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MAXIMA OF POISSON-LIKE VARIABLES AND RELATED TRIANGULAR ARRAYS[1]

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It is known that maxima of independent Poisson variables cannot be normalized to converge to a nondegenerate limit distribution. On the other hand, the Normal distribution approximates the Poisson distribution for large values of the Poisson mean, and maxima of random samples of Normal variables may be linearly scaled to converge to a classical extreme value distribution. We here explore the boundary between these two kinds of behavior. Motivation comes from the wish to construct models for the statistical analysis of extremes of background gamma radiation over the United Kingdom. The methods extend to row-wise maxima of certain triangular arrays, for which limiting distributions are also derived.

1. Introduction. One result of the public concern aroused by the Chernobyl accident in 1986 was the setting up in several Western European countries of a network of independent monitoring stations for background gamma radiation [the Argus Project: see En Garde (1993)]. In the United Kingdom more than 15 monitoring stations have now been in operation for several years. Each continuously records the arrival of γ-rays and once a day downloads aggregated 10-minute counts to a central data bank. The large volume of data thus accumulated offers an unprecedented opportunity to explore the temporal and spatial patterns of variation of background radiation. Of particular interest in any analysis of the data are the unusually high values of radiation, since the occurrence of exceptionally high levels may be indicative of some further accidental nuclear emission. This motivates the search for statistical models on which to base an analysis of the spatial and temporal characteristics of extremes of background gamma radiation.

The physical laws which govern the behavior of radiation emission suggest that counts over fixed periods should follow a Poisson law. Fluctuations in both meteorology and atmospheric conditions as well as imperfections in recording devices cause modifications in this basic law, resulting in nonstationarity and perturbations in the marginal Poisson behavior. The aim of this paper is to develop a framework for modelling the extremal behavior of a sequence of Poisson variables, which is robust to misspecification of the marginal Poisson distribution. This falls in the ambit of classical extreme value theory: given a sequence of independent variables $X_1, \dots, X_n$ with common distribution func-

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tion F, a sequence of normalizing functions $u_n(x)$ is sought such that

$$P\Big(\max_{1\le i\le n} X_i \le u_n(x)\Big) \to G(x), \tag{1}$$

where G is a nondegenerate distribution function. In applications, F is generally unknown, but the class of possible limits G, usually referred to as the extreme value family, is sufficiently narrow to permit modelling of $\max_{1\le i\le n} X_i$ directly as G.

In the case of Poisson variables with mean λ, this argument fails. Anderson (1970, 1980) studied the case where the X_i are Poisson variables and found that there is a sequence of integers I_n for which $\lim_{n\to\infty} P(\max_{1\le i\le n} X_i = I_n \text{ or } I_n + 1) = 1$, so that no normalizing functions $u_n(x)$ can be found which lead to nondegenerate limits in (1). Figure 1 illustrates this behavior for $\lambda = 2$. As n increases, the distribution of $\max_{1\le i\le n} X_i$ concentrates increasingly on a pair of consecutive integers. The asymptotic properties of the sequence of integers I_n have been characterized by Kimber (1983). This argument gives no justification therefore for modelling Poisson maxima through extreme value distributions.

The theme of this paper is the presentation of an argument which nevertheless justifies the use of extreme value distributions for modelling Poisson maxima when λ is sufficiently large. Our reasoning is as follows: for large λ the Poisson distribution can be approximated by a Normal distribution. If

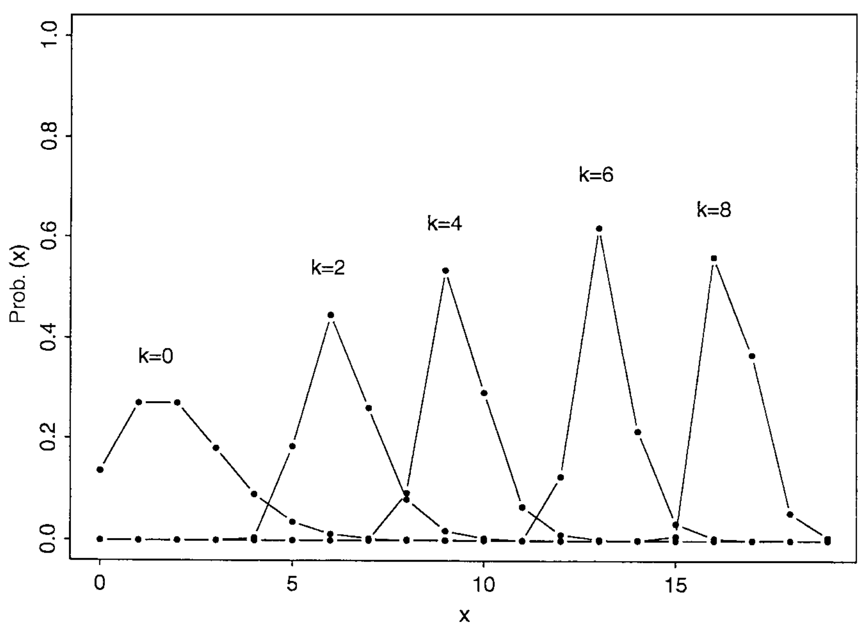

FIG. 1. *Distribution of the maximum of $n = 10^k$ independent Poisson random variables with mean 2.*

the X_i in (1) are Normal, it is well known that u_n can be found so that the Gumbel limit $G(x) = \exp(-e^{-x})$ is obtained. Consequently, using the Normal approximation first and then applying (1), we obtain a Gumbel approximation to the distribution of Poisson maxima. This limiting behavior is formalized in Sections 2 and 3 and supported by numerical calculations in Section 5. The argument hinges critically on the relative rates of convergence of the Poisson–Normal limit and the extreme value limit in (1). The sharpness of the condition for a Gumbel limit is shown to depend on the number, r, of terms used in a series expansion for the tail behavior of the Poisson distribution. In Section 6 it is shown too that if the Normal convergence is too slow, then the standard degenerate behavior of Poisson maxima persists.

Our arguments for a Gumbel limit do not in fact depend critically on the random variables being Poisson. In Section 4 we show how they extend to row-wise maxima of certain triangular arrays of variables, each converging in distribution to normality. We also present some further results about maxima of triangular arrays, which show that in non-Poisson heavy-tailed cases, both the Gumbel and Fréchet extreme value distributions and a related nonextreme value distribution may also arise as limits.

2. The main limit result.

For each positive integer n, let $R_{n,i}, i = 1, \dots, n$ denote independent Poisson random variables with mean λ_n growing with n. We study $\max_{1 \le i \le n} R_{n,i}$ as $n \to \infty$. As λ_n grows, the Poisson distribution of each $R_{n,i}$ approaches normality, and so we might expect that for rapidly increasing λ_n, normality would set in quickly enough for the distribution of $\max_{1 \le i \le n} R_{n,i}$ to resemble that of the maximum of independent Normal variables. The results to follow show that this is indeed the case, and they give appropriate growth rates for λ_n which guarantee it.

The question we address is the following: when is it possible to find functions $u_n(x)$ and a nondegenerate distribution function $G(x)$ such that

$$P\Big(\max_{1 \le i \le n} R_{n,i} \le u_n(x)\Big) \to G(x), \tag{2}$$

as $n \to \infty$, and what forms do u_n and G then take?

Since the $R_{n,i}$ are independent, (2) is equivalent to

$$\lim_{n\to\infty} nP(R_{n,1} > u_n(x)) = -\log G(x), \tag{3}$$

and this is the expression we mainly work with. An estimate of the probability in (3) may be obtained from the large deviations results of central limit theory. The main result used here, Cramér's theorem [see, for example, Petrov (1975), page 218], applies to independent identically distributed random variables X_i whose moment generating function exists in a neighborhood of the origin. If $E(X_i) = 0$, $\operatorname{Var} X_i = \sigma^2$ and $S_n = \sum_1^n X_i$, then for x varying with n in such a way that $x \to \infty$ and $x = o(n^{1/2})$,

$$\frac{P(S_n/\sigma n^{1/2} > x)}{1 - \Phi(x)} = \exp\left(x^2 C\left(\frac{x}{n^{1/2}}\right)\right)\left[1 + O\left(\frac{x}{n^{1/2}}\right)\right], \tag{4}$$

where $C(\cdot)$ is a power series

$$C(z) = c_1 z + c_2 z^2 + \cdots$$

whose coefficients are determined by the moments of the X_i, c_j being a function of moments of order $j+2$ and lower. We apply this result initially to centered unit Poisson variables X_i, replacing n by λ_n, so that S_n in (4) follows the same centered Poisson distribution as $R_{n,i} - \lambda_n$. Thus

$$P\left(\frac{R_{n,1} - \lambda_n}{\lambda_n^{1/2}} > x_n\right) \sim (1 - \Phi(x_n)) \exp(x_n^2 C(x_n/\lambda_n^{1/2})), \tag{5}$$

when $x_n = o(\lambda_n^{1/2})$. The first coefficient of $C(\cdot)$ in this case, for example, is $c_1 = \mu_3/6\sigma^3 = 1/6$, μ_3 being the third moment of X_i. This argument appears to require that the sequence $\{\lambda_n\}$ should be integer valued, but in fact a check on the proof of Cramér's theorem shows that it works also for continuously varying λ_n.

By taking the first $r \geq 0$ terms of the $C(\cdot)$ series, we see from (5) that

$$P\left(\frac{R_{n,1} - \lambda_n}{\lambda_n^{1/2}} > x_n\right) \sim (1 - \Phi(x_n)) \exp\left\{c_1 \frac{x_n^3}{\lambda_n^{1/2}} + \cdots + c_r \frac{x_n^{r+2}}{\lambda_n^{r/2}}\right\}, \tag{6}$$

for $x_n^{r+3} = o(\lambda_n^{(r+1)/2})$. Thus the convergence (2) will certainly occur if it is possible to find $u_n(x)$ satisfying

$$\frac{u_n(x) - \lambda_n}{\lambda_n^{1/2}} = x_n = o(\lambda_n^{(r+1)/(2(r+3))}) \tag{7}$$

and

$$n(1 - \Phi(x_n)) \exp\left\{c_1 \frac{x_n^3}{\lambda_n^{1/2}} + \cdots + c_r \frac{x_n^{r+2}}{\lambda_n^{r/2}}\right\} \to -\log G(x). \tag{8}$$

We prove the existence of x_n and G satisfying (7) and (8) in two steps. First we show that if there is a sequence β_n for which

$$n(1 - \Phi(\beta_n)) \exp\left\{c_1 \frac{\beta_n^3}{\lambda_n^{1/2}} + \cdots + c_r \frac{\beta_n^{r+2}}{\lambda_n^{r/2}}\right\} \to 1, \tag{9}$$

then there is a positive sequence α_n for which (8) holds with $x_n = \alpha_n x + \beta_n$ for each fixed x. Then we show that a β_n satisfying (9) can always be found. Proofs of these facts are contained in Lemmas 1 and 2 in Section 3. It is shown moreover in Lemmas 1 and 2 that, whatever the value of r, for x_n of this form, the limit in (8) is

$$-\log G(x) = e^{-x},$$

so that G is a Gumbel distribution; and for each fixed x,

$$x_n \sim \beta_n \sim (2 \log n)^{1/2}.$$

It follows that (7) is satisfied provided

$$\log n = o(\lambda_n^{(r+1)/(r+3)}),$$

and so the convergence in (2) occurs when λ_n grows faster than $(\log n)^{1+2(r+1)^{-1}}$.

Our main technical findings may therefore be summarized in the following.

PROPOSITION 1. *Let $R_{n,i}$ denote independent Poisson random variables with mean λ_n, and suppose that λ_n grows with n in such a way that for some integer $r \geq 0$,*

$$\log n = o(\lambda_n^{(r+1)/(r+3)}).$$

Then there is a linear normalization

$$u_n(x) = \lambda_n + \lambda_n^{1/2}(\beta_n^{(r)} + \alpha_n x)$$

such that

$$\lim_{n\to\infty} P\Big(\max_{1\leq i\leq n} R_{n,i} \leq u_n(x)\Big) = \exp(-e^{-x}).$$

The constants α_n and $\beta_n^{(r)}$ are specified more fully in the following section.

For the special case when $r = 0$, these facts may be deduced from the large deviations result (6) and known extreme value properties of the Normal distribution. We briefly outline the details for later use. When $r = 0$, relation (6) becomes

$$P\left(\frac{R_{n,1} - \lambda_n}{\lambda_n^{1/2}} > x_n\right) \sim (1 - \Phi(x_n)),$$

for $x_n = o(\lambda_n^{1/6})$, and it is well known [see, e.g., Leadbetter, Lindgren and Rootzén (1983), page 14] that

$$\lim_{n\to\infty} n(1 - \Phi(\alpha_n x + \beta_n^{(0)})) = e^{-x},$$

for

$$\alpha_n = (2\log n)^{-1/2}$$

and

$$\beta_n^{(0)} = (2\log n)^{1/2} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{1/2}}. \qquad (10)$$

Thus, provided

$$(2\log n)^{1/2} \sim \alpha_n x + \beta_n^{(0)} = o(\lambda_n^{1/6}),$$

convergence to $G(x) = \exp(-e^{-x})$ in (2) will hold. The condition on λ_n for this case is evidently that it should grow faster than $(\log n)^3$.

3. Proofs.

LEMMA 1. *Suppose that for fixed r there exists a sequence $\beta_n = o(\lambda_n^{1/2})$ for which (9) is true. Then there exists a sequence $\alpha_n > 0$ for which*

$$n(1-\Phi(x_n))\exp\left\{c_1\frac{x_n^3}{\lambda_n^{1/2}}+\cdots+c_r\frac{x_n^{r+2}}{\lambda_n^{r/2}}\right\}\to e^{-x}, \tag{11}$$

for each fixed x, with $x_n = \alpha_n x + \beta_n$.

PROOF. On taking logs in (11) and using the fact that, as $x \to \infty$, $(1 - \Phi(x)) \sim \exp(-x^2/2)/(x\sqrt{2\pi})$, we see that it will be enough to prove

$$\lim_{n\to\infty}\frac{x_n^2}{2}+\log x_n+\frac{1}{2}\log 2\pi - x_n^2C_r(x_n/\lambda_n^{1/2})-\log n = x, \tag{12}$$

where $C_r(\cdot)$ denotes the truncated series

$$C_r(z)=\sum_{j=1}^{r}c_jz^j.$$

Write $h_n(x) = x^2/2 + \log x + (1/2)\log 2\pi - x^2C_r(x/\lambda_n^{1/2})$. By assumption

$$\lim_{n\to\infty} h_n(\beta_n) - \log n = 0,$$

so (12) will be proved if we show

$$\lim_{n\to\infty} h_n(\alpha_n x+\beta_n) - h_n(\beta_n) = x. \tag{13}$$

To prove (13) let $\mu_n(x) = h_n'(x)$. Then

$$\mu_n(x) = x + x^{-1} - xp_n(x),$$

where $p_n(x) = \sum_1^r (j+2)c_j(x/\lambda_n^{1/2})^j$, and so, since $\beta_n \to \infty$ and $\beta_n = o(\lambda_n^{1/2})$,

$$\mu_n(\beta_n)\sim\beta_n.$$

It follows that

$$\beta_n+\frac{x}{\mu_n(\beta_n)}\sim\beta_n,$$

and that

$$\mu_n\left(\beta_n+\frac{x}{\mu_n(\beta_n)}\right)\sim\mu_n(\beta_n) \tag{14}$$

each uniformly over compact sets of x.

From the mean value theorem, for fixed x,

$$h_n\left(\beta_n+\frac{x}{\mu_n(\beta_n)}\right)-h_n(\beta_n)=x\frac{\mu_n(\beta_n+(\xi_n/\mu_n(\beta_n)))}{\mu_n(\beta_n)},$$

for some ξ_n lying between 0 and x. By the uniformity in (14) the right-hand side converges to x, and (11) is proved, with $\alpha_n = 1/\mu_n(\beta_n)$. □

LEMMA 2. *If* $\log n = o(\lambda_n)$, *then for each* $r \geq 0$ *the equation*

$$h_n(x) = \frac{x^2}{2} + \log x + \frac{1}{2}\log(2\pi) - x^2 \sum_{j=1}^{r} c_j \left(\frac{x}{\lambda_n^{1/2}}\right)^j = \log n \tag{15}$$

has a solution $\beta_n^{(r)}$ *with the property*

$$\beta_n^{(r)} \sim (2\log n)^{1/2}, \qquad n \to \infty. \tag{16}$$

Moreover $\beta_n^{(r)}$ *is the only solution of (15) in the region* $x = o(\lambda_n^{1/2})$.

PROOF. We first prove existence of a solution $\beta_n^{(r)}$ of $h_n(x) = \log n$ in the region $x = o(\lambda_n^{1/2})$, and then show that $\beta_n^{(r)}$ must satisfy (16) and is unique.

To start, note that if $x = o(\lambda_n^{1/2})$ and $x \to \infty$ then $h_n(x) \sim x^2/2$. Let $\rho_n = (\log n)/\lambda_n$. Then $\lambda_n \rho_n^{1/2} = o(\lambda_n)$ and $\log n = o(\lambda_n \rho_n^{1/2})$, so that

$$h_n(\lambda_n^{1/2}\rho_n^{1/4}) \sim \lambda_n \rho_n^{1/2}/2 > \log n,$$

for large enough n. On the other hand, for any fixed x_0, if n is large enough,

$$h_n(x_0) < \log n.$$

But $h_n(x)$ is continuous, so for each large enough n, the equation $h_n(x) = \log n$ has a solution in the interval $(x_0, \lambda_n^{1/2}\rho_n^{1/4})$. Moreover, any such solution $\beta_n^{(r)}$ must satisfy

$$\log n = h_n(\beta_n^{(r)}) \sim (\beta_n^{(r)})^2/2,$$

and so (16) must hold.

Uniqueness is established by showing that, for n large enough, $h_n(x)$ is strictly increasing in any interval of the form $(x_0, \lambda_n^{1/2}\varepsilon_n)$, where $x_0 > 0$ is arbitrary and $\varepsilon_n > 0$ converges to 0. This follows, for example, from the easily verifiable fact that

$$\lim_{n\to\infty} \quad \frac{h_n'(x)}{x} - 1 - \frac{1}{x^2} = 0$$

uniformly in such an interval. □

REMARK 1. We note from (16) that the scaling constants α_n defined in Lemma 1 satisfy

$$\alpha_n = 1/\mu_n(\beta_n^{(r)}) \sim (2\log n)^{-1/2}. \tag{17}$$

REMARK 2. For explicit expressions for $\beta_n^{(r)}$ we can argue as follows. Let $\beta_n^{(0)}$ denote the location constant (11) for the $r = 0$ case, and set $\beta_n^{(r)} = \beta_n^{(0)} + \delta_n$. Then, expanding $h_n(\beta_n^{(0)} + \delta_n) - \log n$ about $\beta_n^{(0)}$, we find

$$-(\beta_n^{(0)})^2 \sum_1^r c_j \left(\frac{\beta_n^{(0)}}{\lambda_n^{1/2}}\right)^j + \delta_n \left(\beta_n^{(0)} + \frac{1}{\beta_n^{(0)}} - \beta_n^{(0)} \sum_1^r (j+2) c_j \left(\frac{\beta_n^{(0)}}{\lambda_n^{1/2}}\right)^j\right) + o(\delta_n) = 0,$$

whence

$$\begin{aligned} \delta_n &= \frac{\beta_n^{(0)} \sum_1^r c_j (\beta_n^{(0)}/\lambda_n^{1/2})^j}{1 - \sum_1^r (j+2) c_j (\beta_n^{(0)}/\lambda_n^{1/2})^j + o(1/\beta_n^{(0)})} \\ &= \beta_n^{(0)} \sum_1^r c_j \left(\frac{\beta_n^{(0)}}{\lambda_n^{1/2}}\right)^j \left(1 + \sum_1^r (j+2) c_j \left(\frac{\beta_n^{(0)}}{\lambda_n^{1/2}}\right)^j + o\left(\frac{1}{\beta_n^{(0)}}\right)\right). \end{aligned} \tag{18}$$

Only terms nonnegligible in comparison to $(\log n)^{-1/2}$ need be retained in this expression, since terms of order $o(\alpha_n)$ will not affect the limiting distribution [Feller (1971), page 253]. The above gives a first correction to $\beta_n^{(0)}$. In principle, further correction terms may be found by expanding around the new approximate $\beta_n^{(r)}$ and retaining only terms nonnegligible in comparison to $(\log n)^{-1/2}$. For $r = 1$ we find

$$\beta_n^{(1)} = (2 \log n)^{1/2} - \frac{\log \log n + \log 4\pi}{2(2 \log n)^{1/2}} + c_1 \frac{2 \log n}{\lambda_n^{1/2}},$$

and for $r = 2$,

$$\begin{aligned} \beta_n^{(2)} &= (2 \log n)^{1/2} - \frac{\log \log n + \log 4\pi}{2(2 \log n)^{1/2}} \\ &\quad + (2 \log n)^{1/2} \left(c_1 \frac{(2 \log n)^{1/2}}{\lambda_n^{1/2}} + (c_2 + 3c_1^2) \frac{(2 \log n)}{\lambda_n}\right). \end{aligned}$$

For the unit Poisson, $c_1 = 1/6$ and $c_2 = -1/8$.

4. Maxima of triangular arrays.

The result in Proposition 1 may usefully be viewed in the wider context of the general theory of maxima of triangular arrays. A central problem in this theory is as follows. Suppose that we are given a triangular array of random variables $\{S_{n,i}: i = 1, \ldots, n;\ n = 1, 2, \ldots\}$, independent and identically distributed in each row, and with common distribution function F_n in the nth row. If the row distributions F_n converge weakly to some nondegenerate limit H as $n \to \infty$, what are the possible nondegenerate limit distributions, G, say, for

$$\max_{i \le n} (S_{n,i} - b_n)/a_n$$

for suitable constants $a_n > 0$ and b_n, and when does convergence to a specific G occur?

For this general problem, it is clear that the class of limit distributions G contains the extreme value distributions. Also, a simple sufficient condition for convergence of $\max_{i\le n}(S_{n,i} - b_n)/a_n$ to an extreme value limit G is evidently that H should belong to the max domain of attraction of G and that convergence of F_n to H should be fast enough in the upper tail. A specific condition for the latter (by an argument similar to that in the $r = 0$ case at the end of Section 2) is that, for each $\tau > 0$,

$$1 - F_n(s_n) \sim 1 - H(s_n) \tag{19}$$

for sufficiently large $s_n \le y_n(\tau)$, where $y_n(\tau)$ satisfies $1 - H(y_n(\tau)) \sim \tau/n$ as $n \to \infty$.

Proposition 1 goes beyond this simple result in the special case of scaled Poisson variables $S_{n,i}$ by showing that a weaker condition than (19) can hold for them and still be sufficient for a Gumbel limit G. Moreover the argument leading to Proposition 1 uses the Poisson nature of the variables only to guarantee the applicability of Cramér's theorem (4), and so the conclusion of Proposition 1 can be expected to hold in other cases when F_n is a convolution. What is required is that each variable $S_{n,i}$ should be representable as a sum, suitably scaled, of independent and identically distributed random variables whose moment generating function exists in a neighborhood of the origin. Specifically, let $U_j,\ j \ge 1$ denote i.i.d. random variables whose moment generating function exists in an open neighborhood of the origin, and suppose that for some sequence of integers k_n,

$$S_{n,i} \overset{d}{=} \left(\sum_{j\le k_n} U_j - c_{k_n} \right) \Big/ d_{k_n}$$

where $c_k = k\mu$ and $d_k = \sigma k^{1/2}$ with $\mu = E(U_1)$ and $\sigma^2 = \text{Var}(U_1)$. Then, by the same arguments as led to Proposition 1, we have Proposition 2.

PROPOSITION 2. *For each positive integer n, let $S_{n,i},\ i = 1, \dots, n$ denote independent random variables, each of which is a sum, scaled to zero mean and unit variance, of k_n independent and identically distributed random summands whose moment generating function exists in an open interval containing the origin. If $\log n = o(k_n^{(r+1)/(r+3)})$ for some integer $r \ge 0$, then*

$$\lim_{n\to\infty} P\Big(\max_{1\le i\le n} S_{n,i} \le \alpha_n x + \beta_n^{(r)} \Big) = \exp(-e^{-x}),$$

where α_n and $\beta_n^{(r)}$ are the normalizing constants defined in the previous section.

Though Proposition 2 suffices for the immediate needs of our modelling problem (see Section 6 below), it is interesting to explore the limiting behavior of maxima of triangular arrays more generally. In Proposition 2 the existence of a moment generating function is somewhat restrictive, excluding heavy-tailed distributions. To investigate a heavy-tailed case in which the $S_{n,i}$ still converge to normality, suppose that $E|U_1|^{2+\delta} < \infty$ for some $\delta > 0$ and that the distribution function, K, say, of the $(U_j - \mu)/\sigma$ has regularly varying tail, $\bar{K} \in \mathscr{R}_{-\alpha}$ for some $\alpha > 2$. Then, for each i, $S_{n,i}$ properly normalized converges still to a normal random variable, but the moment generating function condition is not satisfied.

To study this case we use a large deviations result of A. V. Nagaev (1969a) [see also S. Nagaev (1979)], which shows under the conditions above that

$$P(S_{n,1} > x) = P\left(\sum_1^{k_n} U_i > k_n\mu + k_n^{1/2} x\sigma\right) = (1 - \Phi(x))(1 + o(1)) + k_n \bar{K}(k_n^{1/2} x)(1 + o(1)) \tag{20}$$

for $k_n \to \infty$ and $x \geq 1$.

The following heuristic argument based on (20) indicates the kind of limit distributions now to be expected for $\max_{i \leq n} S_{n,i}$. Multiplying (20) by n and taking exponentials suggests that for large x

$$P\left(\max_{i \leq n} S_{n,i} \leq x\right) \approx \Phi^n(x)\, K^{nk_n}(k_n^{1/2} x), \tag{21}$$

so that a limiting distribution for $\max S_{n,i}$ might be expected to coincide with a limit distribution for the maximum of two independent random variables, one of which is the maximum of n independent standard normal variables and the other the maximum of nk_n independent copies of $(U_j - \mu)/\sigma k_n^{1/2}$. Let the sequence b_k^* be such that $k\bar{K}(b_k^*) \to 1$ as $k \to \infty$. Loosely speaking, b_k^* is a measure of the location of the distribution of $\max_{i \leq k}(U_i - \mu)/\sigma$. It follows that $b_{nk_n}^*/k_n^{1/2}$ is a measure of the magnitude of $\max_{j \leq nk_n}(U_j - \mu)/\sigma k_n^{1/2}$, the second random variable in our informal interpretation above. Similarly $(2\log n)^{1/2}$ is approximately the order of magnitude of the maximum of n i.i.d. standard normal random variables, the first term in the informal interpretation. Suppose

$$\frac{(2\log n)^{1/2}}{b_{nk_n}^*/k_n^{1/2}} \to x_0 \leq \infty, \tag{22}$$

as $n \to \infty$. A value of $x_0 = \infty$ suggests that in (21) the normal maximum will dominate, and so $\max S_{n,i}$ will converge to a Gumbel distribution: a value $x_0 = 0$ on the other hand suggests that the term based on the maximum U_j will dominate, and so a Fréchet limit distribution will result. The following proposition makes these rough arguments precise, and clarifies the behavior when $0 < x_0 < \infty$. In the latter case a nonextreme value limit is found.

PROPOSITION 3. *For each positive integer n, let $S_{n,i}, i = 1, \dots, n$ denote independent random variables, each of which is a sum, scaled to zero mean and unit variance, of k_n independent and identically distributed random summands U_j with distribution function K. Suppose that $E|U_j|^{2+\delta} < \infty$ for some $\delta > 0$, that $\bar{K} \in \mathscr{R}_{-\alpha}$ for some $\alpha > 2$, and that (22) holds for some $x_0 \leq \infty$.*

(i) *If* $x_0 = \infty$, *then*

$$\lim_{n\to\infty} P\Big(\max_{i\le n} S_{n,i} \le \alpha_n x + \beta_n^{(0)}\Big) = \exp(-e^{-x}).$$

(ii) *If* $0 \le x_0 < \infty$, *then*

$$\lim_{n\to\infty} P\Big(\max_{i\le n} S_{n,i} \le b^*_{nk_n} x/k_n^{1/2}\Big) = \begin{cases} \exp(-x^{-\alpha}), & \text{for } x \ge x_0, \\ 0, & \text{for } x < x_0. \end{cases}$$

PROOF. (i) Suppose $x_0 = \infty$. Then for any real x and any $B > 0$,

$$\alpha_n x + \beta_n^{(0)} \sim \beta_n^{(0)} = (2\log n)^{1/2} > Bb^*_{nk_n}/k_n^{1/2}$$

eventually. From (20),

$$nP\big(S_{n,1} > \alpha_n x + \beta_n^{(0)}\big) = n(1-\Phi(\alpha_n x + \beta_n^{(0)}))(1+o(1)) \\ + nk_n\bar{K}(k_n^{1/2}(\alpha_n x + \beta_n^{(0)}))(1+o(1)).$$

The first term on the right here converges to e^{-x} while the second is bounded above by

$$nk_n\bar{K}(Bb^*_{nk_n}) \sim B^{-\alpha},$$

which can be made arbitrarily small by choice of B. Thus

$$\lim_{n\to\infty} nP\big(S_{n,1} > \alpha_n x + \beta_n^{(0)}\big) = e^{-x}$$

which proves the assertion.

(ii) Suppose $x_0 < \infty$. If $x > x_0$, then $xb^*_{nk_n}/k_n^{1/2} \ge B(2\log n)^{1/2}$ eventually for some $B > 1$, and so $n(1-\Phi(xb^*_{nk_n}/k_n^{1/2}))$ is eventually bounded above by $n(1-\Phi(B(2\log n)^{1/2}))$, which tends to 0 as $n \to \infty$. It follows that the second term in (20) dominates, and that

$$nP\big(S_{n,1} > xb^*_{nk_n}/k_n^{1/2}\big) \sim nk_n\bar{K}(b^*_{nk_n}x) \sim x^{-\alpha}.$$

If $x < x_0$, then $xb^*_{nk_n}/k_n^{1/2} \le \theta(2\log n)^{1/2}$ eventually for some $\theta < 1$, and so, by (20) again, eventually

$$nP\big(S_{n,1} > xb^*_{nk_n}/k_n^{1/2}\big) \ge n\big(1-\Phi(\theta(2\log n)^{1/2})\big) \to \infty.$$

These two limits prove the assertion. □

For a lighter-tailed case intermediate between those of Propositions 2 and 3, suppose that the U_j are nonnegative and that $(U_j - \mu)/\sigma$ has an absolutely continuous distribution with probability density function satisfying

$$K'(x) \sim \exp(-x^{1-\varepsilon}) \tag{23}$$

as $x \to \infty$, for some $\varepsilon \in (0,1)$. Then

$$\bar{K}(x) \sim x^{\varepsilon}\exp(-x^{1-\varepsilon})/(1-\varepsilon)$$

so that the tail decays faster than a regularly varying function, but the moment generating function condition of Proposition 2 is not satisfied. The function K is a member of the subexponential family of distributions. A defining property of this family [Embrechts and Goldie (1980)] shows that for each fixed k,

$$P\left(\sum_{j=1}^{k} U_j - k\mu > k^{1/2}x\sigma\right) \sim P\left(\max_{1\le j\le k} U_j - \mu > x\sigma\right) \sim k\bar{K}(x) \tag{24}$$

as $x \to \infty$. Nagaev (1969b), Theorem 3, proves that for distributions (23) the relation (24) continues to hold when k increases as $x \to \infty$, provided that $k^{(1-\varepsilon)/2\varepsilon} = o(x)$. From these facts we get Proposition 4.

PROPOSITION 4. *For each positive integer n let $S_{n,i}$, $i = 1, \dots, n$, denote independent random variables, each of which is a sum, scaled to zero mean and unit variance, of k_n independent and identically distributed nonnegative summands U_j with finite mean μ and variance σ^2. Suppose that the distribution function K of $(U_j - \mu)/\sigma$ is absolutely continuous and has density satisfying (23). Suppose also that $k_n = o(\log n)^{2\varepsilon/(1-\varepsilon)}$. Then*

$$\lim_{n\to\infty} P\left(\max_{i\le n} S_{n,i} \le u_n(x)\right) = \exp(-e^{-x}),$$

where

$$u_n(x) = (\log nk_n)^{1/(1-\varepsilon)} + (\log nk_n)^{\varepsilon/(1-\varepsilon)}\left[\frac{\varepsilon}{(1-\varepsilon)^2}\log_2 nk_n - \frac{1}{1-\varepsilon}\log(1-\varepsilon) + \frac{x}{1-\varepsilon}\right].$$

PROOF. Under the condition on the rate of growth of k_n, $u_n(x) \sim (\log n)^{1/(1-\varepsilon)}$ so that $k_n^{(1-\varepsilon)/2\varepsilon}/u_n(x) = o(1)$, and therefore, by Theorem 3 of Nagaev (1969b), (24) holds with x replaced by $u_n(x)$. The conclusion of Proposition 4 follows directly from this by a short calculation. □

The Gumbel limiting distribution in Proposition 4 arises because, for k_n growing slowly enough, the tail of $S_{n,i}$ resembles that of its summands, the U_j. When k_n grows much faster [in fact, so that $(\log n)^{(1+\varepsilon)/(1-\varepsilon)} = O(k_n)$], the tail of $S_{n,i}$ ultimately resembles that of the Normal distribution [Nagaev (1969b), Theorem 1], and accordingly we would again expect a Gumbel limit for $\max S_{n,i}$, though with different normalizing constants. This fact may be established rigorously by arguments similar to those in Proposition 2 above. For intermediate rates of growth of k_n there are subtle combinations of the two dominant forms of tail behavior of the $S_{n,i}$. Nagaev (1969b) gives a comprehensive discussion, from which further limiting results for $\max_{i\le n} S_{n,i}$ may be deduced. In all cases, though for the different reasons outlined above, it is found that the Gumbel limiting distribution $\exp(-e^{-x})$ persists.

Suppose now that we drop the assumption of a normal limiting distribution for $S_{n,i}$ as $n \to \infty$, and instead assume that $S_{n,i} = (\sum_{j \le k_n} U_j - c_{k_n})/d_{k_n}$ converges, for suitable normalizing constants, to a stable distribution $G_{\alpha\beta}$ as $n \to \infty$, where $0 < \alpha < 2$ and the noncentrality parameter β lies in $[-1, 1]$. Under certain conditions on the pseudomoments of order r of the U_j [see Christoph and Wolf (1992), Section 5.2], we have the following large deviations result, which we assume to hold.

For $\alpha < 2$ and $\alpha < r < 1 + \alpha$,

$$P\biggl(\biggl(\sum_{j \le k} U_j - c_k\biggr)\Big/ d_k > x\biggr) = 1 - G_{\alpha\beta}(x) + O(k^{-(r-\alpha)/\alpha} x^{-r}) \tag{25}$$

as $x \to \infty$.

The stable distribution $G_{\alpha\beta}$ has the well-known tail behavior:

$$1 - G_{\alpha\beta}(x) \sim c x^{-\alpha} \tag{26}$$

as $x \to \infty$, for some constant c.

A direct calculation based on (25) and (26) gives Proposition 5.

PROPOSITION 5. *Under the assumption (25) with $\alpha < 2$, we have for any sequence $k_n \to \infty$,*

$$\lim_{n\to\infty} P\Bigl(\max_{i \le n} S_{n,i} \le (cn)^{1/\alpha} x\Bigr) = \exp(-x^{-\alpha})$$

for $x > 0$.

5. Numerical results for Poisson maxima. Figure 2 compares the distribution function of a scaled version of the Poisson maximum $\max_{1 \le i \le n} R_{n,i}$ with its limiting Gumbel distribution, for $n = 10, 100$, 1,000 and 10,000. The Poisson means are taken to be $\lambda_n = (\log n)^{7/2}$, giving a rate of growth in the $r = 0$ region of Proposition 1. Accordingly, the normalizing constants α_n and $\beta_n^{(0)}$ are used. The comparison is made for clarity on a double log scale, so that what is actually plotted is

$$-\log\Bigl(-\log P\Bigl(\max_{1 \le i \le n} R_{n,i} \le u_n(x)\Bigr)\Bigr) \quad \text{vs. } x,$$

where $u_n(x) = \lambda_n + \lambda_n^{1/2}(\beta_n^{(0)} + \alpha_n x)$. According to Proposition 1, the plotted step function should converge as $n \to \infty$ to the line $y = x$. What is evident from the figure is that the convergence is slow, as might have been expected from the known slowness of convergence of normal maxima to a Gumbel limit. However, over the central range $-1.53 \le x \le 4.6$, which contains 98% of the limit distribution, the agreement is remarkably good, even for $n = 10$.

Figure 3 illustrates a case when λ_n, here taken to be $(\log n)^{5/2}$, grows more slowly with n. This corresponds to an $r = 1$ region in Proposition 1. The step function plotted in this case is based on the normalizing constants α_n and $\beta_n^{(1)}$. The behavior is similar to that in Figure 2, though λ_n reaches only about a tenth of the size.

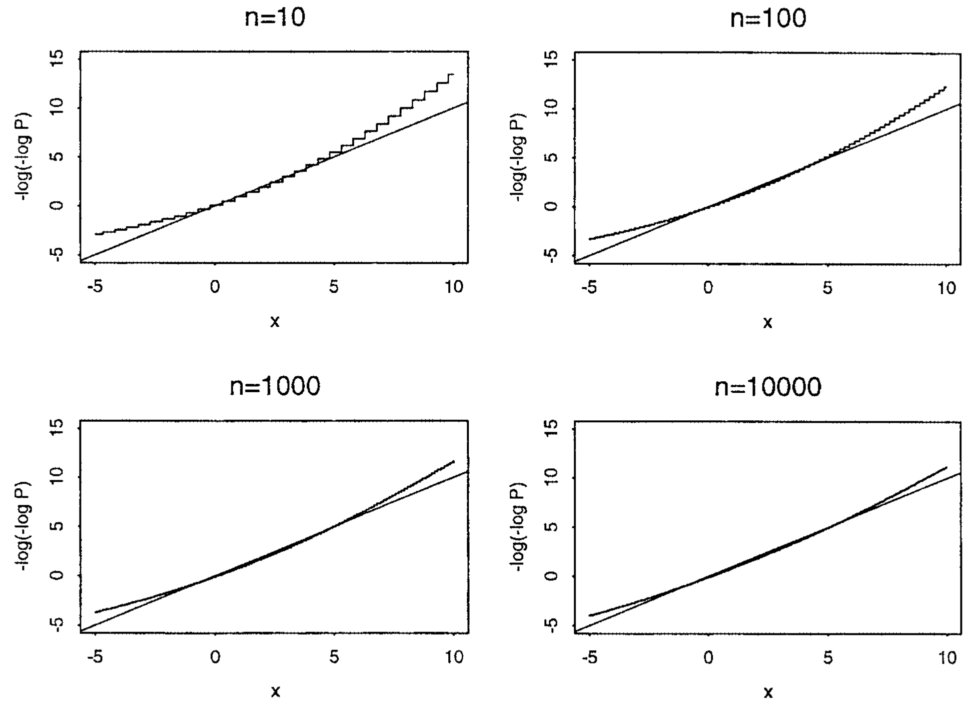

FIG. 2. *Distribution function of the normalized maximum of n independent Poisson random variables with mean $\lambda_n = (\log n)^{7/2}$. Double log vertical scale. Normalization corresponding to $r = 0$ case of Proposition* 1.

Figure 4 illustrates the need for modified normalizing constants in the $r = 1$ case. For $\lambda_n = (\log n)^{5/2}$ as in Figure 3 and for $n = 1,000$ it compares the distribution of $\max_{1\le i\le n} R_{n,i}$ normalized by $\alpha_n, \beta_n^{(1)}$, with the same distribution normalized by $\alpha_n, \beta_n^{(0)}$. Though at this value of n neither normalization gives a perfect correspondence, the $r = 1$ normalization does appear preferable, as would be expected from Proposition 1.

Figure 5 shows the quality of convergence in relation to the growth of λ_n for the $r = 1$ case with $\lambda_n = (2\log n)^{5/2}$. The plotted points are $-\log(-\log P(\max_{1\le i\le n} R_{n,i} \le u_n(x)))$ for $x = -1.53$, 0.37, 4.60, the first, fiftieth and ninty-ninth percentiles of the Gumbel distribution (indicated by dashed lines on the plot). The same normalizing sequence $u_n(x) = \lambda_n + \lambda_n^{1/2}(\beta_n^{(1)} + \alpha_n x)$ is used as in Figures 3 and 4. Only probabilities for values of λ at intervals of 20 are plotted, but the results show both the slowness of convergence and its oscillatory character, a consequence of discreteness. Again it is clear that the agreement is closer in central parts of the distribution than in the tails. This suggests that a penultimate approximation by a non-Gumbel extreme value distribution with shape parameter converging to 0 as $n \to \infty$ is likely to improve the approximation, as it does

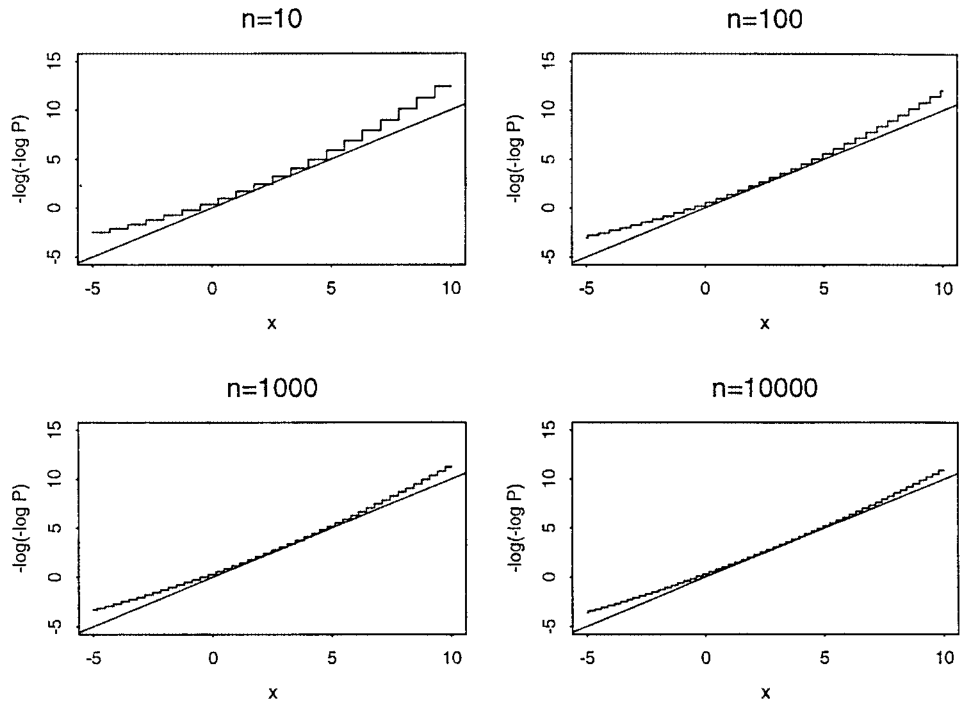

FIG. 3. *Distribution function of the normalized maximum of n independent Poisson random variables with mean $\lambda_n = (\log n)^{5/2}$. Double log vertical scale. Normalization corresponding to $r = 1$ case of Proposition* 1.

for normal and other maxima [Fisher and Tippett (1928), Gomes (1994)]. This can in fact be shown to be the case by arguments similar to those in Section 3.

6. The Poisson case when $\boldsymbol{\lambda}_n = o(\log n)$. When λ_n grows more slowly than $\log n$, we now show that no limiting distribution is possible for Poisson maxima. Thus the growth condition in Proposition 1 is close to being necessary as well as sufficient for a Gumbel limit. To see this we need some further notation. Let $R_{n,i}$ denote independent Poisson variables as in Section 2 and let F_n denote their distribution function and $\mathscr{F}_n$ their survivor function $1 - F_n$. We introduce a continuous distribution function $F_{c,n}$ which agrees with F_n on the integers and is defined by linear interpolation in $-\log \mathscr{F}_n$ elsewhere. Thus for any x,

$$\mathscr{F}_n(x+1) \le \mathscr{F}_{c,n}(x) \le \mathscr{F}_n(x), \tag{27}$$

where $\mathscr{F}_{c,n}$ is the survivor function of $F_{c,n}$. Since $\mathscr{F}_{c,n}$ is strictly decreasing, we may define a sequence of constants $\gamma_{c,n}$ by the equation

$$n\mathscr{F}_{c,n}(\gamma_{c,n}) = 1 \tag{28}$$

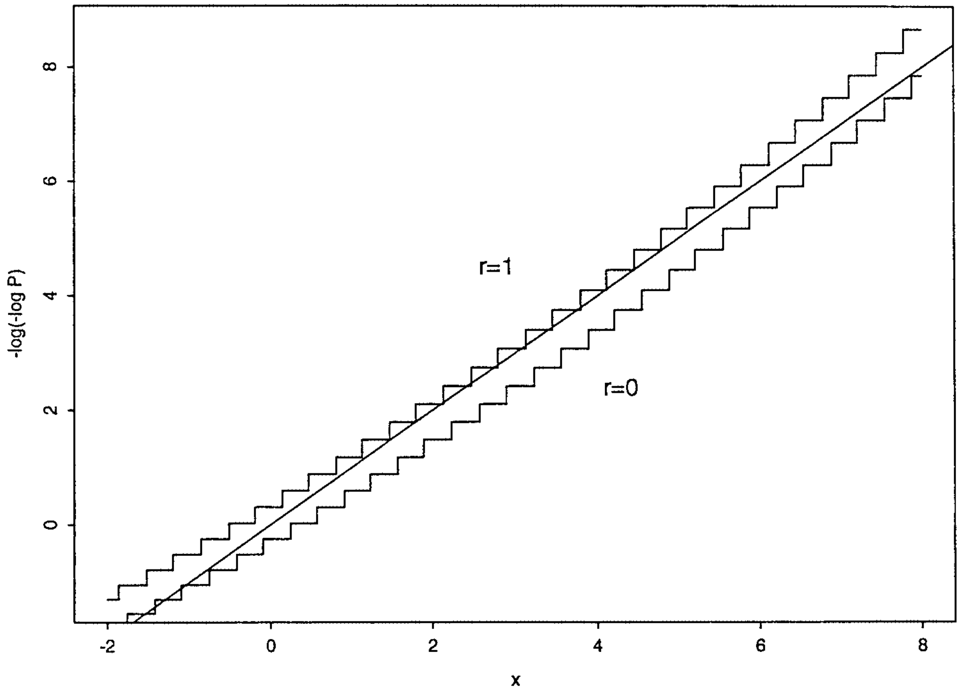

FIG. 4. *Effect of different normalizations on the distribution function of the normalized maximum of n independent Poisson random variables with mean $\lambda_n = (\log n)^{5/2}$. Double log scale; $n = 1000$. Step function labelled $r = 1$ based on normalizing constants asserted by Proposition 1 to be appropriate for this case; function labelled $r = 0$ based on normalization appropriate for faster-growing λ_n.*

for each positive integer n. It is easy to see that for large n the $\gamma_{c,n}$ are the approximate e^{-1} quantiles of the distribution of the maximum of n independent random variables with distribution function $F_{c,n}$, and so they provide an approximation to the location of the distribution of $\max_{1\le i\le n} R_{n,i}$.

We first establish a lower bound on the rate of growth of $\gamma_{c,n}$ when $\lambda_n = o(\log n)$. From (27) and (28) we have

$$(29)\quad 1 = n\mathscr{F}_{c,n}(\gamma_{c,n}) \ge n\mathscr{F}_n(\gamma_{c,n}+1) \ge n\,\frac{\lambda_n^{[\gamma_{c,n}+2]}}{[\gamma_{c,n}+2]!}\,e^{-\lambda_n} = n\,\frac{\lambda_n^g}{g!}\,e^{-\lambda_n},$$

say, where $g = [\gamma_c + 2]$. On taking logs of (29) and using Stirling's approximation $g! \sim (2\pi)^{1/2} e^{-g} g^{g+1/2}$ we find that

$$(30)\qquad \log n - \lambda_n + g\{\log(e\lambda_n) - \log g\} - \tfrac{1}{2}\log g$$

must be bounded above. Necessarily therefore $g > e\lambda_n$ eventually. Note that in this region, (30) is monotonic decreasing in g. To obtain a better bound suppose that $g = [(\lambda_n \log n)^{1/2}] \le (\lambda_n \log n)^{1/2}$. Since $\lambda_n = o(\log n)$, such a g

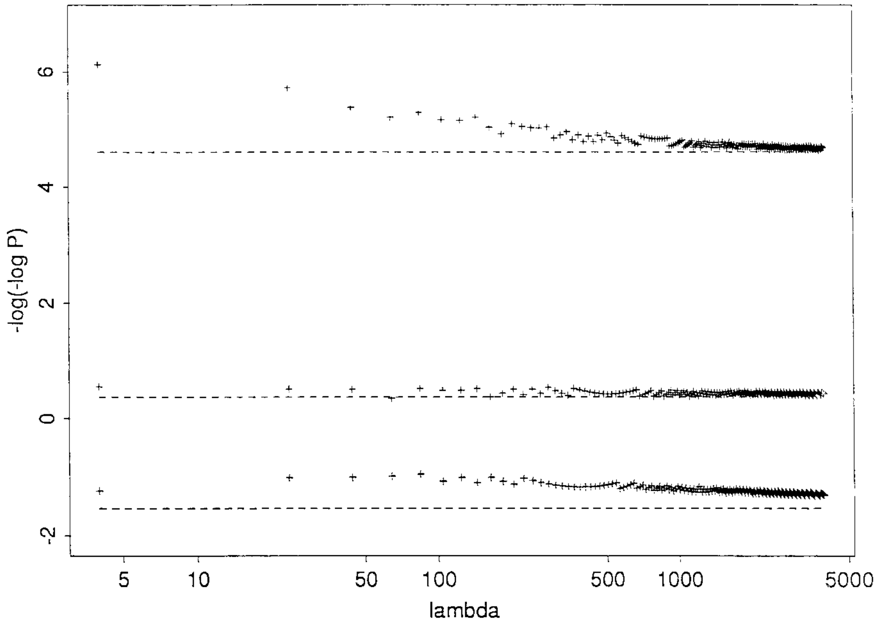

FIG. 5. *The quality of convergence in relation to growth of λ_n for $x = -1.53, 0.37, 4.60$ when $\lambda_n = (2\log n)^{5/2}$.*

is certainly in the region $g > e\lambda_n$ for large enough n. So expression (30) is no less than

$$\log n - \lambda_n + (\lambda_n \log n)^{1/2}\left\{1 + \log\left(\frac{\lambda_n}{\log n}\right)^{1/2}\right\} - \frac{1}{4}\log(\lambda_n \log n)$$
$$\geq \log n\left\{1 - \frac{\lambda_n}{\log n} + \left(\frac{\lambda_n}{\log n}\right)^{1/2}\log\left(\frac{\lambda_n}{\log n}\right)^{1/2} - \frac{\log(\lambda_n \log n)}{4\log n}\right\},$$

which goes to ∞ with n. Thus it must be true that

$$g > (\lambda_n \log n)^{1/2}$$

for all large enough n.

However, for any sequence γ growing in such a way that $\gamma/(\lambda_n \log n)^{1/2}$ is bounded away from 0 it is true that

$$\lim_{n\to\infty} \frac{\mathscr{F}_n(\gamma+1)}{\mathscr{F}_n(\gamma)} = 0. \tag{31}$$

To prove (31) note that

$$e^{-\lambda_n}\frac{\lambda_n^{m+1}}{(m+1)!} < \mathscr{F}_n(m) < e^{-\lambda_n}\frac{\lambda_n^{m+1}}{(m+1)!}\left(1 - \frac{\lambda_n}{m+2}\right)^{-1}$$

for integer $m > \lambda_n - 2$. Thus

$$\frac{\mathscr{F}_n(\gamma+1)}{\mathscr{F}_n(\gamma)} < \frac{\lambda_n}{[\gamma+2]}\left(1 - \frac{\lambda_n}{[\gamma+2]}\right)^{-1} < \frac{\lambda_n}{\gamma}\left(1 - \frac{\lambda_n}{\gamma}\right)^{-1},$$

which goes to 0 as n increases.

It is easily verified from (31) and the definition of $\mathscr{F}_{c,n}$ that for any positive ε and for $\gamma/(\lambda_n \log n)^{1/2}$ bounded away from 0,

$$\lim_{n\to\infty} \frac{\mathscr{F}_{c,n}(\gamma+\varepsilon)}{\mathscr{F}_{c,n}(\gamma)} = 0. \tag{32}$$

In particular (32) holds for $\gamma = \gamma_{c,n}$ and $\gamma = \gamma_{c,n} - \varepsilon$, so that

$$n\mathscr{F}_{c,n}(\gamma_{c,n}+\varepsilon) = \frac{\mathscr{F}_{c,n}(\gamma_{c,n}+\varepsilon)}{\mathscr{F}_{c,n}(\gamma_{c,n})} \to 0$$

and

$$n\mathscr{F}_{c,n}(\gamma_{c,n}-\varepsilon) = \frac{\mathscr{F}_{c,n}(\gamma_{c,n}-\varepsilon)}{\mathscr{F}_{c,n}(\gamma_{c,n})} \to \infty,$$

as $n \to \infty$. It follows that

$$\lim_{n\to\infty} F_{c,n}^n(\gamma_{c,n}+\varepsilon) = 1,$$
$$\lim_{n\to\infty} F_{c,n}^n(\gamma_{c,n}-\varepsilon) = 0$$

and so that

$$\lim_{n\to\infty} F_n^n(\gamma_{c,n}+1+\varepsilon) = 1,$$
$$\lim_{n\to\infty} F_n^n(\gamma_{c,n}-\varepsilon) = 0.$$

Thus we have proved the following proposition.

PROPOSITION 6. *When $\lambda_n = o(\log n)$, there is a sequence of integers I_n such that*

$$\lim_{n\to\infty} P\Big(\max_{1\le i\le n} R_{n,i} = I_n \text{ or } I_n + 1\Big) = 1.$$

In this case, therefore, $\max_{1\le i\le n} R_{n,i}$ behaves in the same way as when λ_n is constant.

7. Discussion.

For practical applications we require an approximate family for the distribution of $\max_{1\le i\le n} R_i$ where the R_i have unknown distribution. What we have shown in this paper is that if the R_i are Poisson with mean λ and we consider $\max_{1\le i\le n} R_i$ as a point on a suitable path of variables of the form $\max_{1\le i\le n} R_{n,i}$, where the $R_{n,i}$ are Poisson with mean λ_n, a Gumbel approximation is valid. This is supported also by numerical calculations. As shown in Propositions 2, 3 and 4, our results do not depend critically on the R_i being Poisson variables; the Gumbel limit for maxima is found to be valid for the entire class of distributions satisfying the conditions of Cramér's theorem, and also for some distributions with heavier tails. This robustness is crucial for statistical applications where the parent population is unknown.

Returning specifically to the case of Poisson variables, we obtain also that the Gumbel approximation for maxima will not be good if λ is so small relative to n that the degenerate limit of Proposition 6 is dominant. In the case of gamma radiation counts, λ is typically of the order $\lambda \approx 1000$ for 10-minute counts, of which there are approximately 53,000 in a year. The relative magnitude of these particular values suggests it is entirely reasonable to model annual maxima of such counts using an extreme value distribution.

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