduce a certain amount of notation and terminology. We will also recall some well-known facts about the statistical mechanics of lattice systems and the Ising model, and we refer the reader to [1] and [5] for their proofs.

We will consider graphs with countably many vertices and locally bounded degree (meaning that each vertex belongs to a finite number of edges). This

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class of graphs will be denoted by $\mathscr{C}$. No assumption that the graph is connected or infinite is being made, but these are the typical cases in which one is interested. Given a graph $G \in \mathscr{C}$, we will use $V(G)$ to denote its set of vertices (also called sites) and $\mathscr{E}(G)$ to denote its set of edges. When there is no risk of confusion, G will be omitted in this notation. Two vertices are called neighbors if they belong to a common edge.

An isomorphism between two graphs, G_1 and G_2, is a one-to-one mapping from $V(G_1)$ onto $V(G_2)$, which preserves the graph structure, that is, such that the set of edges of G_2 can be obtained as the set of pairs of images of vertices of G_1 which form edges. An isomorphism between a graph G and itself is called an automorphism of G. Two vertices of G are said to be of the same type, if each one can be mapped into the other one by an automorphism of G. Graphs which have a single type of vertex will be called homogeneous graphs (sometimes the term transitive graphs is used in the literature). Graphs which have a finite number of types of vertices will be called almost-homogeneous graphs (sometimes the term almost-transitive graphs is used in the literature).

Configurations are elements of the set $\Omega = \{-1, +1\}^V$, interpreted as the assignment of a spin -1 or $+1$ to each site in V. Given $\sigma \in \Omega$ and $x \in V$, we use $\sigma(x)$ for the value of the spin at x. We will consider the formal ferromagnetic Ising Hamiltonian:

$$\mathscr{H}_h(\sigma) = -\sum_{\{x, y\} \in \mathscr{E}} \sigma(x)\sigma(y) - h \sum_x \sigma(x), \tag{1.1}$$

where $h \in \mathbb{R}$ is the external field and $\sigma \in \Omega$ is a generic configuration. In order to give precise definitions, we consider finite subsets of V. The expression $\Lambda \subset\subset V$ will mean that Λ is a finite subset of V. Given a set $\Lambda \subset\subset V$ we define also

$$\mathscr{E}_\Lambda = \{\{x, y\} \in \mathscr{E}: x, y \in \Lambda\},$$
$$\partial\mathscr{E}_\Lambda = \{\{x, y\} \in \mathscr{E}: x \in \Lambda, y \notin \Lambda\}.$$

Given also a configuration $\eta \in \Omega$, we define the following set of configurations:

$$\Omega_{\Lambda, \eta} = \{\sigma \in \Omega: \sigma(x) = \eta(x) \text{ for all } x \notin \Lambda\}.$$

For each set $\Lambda \subset\subset V$ and each boundary condition $\eta \in \Omega$, we define

$$\mathscr{H}_{\Lambda, \eta, h}(\sigma) = -\sum_{\{x, y\} \in \mathscr{E}_\Lambda} \sigma(x)\sigma(y) - \sum_{\substack{\{x, y\} \in \partial\mathscr{E}_\Lambda \\ y \notin \Lambda}} \sigma(x)\eta(y) - h \sum_{x \in \Lambda} \sigma(x). \tag{1.2}$$

The Gibbs (probability) measure in Λ with boundary condition η, under external field h and at temperature $T > 0$ is now defined on Ω as

$$\mu_{\Lambda, \eta, T, h}(\sigma) = \begin{cases} \dfrac{\exp(-\beta \mathscr{H}_{\Lambda, \eta, h}(\sigma))}{Z_{\Lambda, \eta, T, h}}, & \text{if } \sigma \in \Omega_{\Lambda, \eta}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\beta = 1/T$ and

$$Z_{\Lambda,\eta,T,h} = \sum_{\sigma\in\Omega_{\Lambda,\eta}} \exp(-\beta\mathscr{H}_{\Lambda,\eta,T,h}(\sigma)).$$

A Gibbs measure for the system on the graph G (possibly infinite) is defined as any probability measure μ which satisfies the DLR equations in the sense that for every $\Lambda \subset\subset V$ and μ-almost all $\eta \in \Omega$,

$$\mu(\,\cdot\,|\Omega_{\Lambda,\eta}) = \mu_{\Lambda,\eta,T,h}(\,\cdot\,). \tag{1.3}$$

It is easy to check and very important that this definition is self-consistent in case V is finite.

Alternatively and equivalently, Gibbs measures on a possibly infinite graph can be defined as limits of the corresponding Gibbs measures on finite subsets of the system, with arbitrary boundary conditions. For this purpose one says that a sequence of probability measures, $(\mu_n)_{n=1,2,\ldots}$, converges weakly to the probability measure ν in case

$$\lim_{n\to\infty} \int f\,d\mu_n = \int f\,d\nu, \tag{1.4}$$

for each $f\colon \Omega \to \mathbb{R}$ which depends only on the value of the spins on a finite set of sites. The set of Gibbs measures for the system on G coincides with the closed convex hull of the set of weak limit points of sequences of the form $(\mu_{\Lambda_i,\eta_i,T,h})_{i=1,2,\ldots}$, where each Λ_i is finite and $\Lambda_i \to V$, as $i\to\infty$, in the sense that $\bigcup_{i=1}^{\infty}\bigcap_{j=i}^{\infty}\Lambda_j = V$.

The set of Gibbs measures will be denoted by $\mathscr{G}_{T,h}$. The subset of the $h\times T$ half-plane where $\mathscr{G}_{T,h}$ has a single element is called the phase-uniqueness region, and its complement is called the phase-coexistence region. From Dobrushin's uniqueness condition (see, e.g., Chapter 8 of [1]), it is easy to see that if our graph G has the degree of all the vertices bounded by a common constant κ, then there are finite positive constants $h(\kappa)$ and $T(\kappa)$ such that if $|h| > h(\kappa)$, or $T > T(\kappa)$, the system is in the phase-uniqueness region.

For the expected value corresponding to a Gibbs measure $\mu_{\ldots}$, in finite or infinite volume, we will use the notation

$$\langle f\rangle_{\ldots} = \int f\,d\mu_{\ldots},$$

where $\cdots$ stands for arbitrary subscripts. We will use a common and convenient form of abuse of notation: $\sigma(x)$ will be used to denote the function which associates to each configuration the value of the spin at the site x in that configuration.

The Gibbs measures satisfy the following monotonicity relations, which we will refer to as the FKG-Holley inequalities:

$$\text{If } \eta \le \zeta \text{ and } h_1 \le h_2, \text{ then, for each } \Lambda \subset\subset V,\ \mu_{\Lambda,\eta,T,h_1} \le \mu_{\Lambda,\zeta,T,h_2}.$$

In what follows we will abbreviate by $+$ (resp. $-$) the configuration with all spins $+1$ (resp. -1). We will also use the notation $\pm$ in a standard way, with each equation in which it appears representing the two equations obtained by replacing this symbol consistently with $+$ or consistently with $-$.

A consequence of the FKG-Holley inequalities in combination with the DLR equations is that

$$\mu_{\Lambda,\pm,T,h} \to \mu_{\pm,T,h} \quad \text{weakly as } \Lambda \to V. \tag{1.5}$$

The Gibbs distributions $\mu_{+,T,h}$ and $\mu_{-,T,h}$ so obtained are called, respectively, the $(+)$-phase and the $(-)$-phase of the Ising model on G. Moreover, the following three statements are equivalent:

(U1) $|\mathscr{G}_{T,h}| = 1$;
(U2) $\mu_{-,T,h} = \mu_{+,T,h}$;
(U3) $\langle\sigma(x)\rangle_{-,T,h} = \langle\sigma(x)\rangle_{+,T,h}$, for each site $x \in V$.

In case the graph G is not connected, the equality in (U3) can be satisfied for some sites x while it fails for others. On the other hand, if G is connected, that equality is either true for all sites x or false for all sites x. To prove this last statement one can proceed as follows. Suppose that the equality fails for one given site x. Consider next a site y which is neighbor to x. By conditioning on the configuration in the neighbors of y and using the Holley-FKG inequalities, one can readily see that the equality in (U3) also fails for y. By proceeding inductively, one concludes then that once the equality in (U3) fails for one site, it will fail for all sites which belong to the connected component of the graph to which this site belongs.

The Ising model has been mostly studied on the cubic lattices, $V = \mathbb{Z}^d$, with edges connecting sites which are separated by Euclidean distance 1. In this fundamental case, the phase-coexistence region is contained in the T-axis. In case $d = 1$, it is empty, while for $d \geq 2$, it is a nondegenerate interval contained in this axis and of the form $(0, T_c]$. The proofs that there is uniqueness of the Gibbs distribution for the Ising model on these graphs when $h \neq 0$ depend only on the facts that these graphs are homogeneous, and the number of sites at distance N from a fixed site grows slower than the number of sites at distances smaller than N from this site, as $N \to \infty$.

Also frequently studied is the Ising model on a homogeneous tree, $\mathbb{T}_b$. In this notation, b is the branching number, so that each site has $b+1$ neighbors. Note that $\mathbb{T}_1 = \mathbb{Z}^1$. In the case $b \geq 2$, the phase-coexistence region is not confined to the T-axis; it is given by the set

$$\big\{(h, T)\colon 0 < T < T_c,\ -h_c(T) \leq h \leq h_c(T)\big\}, \tag{1.6}$$

where

$$h_c(T) = T \max_{t \geq 0}(b\varphi_\beta(t) - t), \tag{1.7}$$

with

$$\varphi_\beta(t) = \frac{1}{2}\log\left(\frac{\cosh(t+\beta)}{\cosh(t-\beta)}\right) = \tanh^{-1}(\tanh(\beta)\tanh(t)), \tag{1.8}$$

and

$$T_c = \sup\{T > 0\colon h_c(T) > 0\} = \frac{1}{\coth^{-1}(b)} = \frac{2}{\log((b+1)/b-1)}. \tag{1.9}$$

The following properties of the function φ_β are important. It is a continuous odd function, which is strictly increasing and has range $(-\beta, \beta)$. It is also strictly convex on $(-\infty, 0]$ and strictly concave on $[0, \infty)$.

Before proceeding, we recall the well-known argument to the effect that if there is a unique Gibbs measure at the point $(0, T_1)$ of the phase diagram, then the same is also true at all the points $(0, T)$, with $T > T_1$. For this purpose one uses the symmetry between +s and −s in the case $h = 0$, and writes

$$\begin{aligned} 0 &\leq \langle\sigma(x)\rangle_{+,\Lambda,T,0} - \langle\sigma(x)\rangle_{-,\Lambda,T,0} = 2\langle\sigma(x)\rangle_{+,\Lambda,T,0} \\ &\leq 2\langle\sigma(x)\rangle_{+,\Lambda,T_1,0} = \langle\sigma(x)\rangle_{+,\Lambda,T_1,0} - \langle\sigma(x)\rangle_{-,\Lambda,T_1,0}. \end{aligned}$$

Here the first inequality is an instance of the FKG-Holley inequalities, while the second inequality is an instance of one of Griffiths' inequalities. Letting $\Lambda \to V$, and using the equivalence between (U1) and (U3), one then obtains phase uniqueness at $(T, 0)$ from phase uniqueness at $(T_1, 0)$.

A similar well-known argument can be used to show that if we delete edges from a graph on which there is phase uniqueness at a certain point $(0, T)$, then at the same point of the phase diagram, there will be phase uniqueness for the new graph.

Note nevertheless that the absence of symmetry between $+s$ and $-s$ when $h \neq 0$ spoils the argument above, in that case.

We describe next a tree with only two types of sites (an almost-homogeneous tree), on which for certain values of $h \neq 0$, phase uniqueness at certain values of T does not imply phase uniqueness at larger values of T.

Our tree can be constructed from $\mathbb{T}_b$, by adding vertices and edges to it. In this procedure, each vertex of $\mathbb{T}_b$ is connected to A new vertices, by means of A new edges. These new vertices are not connected to any other vertex, so that they are leaves of the graph. This completes the construction of the graph, which we denote by $\mathbb{T}_{b,A}$. This graph has two types of vertices; those of first type are the ones with which we started; the ones we added are of the second type. Each vertex of first type is connected to $b+1$ other vertices of the same type and to A vertices of the second type. Each vertex of the second type is connected to exactly one vertex of the first type and to no vertex of the second type.

PROPOSITION 1. *For the tree $\mathbb{T}_{b,A}$ with a proper choice of b and A (e.g., $A = 2b$ and b large enough) for h in a nondegenerate interval which contains* 1 *the following happens. There are values of T for which there is phase coexistence, but for T either large enough or small enough there is phase uniqueness.*

In the proof of this proposition, we will see that when $A > 0$ the phase-coexistence region of $\mathbb{T}_{b,A}$ is a strict subset of that of $\mathbb{T}_b$. It is clear that the graph that has the same set of vertices as $\mathbb{T}_{b,A}$ but only has edges connecting the sites corresponding to sites of the first type in $\mathbb{T}_{b,A}$ has the same phase diagram as $\mathbb{T}_b$ (the other sites are not connected to anything, and in particular the graph is not connected, but still it is in $\mathscr{C}$). Hence the conclusion that the addition of edges to a graph may reduce the phase-coexistence region, contrary to what happens with the intersection of this region and the T-axis.

The lack of monotonicity in T exemplified in Proposition 1 contrasts with the monotonicity in $|h|$ expressed in the next one.

PROPOSITION 2. *For all graphs in $\mathscr{C}$, if there is a unique Gibbs measure at the point (h_1, T) of the phase diagram, then the same is also true at all the points (h, T), with $|h| > |h_1|$.*

PROOF. With no loss of generality we can take $0 \le h_1 \le h$. The following correlation inequalities are then available, where $\Lambda \subset\subset V$ and $x \in \Lambda$,

$$(1.10)\qquad 0 \le \langle\sigma(x)\rangle_{+,\Lambda,T,h} - \langle\sigma(x)\rangle_{-,\Lambda,T,h} \le \langle\sigma(x)\rangle_{+,\Lambda,T,h_1} - \langle\sigma(x)\rangle_{-,\Lambda,T,h_1}.$$

Here the first inequality is an instance of the FKG-Holley inequalities, while the second inequality can be obtained from Lebowitz's inequalities for duplicated spins in [4]. (The derivation appears in the second step of the proof of Theorem 2 in [3], page 6, where only the case $0 = h_1 \le h$ is considered, but the same argument works as well for $0 \le h_1 \le h$. The reader should also beware that Higuchi uses a slight generalization of the work in [4], in which now different external fields act on each copy of the spin system.) Letting $\Lambda \to V$ in (1.10), and using the equivalence between (U1) and (U3), one obtains phase uniqueness at (T, h) from phase uniqueness at (T, h_1). □

Section 2 will be dedicated to the proof of Proposition 1. Here we give a heuristic argument, which makes this proposition at least plausible. The idea in this heuristic argument is to consider the spins at the sites of first type in $\mathbb{T}_{b,A}$ as being the sites of the smaller graph $\mathbb{T}_b$, and the spins at the sites of second type (the leaves of $\mathbb{T}_{b,A}$) as providing an extra effective external field which acts on the sites of first type, in addition to the external field h.

For the heuristic argument, we will need two facts about the phase diagram of the Ising model on $\mathbb{T}_b$. The first one is

$$(1.11)\qquad \lim_{T\to 0} h_c(T) = b - 1.$$

One way to obtain (1.11) is by computing $h_c(T)$ explicitly from its definition (1.7). This is done in [1], where the result appears as equation (12.30). Their J is our β, d is our b and $h(J, d)$ is our $\beta h_c(T)$. The behavior of $h_c(T)$ as $T \to 0$ is also found in [1] [the first display after (12.30)], and in particular one has (1.11).

The second fact is that for fixed T,

$$(1.12)\qquad \lim_{b\to\infty} \frac{h_c(T)}{b} = 1.$$

This is a straightforward consequence of (1.7) and the properties of φ_β.

Suppose that h is slightly larger than 1. At low temperature the spins at the sites of second type will, with overwhelming probability, be aligned with this external field and so be $+1$. This is so because in this state the contribution to the energy from each such spin is lower than in the opposite state, regardless of the state of the spin at the site of first type neighbor to the site that we are considering. But then the spins at the A sites of second type, neighbors to a site of first type x, produce an effective extra external field acting on the spin at x of magnitude close to A. This should be contrasted with (1.11), which tells us that $h_c(T)$ is close to $b-1$. The total effective external field close to $A+1=2b+1$ should therefore be enough to assure phase uniqueness.

We suppose that b is so large that we can take a temperature large compared with 1, while still small compared with b. When the temperature is much larger than 1, there should be a substantial entropy effect affecting the spins at the leaves of the tree and the extra effective field acting on the spins at sites of first type should be just a small fraction of $A=2b$. Contrasting this with (1.12), we see that, if b is large, now the effective field is no longer strong enough to bring the system into the phase-uniqueness region.

At larger temperatures, of course, Dobrushin's uniqueness condition tells us that phase uniqueness will again be restored.

2. Proof of Proposition 1. When studying Gibbs measures on trees, it is natural to look for recursions. In such an approach, success depends on making a good choice of the quantities for which the recursion is written. Our choice of $L_{x,n,\eta}$ below as this quantity was motivated by [2].

For the moment the setting is an arbitrary connected tree G in $\mathscr{C}$. Call one of the vertices of the tree its root, denoted by 0. Each vertex x in the tree has a generation index $g(x)$, defined inductively by setting $g(0)=0$, and giving the generation index $n+1$ to the vertices which are neighbors of a vertex with index n unless they already have index $n-1$. We write $x \to_1 y$ in case x and y belong to a common edge and $g(y)=g(x)+1$. A vertex z is a descendent of a vertex x if there is a sequence of vertices $x=x_0, x_1, \dots, x_i=z$ such that $x_j \to_1 x_{j+1}$, for $j=0,\dots,i-1$.

Given $x \in V$, we define the tree G_x, obtained from the original tree G that we are considering by only keeping the vertex x and its descendents, and keeping all the edges connecting any two of these vertices. Given also $n \ge g(x)$, we define $V_{x,n}$ as the set of vertices containing x and its descendents with generation index not exceeding n. Note that $V_{x,n}$ is a subset of the set of vertices of G_x.

In what follows T and h are fixed, and will be omitted from the new notation being introduced. First define

$$\begin{aligned}\mathscr{H}_{x,n,\eta}(\sigma) = &- \sum_{\substack{\{y,z\}\in\mathscr{E}(G_x)\\ y\in V_{x,n}\ z\in V_{x,n}}} \sigma(y)\sigma(z)\\ &- \sum_{\substack{\{y,z\}\in\mathscr{E}(G_x)\\ y\in V_{x,n}\ z\notin V_{x,n}}} \sigma(y)\eta(z) - h\sum_{y\in V_{x,n}} \sigma(y).\end{aligned} \tag{2.1}$$

Now, given $A \subset \{-1, +1\}^{V_{x,n}}$, define

$$Z_{x,n,\eta}(A) = \sum_{\sigma \in A} \exp\bigl(-\beta \mathscr{H}_{x,n,\eta}(\sigma)\bigr).$$

Next set

$$L_{x,n,\eta} = \frac{1}{2} \log \frac{Z_{x,n,\eta}(\sigma(x) = +1)}{Z_{x,n,\eta}(\sigma(x) = -1)}.$$

We can now write the following recursion:

$$\begin{aligned} L_{x,n,\eta} = \tfrac{1}{2} \log \Biggl\{ \Biggl[e^{\beta h} \prod_{y\colon x \to_1 y} \Bigl\{ & e^{\beta} Z_{y,n,\eta}(\sigma(y) = +1) \\ & + e^{-\beta} Z_{y,n,\eta}(\sigma(y) = -1) \Bigr\} \Biggr] \\ \times \Biggl[e^{-\beta h} \prod_{y\colon x \to_1 y} \Bigl\{ & e^{-\beta} Z_{y,n,\eta}(\sigma(y) = +1) \\ & + e^{\beta} Z_{y,n,\eta}(\sigma(y) = -1) \Bigr\} \Biggr]^{-1} \Biggr\} \\ = \beta h + \sum_{y\colon x \to_1 y} \varphi_\beta(L_{y,n,\eta}), & \end{aligned} \tag{2.2}$$

where φ_β is defined by (1.8).

We will write $l_{n,\eta} = L_{0,n,\eta}$ and $V_{0,n} = V_n$. Note that

$$l_{n,\eta} = \frac{1}{2} \log \frac{\mu_{V_n,\eta,T,h}(\sigma(0) = +1)}{\mu_{V_n,\eta,T,h}(\sigma(0) = -1)}. \tag{2.3}$$

In what follows, we will suppose that η is either the configuration with all spins -1 or that with all spins $+1$. From (1.5) and (2.3) we obtain

$$l_\pm := \lim_{n\to\infty} l_{n,\pm} = \frac{1}{2} \log \frac{\mu_{\pm,T,h}(\sigma(0) = +1)}{\mu_{\pm,T,h}(\sigma(0) = -1)}.$$

A simple computation now yields

$$\langle \sigma(0) \rangle_{\pm,T,h} = \tanh(l_\pm).$$

Therefore the remark in the paragraph after the equivalent conditions (U1)–(U3) were introduced implies that for an arbitrary connected tree $G \in \mathscr{C}$, also

$$|\mathscr{G}_{T,h}| = 1 \quad \Leftrightarrow \quad l_- = l_+. \tag{2.4}$$

We want to use the equivalence in (2.4) to find the coexistence region for the tree $\mathbb{T}_{b,A}$. From this point on we will use the notation above, having in mind that it refers now to the tree $G = \mathbb{T}_{b,A}$.

Before we can study $\mathbb{T}_{b,A}$, we will have to consider a related but somewhat different tree $G' = \mathbb{T}'_{b,A}$, and for this tree we will use similar notation, but distinguished by a prime. The tree $\mathbb{T}'_{b,A}$ is obtained from $\mathbb{T}_{b,A}$ by removing one of the neighbors of its root, as well as all the descendents of this site. All the edges from $\mathbb{T}_{b,A}$ connecting sites which are not being removed are kept.

The tree $G' = \mathbb{T}'_{b,A}$ has the particularly nice feature that for each of its vertices x, the corresponding tree G'_x [as defined above, before (2.1) was introduced] is isomorphic to $\mathbb{T}'_{b,A}$ itself. This makes the recursion (2.2) become particularly simple and yields

$$l'_{n,\pm} = \beta h + A\varphi_\beta(\beta h) + b\varphi_\beta(l'_{n-1,\pm}), \tag{2.5}$$

where the second term comes from summing over the leaves among the neighbors of the root. We will introduce the notation

$$H^A_{\mathrm{eff}}(\beta, h) = \beta h + A\varphi_\beta(\beta h), \tag{2.6}$$

so that (2.5) turns into

$$l'_{n,\pm} = H^A_{\mathrm{eff}}(\beta, h) + b\varphi_\beta(l'_{n-1,\pm}). \tag{2.7}$$

From the last display and the continuity of φ_β, it follows that $l'_- = \lim_{n\to\infty} l'_{n,-}$ and $l'_+ = \lim_{n\to\infty} l'_{n,+}$ are solutions of the following equation in t

$$t = H^A_{\mathrm{eff}}(\beta, h) + b\varphi_\beta(t). \tag{2.8}$$

This equation is precisely equation (12.22) of [1], where the case of the homogeneous trees is studied. One simply has to replace the variable J in [1] with our β, d in [1] with our b and the h in [1] with our $H^A_{\mathrm{eff}}(\beta, h)$. From the analysis in [1] of this equation, we know that (2.8) has exactly one solution if $T \geq T_c$ or $|H^A_{\mathrm{eff}}(\beta, h)| > \beta h_c(T)$, where T_c is given by (1.9) and $h_c(T)$ is given by (1.7). It has exactly two solutions if $T < T_c$ and $|H^A_{\mathrm{eff}}(\beta, h)| = \beta h_c(T)$. And it has exactly three solutions in case $T < T_c$ and $|H^A_{\mathrm{eff}}(\beta, h)| < \beta h_c(T)$. We will denote by t_- the smallest solution of (2.8) and by t_+ the largest solution of that equation.

Next we want to argue that

$$l'_\pm = t_\pm. \tag{2.9}$$

For this purpose, we extend the definition of φ_β by continuity, setting

$$\varphi_\beta(\pm\infty) = \lim_{t\to\pm} \varphi_\beta(t) = \pm\beta.$$

We also define

$$l'_{-1,\pm} = \pm\infty,$$

so that a direct computation shows that

$$l'_{0,\pm} = H^A_{\mathrm{eff}}(\beta, h) \pm b\beta = H^A_{\mathrm{eff}}(\beta, h) + b\varphi_\beta(l'_{-1,\pm}).$$

In other words, (2.7) is now satisfied for $n = 0, 1, 2, \dots$.

For $t > t_+$ we have $t > H^A_{\mathrm{eff}}(\beta, h) + b\varphi_\beta(t)$ (compare the limits of both sides as $t \to \infty$ and note that by continuity of the functions on both sides, the inequality between them must be the same for all $t > t_+$). It is therefore clear that the recursion (2.7), started from $l'_{-1,+} = +\infty$, produces a decreasing sequence $(l'_{n,+})_{n\geq -1}$, bounded below by t_+. This decreasing sequence must then converge to a fixed point of the recursion, not smaller than t_+, and hence it must converge to t_+. This proves (2.9) in case $\pm$ is replaced with $+$. The case in which it is replaced with $-$ is analogous.

From (2.4), (2.9) and the behavior of the solutions of (2.8) for different values of T and h, we conclude that the phase-coexistence region for the Ising model on $\mathbb{T}'_{b,A}$ is the set

$$\{(h,T)\colon t_- < t_+\} = \{(h,T)\colon 0 < T < T_c,\ -h_c^A(T) \leq h \leq h_c(T)\}, \tag{2.10}$$

where T_c is still given by (1.9), and

$$h_c^A(T) = \max\{h \geq 0\colon H^A_{\text{eff}}(\beta,h) \leq \beta h_c(T)\}, \tag{2.11}$$

with $h_c(T)$ given by (1.7). In justifying the equality in (2.10), note that $H^A_{\text{eff}}(\beta,h)$ is a strictly increasing function of h and $H^A_{\text{eff}}(\beta,0) = 0$.

Regarding the phase-coexistence region for the Ising model on the tree $G = \mathbb{T}_{b,A}$, we can readily see that it is the same set (2.10) above. For this, note that the recursion (2.2), applied in case x is the root of this graph, gives

$$l_{n,\pm} = \beta h + A\varphi_\beta(\beta h) + (b+1)\varphi_\beta(l'_{n-1,\pm}) = H^A_{\text{eff}}(\beta,h) + (b+1)\varphi_\beta(l'_{n-1,\pm}).$$

This is so because for each site y different from the origin, G_y is isomorphic to $\mathbb{T}'_{b,A}$. Letting $n \to \infty$ gives

$$l_\pm = H^A_{\text{eff}}(\beta,h) + (b+1)\varphi_\beta(l'_\pm) = H^A_{\text{eff}}(\beta,h) + (b+1)\varphi_\beta(t_\pm).$$

Since φ_β is strictly monotone increasing, we can conclude now that

$$l_- = l_+ \iff t_- = t_+,$$

so that (2.4) implies that indeed the phase-coexistence region for the Ising model on $\mathbb{T}_{b,A}$ is given by (2.10). Incidentally, note that the analysis above includes a review of a proof that (1.6) is the phase coexistence region for the Ising model on the homogeneous tree $\mathbb{T}_b$ (the case $A = 0$). Also, from (2.11) and (2.6), it is clear that the phase-coexistence region shrinks as A grows. This justifies the claim that we made after the statement of Proposition 1.

Our problem of proving Proposition 1 is now reduced to the study of the phase-coexistence region, as described by (2.10), and in particular of the behavior of $h_c^A(T)$, given by (2.11), as a function of T. According to the statement of that proposition, we take $A = 2b$, and we will at some points below also have to choose b large enough.

Note that from the definition (1.9) of T_c, it follows that

$$\lim_{b\to\infty} T_c/b = 1. \tag{2.12}$$

This, of course, shows that

(2.13) If b is large enough, for $T = 2b$ we have $T > T_c$.

We will show next that

(2.14) If b is large enough, for $T = b/2$ we have $T < T_c$ and $h_c^A(T) > 1$.

The claim about T_c in (2.14) is immediate from (2.12). The first step in the derivation of the claim about $h_c^A(T)$ in (2.14) is a sort of mean-field bound. From (1.7) we have

$$\begin{aligned}\beta h_c(T) &= \max_{t\geq 0}(b\varphi_\beta(t) - t)\\ &= \max_{t\geq 0}\big(b\tanh^{-1}(\tanh(2/b)\tanh(t)) - t\big)\\ &\geq \max_{t\geq 0}\big(b\tanh(2/b)\tanh(t) - t\big).\end{aligned}$$

Therefore, for large enough b,

$$\beta h_c(T) \geq \max_{t\geq 0}\big(\tfrac{3}{2}\tanh(t) - t\big) > 0,$$

where the last inequality is straightforward from the fact that at $t = 0$, $d\tanh(t)/dt = 1$. The important fact here is that the factor 3/2 multiplying $\tanh(t)$ is larger than 1. The reader can see in this estimate the connection with the mean field model. Note that the lower bound obtained is uniform for all large b.

On the other hand, for $h = 1$,

$$\begin{aligned}H^A_{\mathrm{eff}}(\beta, h) = H^A_{\mathrm{eff}}(\beta, 1) &= \beta + A\varphi_\beta(\beta)\\ &= 2/b + 2b\varphi_{2/b}(2/b) = 2/b + 2b\tanh^{-1}(\tanh(2/b)\tanh(2/b))\\ &= 2/b + b\,O(1/b^2).\end{aligned}$$

From the last two displays and the definition (2.11) we conclude that for b large enough,

$$h_c^A(T) = \max\{h \geq 0\colon H^A_{\mathrm{eff}}(\beta, h) \leq \beta h_c(T)\} > 1,$$

which is our claim about $h_c^A(T)$ in (2.14).

Next we show that

(2.15) For each b, for T small enough (depending on b) we have $h_c^A(T) < 1$.

For this we first note that for $h = 1$,

$$\frac{H^A_{\mathrm{eff}}(\beta, h)}{\beta} = 1 + A\frac{\varphi_\beta(\beta)}{\beta} \to 1 + A = 1 + 2b \quad \text{as } T \to 0.$$

Comparing this behavior with the behavior of $h_c(T)$, as given by (1.11) (which, as explained, is derived from an explicit computation that can be found in [1]), we obtain the claim (2.15) from the definition (2.11) of $h_c^A(T)$.

Proposition 1 is a consequence of (2.13), (2.14) and (2.15), and the fact that the phase-coexistence region for the the Ising model on $\mathbb{T}_{b,A}$ is given by (2.10). As a remark, it is worth noting that (2.14) and (2.15) are really the important estimates. In the proof of Proposition 1 we could replace the use of (2.13) with the fact that from Dobrushin's uniqueness condition, for each b, for T large enough (depending on b), there is a unique Gibbs distribution regardless of the value of h.

3. Some open problems.

1. The graph $\mathbb{T}_{b,A}$ is almost homogeneous but not homogeneous. Is there any homogeneous graph on which the Ising model presents for some value of the external field and three values of the temperature, $T_1 < T_2 < T_3$, phase uniqueness at temperatures T_1 and T_3 and phase coexistence at temperature T_2?
2. Is there any graph on which at some temperature T_1 the Ising model presents phase coexistence if and only if $h = 0$, but at some larger temperature $T_2 > T_1$ it presents phase coexistence also for some nonnull value of the external field?

Note added in revision. After this paper was completed, problem 2 above was solved by M. Salzano, who obtained a graph with the required property. As far as we know, problem 1 is still open.

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