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MOTION IN A GAUSSIAN INCOMPRESSIBLE FLOW

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We prove that the solution of a system of random ordinary differential equations $d\mathbf{X}(t)/dt = \mathbf{V}(t, \mathbf{X}(t))$ with diffusive scaling, $\mathbf{X}_\varepsilon(t) = \varepsilon\mathbf{X}(t/\varepsilon^2)$, converges weakly to a Brownian motion when $\varepsilon \downarrow 0$. We assume that $\mathbf{V}(t, \mathbf{x})$, $t \in R$, $\mathbf{x} \in R^d$ is a d-dimensional, random, incompressible, stationary Gaussian field which has mean zero and decorrelates in finite time.

1. Introduction. Let us consider a particle undergoing diffusion with a drift caused by an external velocity field $\mathbf{V}(t, \mathbf{x})$, $t \geq 0$, $\mathbf{x} \in R^d$. Its motion is determined by the Itô stochastic differential equation

$$
\begin{aligned}
d\mathbf{X}(t) &= \mathbf{V}(t, \mathbf{X}(t))\, dt + \sigma\, d\mathbf{B}(t), \\
\mathbf{X}(0) &= 0,
\end{aligned} \tag{1}
$$

where $\mathbf{B}(t)$, $t \geq 0$ denotes the standard d-dimensional Brownian motion and σ^2 is the molecular diffusivity of the medium. The particle trajectory $\mathbf{X}(t)$, $t \geq 0$ is assumed, for simplicity, to start at the origin.

We want the velocity field $\mathbf{V}$ to model turbulent, incompressible flows so we assume that it is random, zero mean, divergence free and mixing at macroscopically short space and time scales (see, e.g., [8], [24], [31]). To describe the long time and long distance behavior of the particle trajectory, we introduce macroscopic units in which the time t and space variable $\mathbf{x}$ are of order ε^{-2} and ε^{-1}, respectively, where $0 < \varepsilon \ll 1$ is a small scaling parameter. In the new variables, the Brownian motion is expressed by the formula $\varepsilon\mathbf{B}(t/\varepsilon^2)$, $t \geq 0$. Its law is then identical to that of the standard Brownian motion. In the scaled variables the Itô stochastic differential equation for the trajectory is

$$
\begin{aligned}
d\mathbf{X}_\varepsilon(t) &= \frac{1}{\varepsilon}\mathbf{V}\left(\frac{t}{\varepsilon^2}, \frac{\mathbf{X}_\varepsilon(t)}{\varepsilon}\right) dt + \sigma\, d\mathbf{B}(t), \\
\mathbf{X}_\varepsilon(0) &= 0.
\end{aligned} \tag{2}
$$

We are interested in the asymptotic behavior of the trajectories $\{\mathbf{X}_\varepsilon(t)\}_{t \geq 0}$, $\varepsilon > 0$, when $\varepsilon \downarrow 0$.

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In the early twenties, G. I. Taylor (cf. [34]) argued that the trajectory $\mathbf{X}(t)$, $t \geq 0$ of a particle in a diffusive medium with turbulent advection due to a zero mean random velocity field $\mathbf{V}(t, \mathbf{x})$ will behave approximately like a d-dimensional Brownian motion whose covariance matrix is given by

$$\int_0^{+\infty} v_{i,j}(t)\, dt + \sigma^2 \delta_{i,j}, \qquad i, j = 1, \dots, d, \tag{3}$$

where

$$v_{i,j}(t) = \left\langle V_i(t, \mathbf{X}(t)) V_j(0, \mathbf{0}) + V_j(t, \mathbf{X}(t)) V_i(0, \mathbf{0}) \right\rangle, \qquad i, j = 1, \dots, d \tag{4}$$

is the symmetric part of the Lagrangian covariance function of the random velocity field, with $\langle \cdot \rangle$ denoting averaging over all possible realizations of the medium. As noted by Taylor, the diffusivity is increased by advection in a turbulent flow. The main assumption in [34] was the convergence of the improper integrals in (3). This means that rapid decorrelation of Eulerian velocities $\mathbf{V}(t, \mathbf{x})$ will be inherited by the Lagrangian velocities $\mathbf{V}(t, \mathbf{X}(t))$ of the tracer particle. The mathematical analysis and proof of this fact, under different circumstances regarding $\mathbf{V}(t, \mathbf{x})$, is immensely difficult. In this paper we make a contribution toward the understanding of this issue.

Before stating our results, we will review briefly the status of the mathematical theory of turbulent diffusion. We mention first the papers [21], [26], [25], [23], [21] in which, among other things, it is shown that for an ergodic zero mean velocity field $\mathbf{V}$ with sufficiently smooth realizations and suitably restricted power spectra, when the molecular diffusivity σ is *positive*, then the family of trajectories $\{\mathbf{X}_\varepsilon(t)\}_{t \geq 0}$, $\varepsilon > 0$ converges weakly to Brownian motion in dimensions $d > 2$. Its covariance matrix $\mathbf{K} = [\kappa_{i,j}]_{i,j=1,\dots,d}$ is, in accordance with Taylor's prediction,

$$\kappa_{i,j} = \int_0^{+\infty} \mathbf{ME}\left[V_i(t, \mathbf{X}(t)) V_j(0, \mathbf{0}) + V_j(t, \mathbf{X}(t)) V_i(0, \mathbf{0}) \right] dt + \sigma^2 \delta_{i,j}, \qquad i, j = 1, \dots, d, \tag{5}$$

where $\mathbf{X}(t)$, $t \geq 0$ is the solution of (1), $\mathbf{M}$ and $\mathbf{E}$ denote expectations over the underlying probability spaces for the Brownian motion $\mathbf{B}(t)$, $t \geq 0$, and the random vector field $\mathbf{V}$, respectively. The improper integrals of (5) converge in the Cesaro sense; that is, the limits

$$\lim_{T \uparrow +\infty} \frac{1}{T} \int_0^T d\tau \int_0^\tau \mathbf{ME}\left[V_i(t, \mathbf{X}(t)) V_j(0, \mathbf{0}) + V_j(t, \mathbf{X}(t)) V_j(0, \mathbf{0}) \right] dt$$

exist and are finite. We should stress that the Cesaro convergence of the improper integrals appearing in (5) follows from the proof of weak convergence and is not an assumption of the theorem. Since $\sigma^2 \mathbf{I}$ is the intrinsic diffusivity of the medium, we note that $\mathbf{K} \geq \sigma^2 \mathbf{I}$ (or that $\mathbf{K} - \sigma^2 \mathbf{I}$ is positive definite) so that advection enhances the effective diffusivity $\mathbf{K}$. The proof of the statements made above can be found in [21] and [26] for bounded time independent velocity fields. For unbounded fields (such as Gaussian fields, for example) the proof can be found in [25], [4] and [13]. Diffusion in time dependent zero mean fields can be analyzed using an extension of the averaging technique developed there (see also [7]).

The one-dimensional case, $d = 1$, has also been investigated and the results there are different from the higher-dimensional ones. This is because there are no nontrivial incompressible velocity fields in one dimension. For a Gaussian "white noise" velocity field, typical particle displacements are of order $\ln^2 t$ instead of $\sqrt{t}$, as is expected in the multidimensional case. This localization phenomenon is discussed in [30], [23], [17], [6] and does not occur when the random velocity field is incompressible and integrable in a suitable sense [13].

Since formula (5) for the effective diffusivity $\mathbf{K}$ involves the solution of an Itô stochastic differential equation with stochastic drift, it is far from being explicit. In many applications it is important to consider how $\mathbf{K}$ behaves as a function of σ, especially when $\sigma \downarrow 0$, which is the large Peclét number case where advection is dominant [16]. Although this question is very difficult to deal with analytically, some progress has been made recently by noting that $\mathbf{K}$ is a minimum of an energy-like functional allowing, therefore, extensive use of variational techniques (see [11] for periodic and [12] for certain random flows). We can go one step further and ask about the limiting behavior of the trajectories of (2), as $\varepsilon \downarrow 0$, in the absence of any molecular diffusivity, that is, when $\sigma = 0$. The effective diffusivity, if it exists in this case, is purely due to turbulent advection. It is this aspect of the problem of turbulent diffusion that we will address in this paper.

In previous mathematical studies of advection induced diffusion, the focus has been mainly on velocity fields which besides regularity, incompressibility and stationarity, either in time or both in space and time, have also strong mixing properties in t ([20], [18], [19]) and are slowly varying in $\mathbf{x}$. This means that equation (2) (with $\sigma = 0$) is now an ordinary differential equation with a stochastic right-hand side

$$\frac{d\mathbf{X}_\varepsilon(t)}{dt} = \frac{1}{\varepsilon}\mathbf{V}\left(\frac{t}{\varepsilon^2}, \frac{\mathbf{X}_\varepsilon(t)}{\varepsilon^{1-\alpha}}\right), \qquad \mathbf{X}_\varepsilon(0) = 0 \tag{6}$$

and the parameter α is in the range $0 < \alpha \le 1$, with $\alpha = 1$ typically. One can prove that the solutions of (6) considered as stochastic processes converge weakly to Brownian motion with covariance matrix given by the Kubo formula

$$\kappa_{i,j} = \int_0^{+\infty} \mathbf{E}\left[V_i(t,\mathbf{0})V_j(0,\mathbf{0}) + V_j(t,\mathbf{0})V_i(0,\mathbf{0})\right] dt, \qquad i,j = 1,\dots,d.$$

As before, $\mathbf{E}$ denotes expectation taken over the underlying probability space of the random velocity field $\mathbf{V}$. The convergence of the improper integral follows from the rapid decay of the correlation matrix of the Eulerian velocity field, which is a consequence of the assumed strong mixing properties of the field.

Very little is known when $\alpha = 0$, that is, when the velocity field is not slowly varying in the $\mathbf{x}$ coordinates. However, it is believed that we have again diffusive behavior, in the sense described above, if the velocity field $\mathbf{V}(t,x)$ is sufficiently strongly mixing in time, and then the effective diffusivity is given by (5). In a series of numerical experiments R. Kraichnan [22] observed that the diffusion approximation holds even for zero mean divergence-free velocity fields $\mathbf{V}(\mathbf{x})$ that are independent of t, provided that they are strongly mixing in the spatial variable $\mathbf{x}$ and the dimension d of the space R^d is greater than two. For $d = 2$, there is a "trapping" effect for the advected particles, prohibiting diffusive behavior.

Some theoretical results in this direction have been obtained recently by Allevaneda, Eliot and Apelian [3] for specially constructed two-dimensional, stationary velocity fields. An extensive exposition of turbulent advection from a physical point of view can be found in [16].

In this paper we assume that the field $\mathbf{V}(t,\mathbf{x})$ is stationary both in t and $\mathbf{x}$, has zero mean, is Gaussian and its correlation matrix

$$\mathbf{R}(t,\mathbf{x}) = \left[\mathbf{E}\left[V_i(t,\mathbf{x})V_j(0,0)\right]\right]_{i,j=1,\dots,d}$$

has compact support in the t variable. This means that there exists a $T > 0$ such that

$$\mathbf{R}(t,\mathbf{x}) = \mathbf{0},$$

for $|t| \ge T$, $\mathbf{x} \in R^d$. Such random fields are sometimes called T-dependent (cf., e.g, [9]). We also require that the realizations of $\mathbf{V}$ are continuous in t, C^1 smooth in the $\mathbf{x}$ variable and that

$$\operatorname{div}\mathbf{V}(t,\mathbf{x}) = \sum_{i=1}^{d} \partial_{x_i} V_i(t,\mathbf{x}) \equiv 0.$$

We prove that for such velocity fields the improper integrals appearing in (5) converge and that the family of processes $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ tends weakly to Brownian motion with covariance matrix given by (5). The precise formulation of the result and an outline of its proof is presented in the next section. It is important to note that the result presented here seems to be unattainable by a simple extension of the path-freezing technique used in the proofs of the diffusion approximation for slowly varying velocity fields in [20], [18], [19].

The paper is organized as follows. In Section 2 we introduce some basic notation, formulate the main result and outline the proof. Sections 3–5 contain some preparatory lemmas for proving weak compactness of the family $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$. The proof of tightness is done in Section 6 where we also identify the unique weak limit of the family with the help of the martingale uniqueness theorem.

2. Basic notation, formulation of the main result and outline of the proof. Let $(\Omega, \mathcal{V}, \mathcal{P})$ be a probability space, $\mathbf{E}$ expectation relative to the probability measure P, $\mathcal{A}$ a sub-σ-algebra of $\mathcal{V}$ and $X: \Omega \to R$ a random variable measurable with respect to $\mathcal{V}$. We denote the conditional expectation of X with respect to $\mathcal{A}$ by $\mathbf{E}[X \mid \mathcal{A}]$ and the space of all $\mathcal{V}$-measurable, p-integrable random variables by $L^p(\Omega, \mathcal{V}, P)$. The L^p-norm is

$$\|X\|_{L^p} = \{\mathbf{E}|X|^p\}^{1/p}.$$

We assume that there is a family of transformations on the probability space Ω such that

$$\tau_{t,\mathbf{x}}: \Omega \to \Omega, \qquad t \in R, \mathbf{x} \in R^d$$

and such that

(P1) $\tau_{s,\mathbf{x}}\tau_{t,\mathbf{y}} = \tau_{s+t,\mathbf{x}+\mathbf{y}}$ for all $s, t \in R$, $\mathbf{x}, \mathbf{y} \in R^d$,
(P2) $\tau_{t,\mathbf{x}}^{-1}(A) \in \mathcal{V}$ for all $t \in R$, $\mathbf{x} \in R^d$, $A \in \mathcal{V}$,
(P3) $P[\tau_{t,\mathbf{x}}^{-1}(A)] = P[A]$ for all $t \in R$, $\mathbf{x} \in R^d$, $A \in \mathcal{V}$,
(P4) the map

$$(t, \mathbf{x}, \omega) \mapsto \tau_{t,\mathbf{x}}(\omega)$$

is jointly $\mathcal{B}_R \otimes \mathcal{B}_{R^d} \otimes \mathcal{V}$ to $\mathcal{V}$ measurable. Here we denote by $\mathcal{B}_{R^d}$ the σ-algebra of Borel subsets of R^d.

Suppose now that $\tilde{\mathbf{V}}: \Omega \to R^d$, where $\tilde{\mathbf{V}} = (\tilde{V}_1, \dots, \tilde{V}_d)$ and $\mathbf{E}\tilde{\mathbf{V}} = \mathbf{0}$, is such that the random field

$$\mathbf{V}(t, \mathbf{x}; \omega) = \tilde{\mathbf{V}}(\tau_{\mathbf{t},\mathbf{x}}(\omega)) \tag{7}$$

has all finite-dimensional distributions Gaussian. Thus, $\mathbf{V}(t, \mathbf{x}; \omega)$ is a stationary zero mean Gaussian random field. We denote its components by $(V_1(t, \mathbf{x}), \dots, V_d(t, \mathbf{x}))$. Let $L^2_{a,b}$ be the closure of the linear span of $V_i(t, \mathbf{x})$, $a \le t \le b$, $\mathbf{x} \in R^d$, $1 = 1, \dots, d$, in the L^2-norm. We denote by $\mathcal{V}_{a,b}$ the σ-algebra generated by all random vectors from $L^2_{a,b}$. Let $L^{2,\perp}_{a,b} = L^2_{-\infty,+\infty} \ominus L^2_{a,b}$; that is, $L^{2,\perp}_{a,b}$ is the orthogonal complement of $L^2_{a,b}$ in $L^2_{-\infty,+\infty}$. We denote by $\mathcal{V}^{\perp}_{a,b}$ the σ-algebra generated by all random vectors belonging to $L^{2,\perp}_{a,b}$. According to [29], page 181, Theorems 10.1 and 10.2, $\mathcal{V}_{a,b}$ and $\mathcal{V}^{\perp}_{a,b}$ are independent. Let $\mathbf{V}_{a,b}(t, \mathbf{x})$ be the orthogonal projection of $\mathbf{V}(t, \mathbf{x})$ onto $L^2_{a,b}$; that is, each component of $\mathbf{V}_{a,b}$ is the projection of the corresponding component of $\mathbf{V}$. We denote by $\mathbf{V}^{a,b}(t, \mathbf{x}) = \mathbf{V}(t, \mathbf{x}) - \mathbf{V}_{a,b}(t, \mathbf{x})$ the orthogonal complement of $\mathbf{V}_{a,b}(t, \mathbf{x})$. Of course $\mathbf{V}^{a,b}(t, \mathbf{x})$ is $\mathcal{V}^{\perp}_{a,b}$-measurable while $\mathbf{V}_{a,b}(t, \mathbf{x})$ is $\mathcal{V}_{a,b}$-measurable for any t and $\mathbf{x}$. The correlation matrix of the field $\mathbf{V}$ is defined as

$$\mathbf{R}(t, \mathbf{x}) = \left[\mathbf{E}\left[V_i(t, \mathbf{x}) V_j(0, \mathbf{0})\right]\right]_{i,j=1,\dots,}.$$

We will assume that the Gaussian field $\mathbf{V}$ given by (7) satisfies the following hypotheses.

(A1) $\mathbf{V}(t, \mathbf{x})$ is almost surely continuous in t and C^1 smooth in $\mathbf{x}$. Moreover, for any $a \le b$, all $\mathbf{V}_{a,b}(t, \mathbf{x})$ have the same property.

(A2) $\mathbf{V}(t, \mathbf{x})$ is divergence free, that is, $\operatorname{div} \mathbf{V}(t, \mathbf{x}) = \sum_{i=1}^{d} \partial_{x_i} V_i(t, \mathbf{x}) \equiv 0$.

(A3) $\mathbf{E}\mathbf{V}(0, \mathbf{0}) = \mathbf{0}$.

(A4) $\mathbf{R}(t, \mathbf{x})$ is Lipschitz continuous in $\mathbf{x}$ and so as the first partials in $\mathbf{x}$ of all is entries.

(A5) The field $\mathbf{V}$ is T-dependent, that is, there exists $T > 0$ such that for all $|t| \ge T$ and $\mathbf{x} \in R^d$, $\mathbf{R}(t, \mathbf{x}) = \mathbf{0}$.

REMARK 1. Clearly (A4) implies the continuity of the field in t and $\mathbf{x}$ (see, e.g., [1]).

REMARK 2. Note that although $\mathbf{V}_{a,b}(t, \mathbf{x})$ and $\mathbf{V}^{a,b}(t, \mathbf{x})$ are no longer stationary in t, they are stationary in $\mathbf{x}$, for any t fixed. Moreover

$$\operatorname{div} \mathbf{V}_{a,b}(t, \mathbf{x}) = \operatorname{div} \mathbf{V}^{a,b}(t, \mathbf{x}) \equiv 0,$$

that is, both $\mathbf{V}_{a,b}$ and $\mathbf{V}^{a,b}$ are divergence free.

REMARK 3. We note that the field

$$\mathbf{W}(t, \mathbf{x}) = \frac{\mathbf{V}(t, \mathbf{x})}{\sqrt{t^2 + |\mathbf{x}|^2 + 1}}$$

is a.s. bounded. This can be seen by using some well-known conditions for boundedness of a Gaussian field. We recall that for a Gaussian field $G(\mathbf{t})$, where $\mathbf{t} \in \mathscr{T}$t and $\mathscr{T}$ is some abstract parameter space, a d-ball with center at t and radius ϱ is defined relative to the pseudometric

$$d(\mathbf{t}_1, \mathbf{t}_2) = \left[\mathbf{E}\left|G(\mathbf{t}_1) - G(\mathbf{t}_2)\right|^2\right]^{1/2}.$$

Let $N(\varepsilon)$ be the minimal number of d-balls with radius $\varepsilon > 0$ for the velocity field $\mathbf{W}$ needed to cover R^d. It can be verified that

$$N(\varepsilon) \le K\varepsilon^{-2d},$$

where K is a constant independent of ε. It is also clear that for sufficiently large ε, $N(\varepsilon) = 1$. Thus

$$\int_0^{+\infty} \sqrt{\log N(\varepsilon)}\, d\varepsilon < +\infty.$$

According to Corollary 4.15 of [1], this is all we need to guarantee a.s. boundedness of $\mathbf{W}$.

REMARK 4. Condition (A1) is clearly satisfied when the covariance matrix R satisfies the condition

$$|R(0,\mathbf{0}) - R(t,\mathbf{x})| + \sum_{i,j=1}^{d} \left|R^*_{ij}(0,\mathbf{0}) - R^*_{ij}(t,\mathbf{x})\right| \le \frac{C}{\ln^{1+\eta}\sqrt{t^2+|\mathbf{x}|^2}}, \tag{8}$$

where $\eta > 0$ and

$$R^*_{ij}(t,\mathbf{x}) = \partial^2_{x_i,x_j} R(t,\mathbf{x})$$

for $i, j = 1, \dots, d$. Indeed as is well known (see, e.g., [1], page 62), (8) implies that

$$\begin{aligned} &\mathbf{E}\left|\mathbf{V}(t,\mathbf{x}) - \mathbf{V}(t+h,\mathbf{x}+\mathbf{k})\right|^2 + \mathbf{E}\left|\nabla_{\mathbf{x}}\mathbf{V}(t,\mathbf{x}) - \nabla_{\mathbf{x}}\mathbf{V}(t+h,\mathbf{x}+\mathbf{k})\right|^2 \\ &\qquad \le \frac{C}{\ln^{1+\eta}\sqrt{h^2+|\mathbf{k}|^2}}, \end{aligned} \tag{9}$$

where $\mathbf{k} = (k_1, \dots, k_d)$.

Since the projection operator is a contraction in the L^2-norm, we can also see that $\mathbf{V}_{a,b}(t,\mathbf{x})$ satisfies (9), for any $a \le b$; thus by Theorem 3.4.1 of [1] it is continuous in t and C^1-smooth in $\mathbf{x}$.

Let us consider now the family of processes $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ given by

$$\begin{aligned} \frac{d\mathbf{X}_\varepsilon(t)}{dt} &= \frac{1}{\varepsilon}\mathbf{V}\left(\frac{t}{\varepsilon^2}, \frac{\mathbf{X}_\varepsilon(t)}{\varepsilon}\right), \\ \mathbf{X}_\varepsilon(0) &= 0. \end{aligned} \tag{10}$$

Because $\mathbf{V}(t,\mathbf{x})$ grows at most linearly, both in t and $\mathbf{x}$, (10) can be solved globally in t and the processes $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ are therefore well defined. Note that each such process has continuous (even C^1-smooth) trajectories; therefore, its law is supported in $C([0,+\infty), R^d)$, the space of all continuous functions with values in R^d equipped with the standard Frechét space structure.

DEFINITION 1. We will say that a family of processes $\{Y_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ having continuous trajectories converges weakly to a Brownian motion if their laws in $C([0,+\infty), R^d)$ are weakly convergent to a Wiener measure over that space.

The main result of this paper can now be formulated as follows.

THEOREM 1. *Suppose that a Gaussian random field* $\mathbf{V}(t,\mathbf{x})$ *is stationary in t and* $\mathbf{x}$ [*i.e.*, $\mathbf{V}(t,\mathbf{x};\omega) = \tilde{\mathbf{V}}(\tau_{t,\mathbf{x}}(\omega))$] *for some* $\tilde{\mathbf{V}}(\omega)$ *as in* (7)]. *Assume that it satisfies conditions* (A1)–(A4).

Then:

(i) *The improper integrals*

$$b_{i,j} = \int_0^{+\infty} \mathbf{E}\big[V_i(t,\mathbf{X}(t))V_j(0,\mathbf{0})\big]\,dt, \qquad i,j = 1,\dots,d$$

converge. Here $\mathbf{X}(t) = \mathbf{X}_1(t)$.

(ii) *The processes* $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ *converge weakly, as* $\varepsilon \downarrow 0$, *to Brownian motion whose covariance matrix is given by*

$$\kappa_{i,j} = b_{i,j} + b_{j,i}, \qquad i,j = 1,\dots,d. \tag{11}$$

At this point we would like to present a brief outline of the proof of Theorem 1. As usual, we will first establish tightness of the family $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$. According to [30] it is enough to prove that this family is tight in $C([0,L], R^d)$, the space of continuous functions on $[0,L]$ with values in R^d, for any $L > 0$. In order to show this fact, we need estimates of the type

$$\mathbf{E}|\mathbf{X}_\varepsilon(u) - \mathbf{X}_\varepsilon(t)|^p|\mathbf{X}_\varepsilon(t) - \mathbf{X}_\varepsilon(s)|^2 < C(u-s)^{1+q}, \tag{12}$$

for all $\varepsilon > 0$, $0 \le s \le t \le u \le L$ and some constants $p, q, C > 0$. After establishing (12) we can apply some of the classical weak compactness lemmas of Kolmogorov and Chentzov (see [5]) to conclude tightness. Note that

$$\mathbf{X}_\varepsilon(t) = \varepsilon\mathbf{X}\left(\frac{t}{\varepsilon^2}\right), \qquad \varepsilon > 0,\ t \ge 0.$$

For brevity we shall use the notation

$$\mathbf{V}(t,s,\mathbf{x}) = \mathbf{V}(t,\mathbf{X}^{s,\mathbf{x}}(t))$$

with the obvious extension to the components of the field. Let us also make a convention of suppressing the last two arguments, that is, s and $\mathbf{x}$ in cases when they both vanish. For any nonnegative integer k and $i_1,\dots, i_k \in \{1,\dots,d\}$, we shall use the notation

$$V_{i_1,\dots,i_k}(s_1,\dots,s_k,s,\mathbf{x}) = \prod_{p=1}^{k} V_{i_p}\big(s_p,\mathbf{X}^{s,\mathbf{x}}(s_p)\big).$$

Again we shall suppress writing the last two arguments in case they are both zero.

We now write the equation for the trajectories in integral form

$$\mathbf{X}_\varepsilon(t) - \mathbf{X}_\varepsilon(s) = \varepsilon\int_{s/\varepsilon^2}^{t/\varepsilon^2}\mathbf{V}(\varrho)\,d\varrho$$

and so the left-hand side of (12) is

$$\sum_{i=1}^{d} 2\varepsilon^2\int_{s/\varepsilon^2}^{t/\varepsilon^2} d\varrho' \int_{s/\varepsilon^2}^{\varrho'} \mathbf{E}\big[|\mathbf{X}_\varepsilon(u) - \mathbf{X}\varepsilon(t)|^p V_{i,i}(\varrho',\varrho'')\big]\,d\varrho''. \tag{13}$$

Since, according to Lemma 1, the field $\{\mathbf{V}(t)\}_{t\geq 0}$ is stationary, (13) is equal to

$$\begin{aligned}
&\sum_{i=1}^{d} 2\varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} d\varrho' \int_{s/\varepsilon^2}^{\varrho'} \mathbf{E}\Big[\big|\mathbf{X}_\varepsilon(u-\varepsilon^2\varrho'') - \mathbf{X}_\varepsilon(t-\varepsilon^2\varrho'')\big|^p \\
&\qquad\qquad \times V_{i,i}(\varrho'-\varrho'',0)\Big]\, d\varrho'' \\
&= \sum_{i=1}^{d} 2\varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} d\varrho' \int_{0}^{\varrho'-s/\varepsilon^2} \mathbf{E}\Big[\big|\mathbf{X}_\varepsilon(u-\varepsilon^2\varrho'+\varepsilon^2\varrho'') \\
&\qquad\qquad -\mathbf{X}_\varepsilon(t-\varepsilon^2\varrho'+\varepsilon^2\varrho'')\big|^p \\
&\qquad\qquad \times V_{i,i}(\varrho'',0)\Big]\, d\varrho''.
\end{aligned} \tag{14}$$

The key observation we make in the remark at the end of Section 4 is that for any positive integer k the sequence of Lagrangian velocities

$$\{\mathbf{V}(t_1+NT;\omega),\dots,\mathbf{V}(t_k+NT;\omega)\}_{N\geq 0}$$

has law identical with that of the sequence

$$\{\mathbf{V}(t_1;\Xi_N),\dots,\mathbf{V}(t_k;\Xi_N)\}_{N\geq 0},$$

where $\{\Xi_N\}_{N\geq 0}$ is a certain Markov chain of random environments (i.e., the state space of the chain is $(\Omega,\mathscr{V}_{-\infty,0})]$ defined over the probability space $(\Omega,\mathscr{V}_{-\infty,0}^{\perp},P)$. Now the expectation in (14) can be written as

$$\begin{aligned}
&\mathbf{E}\Big[\big|\mathbf{X}_\varepsilon(u-\varepsilon^2\varrho'+\varepsilon^2(\varrho''-NT);\Xi_N(\omega)) \\
&\quad -\mathbf{X}_\varepsilon(t-\varepsilon^2\varrho'+\varepsilon^2(\varrho''-NT);\Xi_N(\omega))\big|^p \\
&\qquad \times V_i(\varrho''-NT;\Xi_N(\omega))V_i(0;\omega)\Big] \\
&\quad +\mathbf{E}\Big[\big|\mathbf{X}_\varepsilon(u-\varepsilon^2\varrho'+\varepsilon^2(\varrho''-NT)) - \mathbf{X}_\varepsilon(t-\varepsilon^2\varrho'+\varepsilon^2(\varrho''-NT))\big|^p \\
&\qquad\qquad \times V_i(\varrho''-NT)Q^N\tilde{V}_i\Big],
\end{aligned}$$

where $N=[\varrho''/T]$ ($[\cdot]$ is the greatest integer function), Q is a transition probability operator for the Markov chain $\{\Xi_N\}_{n\geq 0}$ and $\tilde{V}_i$ is the ith component of the random vector $\tilde{\mathbf{V}}$ defining the random field [see (7)]. In order to be able to define the chain we will have to develop some tools in Sections 3 and 4 and establish certain technical lemmas about the expectation of a process along the trajectory $\mathbf{X}(t)$, $t\geq 0$, that is, $f(\tau_{t,\mathbf{X}(t;\omega)}(\omega))$ conditioned on the σ-algebra $\mathscr{V}_{-\infty,0}$ representing the past.

The proof of estimate (12) can thus be reduced to the investigation of the rate of convergence of the orbit $\{Q^N\tilde{V}_i\}_{N\geq 0}$ to 0, $i=1,\dots,d$. We will prove in Lemma 10 that the L^1-norm of $Q^N\tilde{V}_i$ satisfies

$$\lim_{n\to+\infty}\frac{\|Q^N\tilde{V}_i\|_{L^1}}{1/N^s}=0, \tag{15}$$

for any $s > 0$. The result in (15) may appear to be too weak to claim (12), yet combined with the fact that $\mathbf{V}(t)$ is a Gaussian random variable for any fixed $t \geq 0$, so that

$$P\big[\,|\mathbf{V}(\varrho'' - NT, \mathbf{X}(\varrho'' - NT))| \geq N\big] \leq \exp(-CN^2),$$

for some constant $C > 0$, we will be able to prove in Section 6 that (12) is estimated by $C(u-s)^{1+q}$, for some $q > 0$. In this way we will establish the tightness of $\{\mathbf{X}_\varepsilon(t)\}_{t \geq 0}$, $\varepsilon > 0$. The limit identification will be done by proving that any limiting measure must be a solution of a certain martingale problem. Thus the uniqueness of the limiting measure will follow from the fact that the martingale problem is well posed in the sense of [33].

REMARK. In what follows we wish to illustrate the idea of an abstract-valued Markov chain $\{\Xi_n\}_{n \geq 0}$ that we have introduced above and shall develop in Section 4. Consider a space–time stationary velocity field $\mathbf{V}(t, \mathbf{x}; \omega)$ on a probability space $(\Omega, \mathscr{V}, P)$ with the group of measure preserving transformations $\tau_{t,\mathbf{x}}$, $t \in R$, $\mathbf{x} \in R^d$. Let us denote by Z_+ the set of all positive integers and

$$\tilde{\Omega} = \prod_{n \in Z_+} \Omega,$$

$$\tilde{P} = \bigotimes_{n \in Z_+} P,$$

$$\tilde{\mathscr{V}} = \bigotimes_{n \in Z_+} \mathscr{V}$$

and define

$$\mathbf{W}\big(t, \mathbf{x}; (\omega_n)_{n \in Z_+}\big) = \mathbf{V}(t - NT, \mathbf{x}; \omega_N) \quad \text{for } NT \leq t \leq (N+1)T.$$

This field is of course not time stationary (actually it can be made so by randomizing the "switching times" NT) and we do not even assume that it is Gaussian. However, we wish to use this case to shed some light on the main points of the construction we carry out in this article.

Let

$$\mathscr{S}\big((\omega_n)_{n \in Z_+}\big) = (\omega_{n+1})_{n \in Z_+}$$

and

$$\mathscr{Z}\big((\omega_n)_{n \in Z}\big) = \mathscr{S}\Big(\big(\tau_{0, \mathbf{X}(T;\, \omega_0)}(\omega_n)\big)_{n \in Z_+}\Big).$$

Here $\mathbf{X}(t)$ denotes the random trajectory generated by $\mathbf{V}$ starting at $\mathbf{0}$ at time $t = 0$. Let us consider the Markov chain with the state space $\tilde{\Omega}$ on the probability space $(\tilde{\Omega}, \tilde{\mathscr{V}}, \tilde{P})$ given by

$$\Xi_0\big((\omega_n)_{n \in Z_+}\big) = (\omega_n)_{n \in Z_+},$$

$$\Xi_{m+1}\big((\omega_n)_{n \in Z_+}\big) = \mathscr{Z}\Big(\Xi_m\big((\omega_n)_{n \in Z_+}\big)\Big).$$

We can easily observe that

$$\mathbf{W}\big(t,\mathbf{Y}\big(t;(\omega_n)_{n\in Z_+}\big);(\omega_n)_{n\in Z_+}\big)$$
$$=\mathbf{W}\big(t-NT,\mathbf{Y}\big(t-NT;\Xi_N\big((\omega_n)_{n\in Z_+}\big)\big);\Xi_N\big((\omega_n)_{n\in Z_+}\big)\big)$$

for all t and $N\in Z_+$. By $\mathbf{Y}$ we have denoted the random trajectory of a particle generated by the flow $\mathbf{W}$ and such that $\mathbf{Y}(0)=\mathbf{0}$. The transition probability operator Q for the chain $\{\Xi_m((\omega_n)_{n\in Z_+})\}_{m\in Z_+}$ defined on $L^1(\tilde\Omega,\tilde{\mathscr V},\tilde P)$ is given by the formula

$$Qf\big((\omega_n)_{n\in Z_+}\big)=\int f\big((\omega_n')_{n\in Z_+}\big)P(d\omega_0), \tag{16}$$

where

$$\omega_0'=\omega_0,$$
$$\omega_n'=\tau_{0,-X(T;\omega_0)}(\omega_n)\quad \text{for } n\ge 1.$$

Let us observe that

$$Q\mathbf{1}=\mathbf{1};$$

that is, Q preserves $\tilde P$. Therefore (13) equals, up to a term of order $0(\varepsilon^2)$,

$$\sum_{i=1}^d 2\varepsilon^2\int_0^{(t-s)/\varepsilon^2}d\varrho'\int_0^{\varrho'-s/\varepsilon^2}\mathbf{E}\Big[|\mathbf{Y}_\varepsilon(u-s)-\mathbf{Y}_\varepsilon(t-s)|^pW_{i,i}(\varrho',\varrho'')\Big]\,d\varrho''.$$

Here

$$W_i\big(t;(\omega)_{n\in Z_+}\big)=W_i\big(t,\mathbf{Y}\big(t;(\omega)_{n\in Z_+}\big);(\omega)_{n\in Z_+}\big),\qquad i=1,\dots,d$$

and the notation concerning multiple products of the above processes can be introduced in analogy with what we have done for $\mathbf{V}$. We also denote

$$\mathbf{Y}_\varepsilon(t)=\varepsilon\mathbf{Y}(t/\varepsilon^2).$$

Consider only the off-diagonal part of the double integral, that is, when $\varrho'-\varrho''>T$. We get, again up to a term of order $O(\varepsilon^2)$,

$$\begin{aligned}
&\sum_{i=1}^d 2\varepsilon^2\int\int_{(t-s)/\varepsilon^2>\varrho'>\varrho''+T}\ \sum_{n=[\varrho'/T]}^{(t-s)/(\varepsilon^2T)}\ \sum_{m=0}^{[\varrho''/T]}\mathbf{E}\Big[\big|\mathbf{Y}_\varepsilon\big(u-s-\varepsilon^2nT;\Xi_{n-m}\big)\\
&\qquad\qquad -\mathbf{Y}_\varepsilon\big(t-s-\varepsilon^2nT;\Xi_{n-m}\big)\big|^p\\
&\qquad\qquad \times W_i\big(\varrho'-\varepsilon^2nT;\Xi_{n-m}\big)W_i\big(\varrho''-\varepsilon^2mT\big)\Big]\,d\varrho'\,d\varrho''\\
&=\sum_{i=1}^d 2\varepsilon^2\int\int_{(t-s)/\varepsilon^2>\varrho'>\varrho''+T}\ \sum_{n=[\varrho'/T]}^{(t-s)/(\varepsilon^2T)}\ \sum_{m=0}^{[\varrho''/T]}\mathbf{E}\Big[\big|\mathbf{Y}_\varepsilon(u-s-\varepsilon^2nT)\\
&\qquad\qquad -\mathbf{Y}_\varepsilon(t-s-\varepsilon^2nT)\big|^p\\
&\qquad\qquad \times W_i(\varrho'-\varepsilon^2nT)Q^{n-m}\big(W_i(\varrho''-\varepsilon^2mT)\big)\Big]\,d\varrho'\,d\varrho''.
\end{aligned}\tag{17}$$

Let us observe that the Lagrangian velocity $W_i(\varrho'' - \varepsilon^2 mT)$ is of zero mean and $\tilde{\mathscr{V}}_0$-measurable, where

$$\tilde{\mathscr{V}}_0 = \mathscr{V} \otimes \mathscr{T} \otimes \mathscr{T} \otimes \cdots$$

and $\mathscr{T} = \{\phi, \Omega\}$ is a trivial σ-algebra. The last term in (17) therefore can be easily shown to be equal to 0 when $n - m \geq 1$, using (16).

We can see thus that for any $p > 0$,

$$\mathbf{E}\Big[|\mathbf{Y}_\varepsilon(u) - \mathbf{Y}_\varepsilon(t)|^p |\mathbf{Y}_\varepsilon(t) - \mathbf{Y}_\varepsilon(s)|^2\Big] \leq C(t - s),$$

which suffices to claim tightness of the family $\{\mathbf{Y}_\varepsilon(t)\}_{t \geq 0}$.

3. Lemmas on stationarity and conditional expectations. The following lemma is a version of a well-known fact about random shifts, shown in [28] and [15], where it was derived with the help of the theory of Palm measures. Before formulating it, let us make a convention concerning the terminology. We say that a random vector field $\mathbf{W}(t, \mathbf{x})$ on the probability space $(\Omega, \mathscr{V}, P)$ is of at most linear growth in t and $\mathbf{x}$ if for almost every $\omega \in \Omega$ there is a constant $K(\omega)$ so that $|\mathbf{W}(t, \mathbf{x}; \omega)| \leq K(\omega)(|t| + |\mathbf{x}|)$.

REMARK. Note that according to Remark 3 in Section 2, $\mathbf{V}(t, \mathbf{x})$ is of at most linear growth.

For a random variable $\tilde{U}$: $\Omega \to R$ let us denote

$$U(t, s, \mathbf{x}; \omega) = \tilde{U}\big(\tau_{t, \mathbf{Y}^s, \mathbf{x}_{(t;\omega)}}(\omega)\big)$$

with the convention of suppressing both s and $\mathbf{x}$ when they are both zero.

LEMMA 1. *Suppose that* $\mathbf{W}(t, \mathbf{x}) = (W_1(t, \mathbf{x}), \dots, W_d(t, \mathbf{x}))$ *is a space–time strictly stationary random d-dimensional vector field on* $(\Omega, \mathscr{V}, P)$; *that is, there exists* $\tilde{\mathbf{W}}$ *so that* $\mathbf{W}(t, \mathbf{x}; \omega) = \tilde{\mathbf{W}}(\tau_{t, \mathbf{x}}(\omega))$, *for all* $\omega, t, \mathbf{x}$. *In addition, assume that* $\mathbf{W}$ *has trajectories continuous in* t, C^1-*smooth in* $\mathbf{x}$,

$$\operatorname{div} \mathbf{W}(t, x) = \sum_{i=1}^{d} \partial_{x_i} W_i \equiv 0$$

and that its growth in $t, \mathbf{x}$ *is at most linear. Then for any* $s, \mathbf{x}$ *and* $\tilde{U} \in L^1(\Omega, \mathscr{V}, P)$, *the random process* $U(t, s, \mathbf{x})$, $t \in R$ *is strictly stationary. Here*

$$\mathbf{Y}^{s,\mathbf{x}}(t) = \mathbf{x} + \int_s^t \mathbf{W}(\sigma, \mathbf{Y}^{s,\mathbf{x}}(\sigma))\, d\sigma.$$

PROOF. Note that for any t, $h \in R$,

$$\mathbf{Y}^{s,\mathbf{x}}(t + h; \omega) = \mathbf{Y}^{s,\mathbf{x}}(h; \omega) + \mathbf{Y}^{0,\mathbf{0}}\big(t; \tau_{h, \mathbf{Y}^s, \mathbf{x}_{(h;\omega)}}(\omega)\big). \tag{18}$$

The transformation of the probability space

$$\theta_h \colon \omega \mapsto \tau_{h, \mathbf{Y}^s, \mathbf{x}_{(h;\omega)}}(\omega)$$

preserves the measure P (see, e.g., [28], Theorem 3, page 501 or the results of [15]). We can write that for any $t_1 \le \cdots \le t_n$ and $A_1, \dots, A_n \in R^d$,

$$\begin{aligned}
&\mathbf{E}\mathbf{1}_{A_1}(U(t_1 + h, s, \mathbf{x})) \cdots \mathbf{1}_{A_n}(U(t_n + h, s, \mathbf{x})) \\
&\quad = \mathbf{E}\mathbf{1}_{A_1}(U(t_1; \theta_h(\omega))) \cdots \mathbf{1}_{A_n}(U(t_n; \theta_h(\omega))) \\
&\quad = \mathbf{E}\mathbf{1}_{A_1}(U(t_1)) \cdots \mathbf{1}_{A_n}(U(t_n)). \qquad \square
\end{aligned}$$

In the next lemma we recall Theorem 3 of [28].

LEMMA 2. *Under the assumptions of Lemma* 1,

$$\mathbf{E}\tilde{U}\big(\tau_{0,\mathbf{Y}^s}, \mathbf{x}_{(t;\,\omega)}(\omega)\big) = \mathbf{E}\tilde{U}.$$

The following lemma is a simple consequence of the fact that in the case of Gaussian random variables, L^2-orthogonality and the notion of independence coincide.

LEMMA 3. *Assume that* $f \in L^2(\Omega, \mathscr{V}_{-\infty,+\infty}, P)$ *and* $-\infty \le a \le b \le +\infty$. *Then there exists* $\tilde{f} \in L^2(\Omega \times \Omega, \mathscr{V}_{a,b} \otimes \mathscr{V}_{a,b}^{\perp}, P \otimes P)$ *such that one of the following hold*:

(i) $$f(\omega) = \tilde{f}(\omega, \omega);$$

(ii) *f and $\tilde{f}$ have the same probability distributions.*

Before stating the proof of this lemma, let us introduce some additional notation. Denote by $\mathscr{C} = C(R^{d+1}; R^d)$ the space of all continuous mappings from R^{d+1} to R^d. For $f, g \in \mathscr{C}$,

$$D(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{|t|+|\mathbf{x}| \le n} |f(t,\mathbf{x}) - g(t,\mathbf{x})|}{1 + \sup_{|t|+|\mathbf{x}| \le n} |f(t,\mathbf{x}) - g(t,\mathbf{x})|}.$$

As is well known, D is a metric on $\mathscr{C}$ and the metric space $(\mathscr{C}, D)$ is a Polish space; that is, it is separable and complete. By $\mathscr{M}$ we denote the smallest σ-algebra containing the sets of the form

$$[f: f(t,\mathbf{x}) \in A],$$

here $(t,\mathbf{x}) \in R^{d+1}$ and $A \in \mathscr{B}_{R^d}$. It can be shown with no difficulty that $\mathscr{M}$ is the Borelian σ-algebra of subsets of $(\mathscr{C}, D)$.

PROOF OF LEMMA 3. Consider the mappings ϕ, $\phi_{a,b}$, $\phi_{a,b}^{\perp}$ from Ω to $\mathscr{C}$ and ψ from $\Omega \times \Omega$ to $\mathscr{C}$ given by the formulas

$$\begin{aligned}
\phi(\omega) &= \mathbf{V}(\cdot, \cdot\,; \omega), \\
\phi_{a,b}(\omega) &= \mathbf{V}_{a,b}(\cdot, \cdot\,; \omega), \\
\phi_{a,b}^{\perp}(\omega) &= \mathbf{V}_{a,b}^{\perp}(\cdot, \cdot\,; \omega), \\
\psi(\omega, \omega') &= \mathbf{V}_{a,b}(\cdot, \cdot\,; \omega) + \mathbf{V}_{a,b}^{\perp}(\cdot, \cdot\,; \omega'),
\end{aligned}$$

where in the last definition the addition is understood as the addition of two vectors in a linear space $\mathscr{C}$. These mappings are $\mathscr{V}_{-\infty,+\infty}$ to $\mathscr{M}$ measurable, $\mathscr{V}_{a,b}$ to $\mathscr{M}$ measurable, $\mathscr{V}^{\perp}_{a,b}$ to $\mathscr{M}$ measurable and $\mathscr{V}_{a,b}\otimes\mathscr{V}^{\perp}_{a,b}$ to $\mathscr{M}$ measurable, respectively.

Suppose that $f\colon \Omega\to R$ is a $\mathscr{V}_{-\infty,+\infty}$-measurable random variable. There exists $g\colon \mathscr{C}\to R$ such that $f = g\circ\phi$. Set

$$\tilde{f}(\omega,\omega') = g(\psi(\omega,\omega')).$$

Obviously it fulfills condition (i) from the conclusion of the lemma.

The fact that f and $\tilde{f}$ have the same probability distributions follows from the observation that the probability distribution laws of the processes

$$\mathbf{V}(t,\mathbf{x},\omega) = \phi(\omega)(t,\mathbf{x})$$

and

$$\mathbf{W}(t,\mathbf{x},\omega,\omega') = \psi(\omega,\omega')(t,\mathbf{x})$$

are identical. □

The following lemma provides us with a simple formula for an expectation of a random process along the trajectory $\mathbf{X}(t)$, $t\ge 0$ [the solution of (10) with $\varepsilon = 1$] conditioned on $\mathscr{V}_{a,b}$. Let $\mathbf{X}^{s,\mathbf{x}}$ be the solution of

$$\mathbf{X}^{s,\mathbf{x}}(t) = \mathbf{x} + \int_s^t \mathbf{V}(\varrho,\mathbf{X}^{s,\mathbf{x}}(\varrho))\,d\varrho,$$

with the convention that we suppress superscripts if $s = 0$ and $\mathbf{x} = \mathbf{0}$.

By $\{\tilde{\mathbf{X}}^{b,\mathbf{x}}_c(t)\}_{t\in R}$ we denote a stochastic process obtained by solving the following system of O.D.E's:

$$\begin{aligned}\frac{d\tilde{\mathbf{X}}^{b,\mathbf{x}}_c(t)}{dt} &= \mathbf{V}_{a,b}\big(t+c,\tilde{\mathbf{X}}^{b,\mathbf{x}}_c(t;\omega,\omega');\omega\big)\\ &\quad+\mathbf{V}^{a,b}\big(t+c,\tilde{\mathbf{X}}^{b,\mathbf{x}}_c(t;\omega,\omega');\omega'\big),\\ \tilde{\mathbf{X}}^{b,\mathbf{x}}_c(b) &= \mathbf{x}.\end{aligned}\tag{19}$$

In addition, when $a = -\infty$ and $b = 0$ we set

$$\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^{0,\mathbf{0}}_0.\tag{20}$$

LEMMA 4. *Assume that $f\in L^2(\Omega,\mathscr{V}_{-\infty,+\infty},P)$. Let $\tilde{f}$ be any random variable which satisfies conditions* (i) *and* (ii) *of Lemma* 3. *Suppose that $-\infty\le a\le b\le+\infty$ and c is any real number.*

Then for any event C that is $\mathscr{V}^{\perp}_{a,b}$ measurable

$$\begin{aligned}&\mathbf{E}\Big\{f\big(\tau_{0,\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega))}(\omega)\big)\mathbf{1}_C(\omega)\Big|\mathscr{V}_{a,b}\Big\}\\ &\qquad= \mathbf{E}_{\omega'}\Big\{\tilde{f}\big(\tau_{0,\tilde{\mathbf{X}}^{b,\mathbf{0}}_c(t;\omega,\omega')}(\omega),\tau_{0,\tilde{\mathbf{X}}^{b,\mathbf{0}}_c(t;\omega,\omega')}(\omega')\big)\mathbf{1}_C(\omega')\Big\}.\end{aligned}$$

Here $\mathbf{E}_{\omega'}$ denotes the expectation over the ω' variable only.

PROOF. Let h be fixed. Consider the difference equation

$$\mathbf{X}_{n+1}^h = \mathbf{X}_n^h + \mathbf{V}_n^h(\mathbf{X}_n^h)h,$$
$$\mathbf{X}_0^h = \mathbf{0},$$

where $\mathbf{V}_n^h(\mathbf{x}) = \mathbf{V}(b + c + nh, \mathbf{x})$. Define also $\{\tilde{\mathbf{X}}_n^h\}_{n \geq 0}$ via the difference equation

$$\tilde{\mathbf{X}}_{n+1}^h = \tilde{\mathbf{X}}_n^h + \mathbf{U}_n^h(\tilde{\mathbf{X}}_n^h)h + \mathbf{W}_n^h(\tilde{\mathbf{X}}_n^h)h,$$
$$\tilde{\mathbf{X}}_0^h = \mathbf{0},$$

where

$$\mathbf{U}_n^h(\mathbf{x}) = \mathbf{V}_{a,b}(b + c + nh, \mathbf{x}),$$
$$\mathbf{W}_n^h(\mathbf{x}) = \mathbf{V}^{a,b}(b + c + nh, \mathbf{x}).$$

The following lemma holds.

LEMMA 5. *For any bounded measurable function* φ: $R^d \to R^d$ *and* $C \in \mathscr{V}_{a,b}^{\perp}$,

$$\mathbf{E}\big[\varphi(\mathbf{X}_n^h)\mathbf{1}_C \big| \mathscr{V}_{a,b}\big] = \mathbf{E}_{\omega'}\big[\varphi(\tilde{\mathbf{X}}_n^h(\omega, \omega'))\mathbf{1}_C(\omega')\big] \quad \textit{for all } n \geq 0. \tag{21}$$

We shall prove this lemma after we finish the proof of Lemma 4. Suppose that $\tilde{f}$ satisfies conditions (i) and (ii) of the conclusion of Lemma 3. Then

$$\mathbf{E}\big[f(\tau_{0,\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega))}(\omega))\mathbf{1}_C(\omega)\big|\mathscr{V}_{a,b}\big]$$
$$= \mathbf{E}\big[\tilde{f}(\tau_{0,\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega))}(\omega), \tau_{0,\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega))}(\omega))\mathbf{1}_C(\omega)\big|\mathscr{V}_{a,b}\big].$$

By a standard approximation argument we may consider $\tilde{f}(\omega, \omega')$ as being of the form $f_1(\omega)f_2(\omega')$, where f_1, f_2 are, respectively, $\mathscr{V}_{a,b}$-measurable and $\mathscr{V}_{a,b}^{\perp}$-measurable. Then the right-hand side of (22) is equal to

$$\mathbf{E}\big[f_1(\tau_{0,\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega))}(\omega))f_2(\tau_{0,\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega))}(\omega)\mathbf{1}_C(\omega)\big|\mathscr{V}_{a,b}\big].$$

Consider two random functions

$$(\mathbf{x}, \omega) \mapsto F_1(\mathbf{x}, \omega) = f_1(\tau_{0,\mathbf{x}}(\omega))$$

and

$$(\mathbf{x}, \omega) \mapsto F_2(\mathbf{x}, \omega) = f_2(\tau_{0,\mathbf{x}}(\omega)),$$

which are, respectively, $\mathscr{B}_{R^d} \otimes \mathscr{V}_{a,b}$-measurable and $\mathscr{B}_{R^d} \otimes \mathscr{V}_{a,b}^{\perp}$-measurable. We wish to prove that

$$\begin{aligned}&\mathbf{E}\big[F_1(\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega)), \omega)F_2(\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega)), \omega)\mathbf{1}_C(\omega)\big|\mathscr{V}_{a,b}\big]\\ &\quad= \mathbf{E}_{\omega'}\big[F_1(\tilde{\mathbf{X}}_c^{b,\mathbf{h}0}(t;\omega,\omega'), \omega)F_2(\tilde{\mathbf{X}}_c^{b,\mathbf{0}}(t;\omega,\omega'), \omega')\mathbf{1}_C(\omega')\big].\end{aligned} \tag{23}$$

Let us consider the functions

$$\begin{aligned}F_1(\mathbf{x}, \omega) &= \Phi_1(\mathbf{x})\mathbf{1}_{C_1}(\omega),\\ F_2(\mathbf{x}, \omega) &= \Phi_2(\mathbf{x})\mathbf{1}_{C_2}(\omega),\end{aligned} \tag{24}$$

where $\Phi_1, \Phi_2\colon R^d \to R_+$ are bounded and continuous, $R_+ = [0, +\infty)$ and $C_1 \in \mathscr{V}_{a,b}$, $C_2 \in \mathscr{V}_{a,b}^{\perp}$. It is enough to prove (23) for such functions only. By a standard approximation argument we can prove (23) for arbitrary measurable F_1, F_2. Equation (23) will be proven if we can show that for any $n \geq 0$ and $h \in R$,

$$
\begin{aligned}
&\mathbf{E}\Big[\Phi_1(\mathbf{X}_n^h)\mathbf{1}_{C_1}\Phi_2(\mathbf{X}_n^h)\mathbf{1}_{C_2}\mathbf{1}_C\Big|\mathscr{V}_{a,b}\Big]\\
(25)\qquad &\quad= \mathbf{E}\Big[\Phi_1(\mathbf{X}_n^h)\Phi_2(\mathbf{X}_n^h)\mathbf{1}_{C_2}\mathbf{1}_C\Big|\mathscr{V}_{a,b}\Big]\mathbf{1}_{C_1}\\
&\quad= \mathbf{E}_{\omega'}\Big[\Phi_1\big(\tilde{\mathbf{X}}_n^h(\omega,\omega')\big)\Phi_2\big(\tilde{\mathbf{X}}_n^h(\omega,\omega')\big)\mathbf{1}_{C_2}(\omega')\mathbf{1}_C(\omega')\Big]\mathbf{1}_{C_1}(\omega).
\end{aligned}
$$

Indeed, passing to the limit with $h \downarrow 0$, we see that the functions

$$\mathbf{X}^h(t) = \mathbf{X}_n^h \quad \text{for } nh \leq t < nh$$

and

$$\tilde{\mathbf{X}}^h(t) = \tilde{\mathbf{X}}_n^h \quad \text{for } nh \leq t < nh$$

converge, for all ω, ω', to $\mathbf{X}(t)$ and $\tilde{\mathbf{X}}(t)$, respectively, uniformly on any compact interval.

The first equality in (25) follows from an elementary property of conditional expectations, while the second is a direct application of Lemma 5.

Hence the only thing remaining to be proven is Lemma 5.

PROOF OF LEMMA 5. Formula (21) is obviously true for $n = 0$. Suppose it holds for a certain n. We get for $n+1$ that

$$
\begin{aligned}
(26)\qquad \mathbf{E}\big\{\varphi(\mathbf{X}_{n+1}^h)\mathbf{1}_C\big|\mathscr{V}_{a,b}\big\} &= \mathbf{E}\big\{\varphi\big(\mathbf{X}_n^h + \mathbf{V}_n^h(\mathbf{X}_n^h)h\big)\mathbf{1}_C\big|\mathscr{V}_{a,b}\big\}\\
&= \mathbf{E}\big\{\varphi\big(\mathbf{X}_n^h + \mathbf{U}_n^h(\mathbf{X}_n^h)h + \mathbf{W}_n^h(\mathbf{X}_n^h)h\big)\mathbf{1}_C\big|\mathscr{V}_{a,b}\big\}.
\end{aligned}
$$

The only thing we need yet to prove is, therefore, that for any continuous function $\psi\colon R^d \times R^d \times R^d \to R_+$,

$$
\begin{aligned}
&\mathbf{E}\big\{\psi\big(\mathbf{X}_n^h, \mathbf{U}_n^h(\mathbf{X}_n^h)h, \mathbf{W}_n^h(\mathbf{X}_n^h)h\big)\mathbf{1}_C\big|\mathscr{V}_{a,b}\big\}\\
(27)\qquad &\quad= \mathbf{E}_{\omega'}\Big\{\psi\Big(\tilde{\mathbf{X}}_n^h(\omega,\omega'), \mathbf{U}_n^h\big(\tilde{\mathbf{X}}_n^h(\omega,\omega');\omega\big)h,\\
&\qquad\qquad \mathbf{W}_n^h\big(\tilde{\mathbf{X}}_n^h(\omega,\omega');\omega'\big)h\Big)\mathbf{1}_C(\omega')\Big\}.
\end{aligned}
$$

Actually it is enough to consider only $\psi(x,y,z) = \psi_1(x)\psi_2(y)\psi_3(z)$. A standard approximation argument will then yield (26) for any ψ. We need to verify that

$$
\begin{aligned}
&\mathbf{E}\big\{\psi_1(\mathbf{X}_n^h)\psi_2\big(\mathbf{U}_n^h(\mathbf{X}_n^h)h\big)\psi_3\big(\mathbf{W}_n^h(\mathbf{X}_n^h)h\big)\mathbf{1}_C\big|\mathscr{V}_{a,b}\big\}\\
(28)\qquad &\quad= \mathbf{E}_{\omega'}\Big\{\psi_1\big(\tilde{\mathbf{X}}_n^h(\omega,\omega')\big)\psi_2\Big(\mathbf{U}_n^h\big(\tilde{\mathbf{X}}_n^h(\omega,\omega');\omega\big)h\Big)\\
&\qquad\qquad \times\psi_3\Big(\mathbf{W}_n^h\big(\tilde{\mathbf{X}}_n^h(\omega,\omega');\omega'\big)h\Big)\mathbf{1}_C(\omega')\Big\}.
\end{aligned}
$$

Consider now the random functions

$$\Psi_2(\mathbf{x};\omega) = \psi_2\big(\mathbf{U}_n^h(\mathbf{x};\omega)h\big)$$

and

$$\Psi_3(\mathbf{x};\omega) = \psi_3(\mathbf{W}_n^h(\mathbf{x};\omega)h).$$

They are $\mathscr{B}_{R^d} \otimes \mathscr{V}_{a,b}$-measurable and $\mathscr{B}_{R_u}d \otimes \mathscr{V}_{a,b}^{\perp}$-measurable, respectively. Applying again an approximation argument, we may reduce the verification procedure to the functions of the form

$$\Psi_2(\mathbf{x};\omega) = \tilde{\Psi}_2(\mathbf{x})\mathbf{1}_{C_2}(\omega)$$

and

$$\Psi_3(\mathbf{x};\omega) = \tilde{\Psi}_3(\mathbf{x})\mathbf{1}_{C_3}(\omega),$$

where $\tilde{\Psi}_2, \tilde{\Psi}_3\colon R^d \to R_+$ are continuous, $C_2 \in \mathscr{V}_{a,b}$, $C_3 \in \mathscr{V}_{a,b}^{\perp}$. Substituting these expressions into (28) we get

$$\begin{aligned}
&\mathbf{E}\Big\{\psi_1(\mathbf{X}_n^h)\tilde{\Psi}_2(\mathbf{X}_n^h)\mathbf{1}_{C_2}\tilde{\Psi}_3(\mathbf{X}_n^h)\mathbf{1}_{C_3}\mathbf{1}_C\Big|\mathscr{V}_{a,b}\Big\}\\
&\quad= \mathbf{E}\Big\{\psi_1(\mathbf{X}_n^h)\tilde{\Psi}_2(\mathbf{X}_n^h)\tilde{\Psi}_3(\mathbf{X}_n^h)\mathbf{1}_{C_3}\mathbf{1}_C\Big|\mathscr{V}_{a,b}\Big\}\mathbf{1}_{C_2}\\
&\quad= \mathbf{E}_{\omega'}\Big\{\psi_1\big(\tilde{\mathbf{X}}_n^h(\omega,\omega')\big)\tilde{\Psi}_2\big(\tilde{\mathbf{X}}_n^h(\omega,\omega')\big)\\
&\qquad\times\tilde{\Psi}_3\big(\tilde{\mathbf{X}}_n^h(\omega,\omega')\big)\mathbf{1}_C(\omega')\mathbf{1}_{C_3}(\omega')\Big\}\mathbf{1}_{C_2}(\omega).
\end{aligned}$$

The last equality holds because we have assumed that Lemma 5 is true for n. Therefore we have verified (28) for functions specified as above and thus the proof of the lemma can be concluded with a help of a standard approximation argument. □

4. The operator Q and its properties. Throughout this section we shall assume that $\tilde{\mathbf{X}}$ has the same meaning as in (20). For fixed ω let $Z_\omega^t\colon \Omega \to \Omega$ be defined by

$$Z_\omega^t(\omega') = \tau_{0,\tilde{\mathbf{X}}(t;\omega,\omega')}(\omega').$$

Let $J^t(\cdot,\omega)$ be a probability measure on $(\Omega, \mathscr{V}_{-\infty,0}^{\perp}, P)$ given by

$$J^t(A,\omega) = P\Big[\big(Z_\omega^t\big)^{-1}(A)\Big].$$

It follows easily from the definition of $\tilde{\mathbf{X}}(t)$, $t \ge 0$, that for any set $A \in \mathscr{V}_{-\infty,0}^{\perp}$ the function $\omega \mapsto J^t(A,\omega)$ is $\mathscr{V}_{-\infty,0}$-measurable. Let $\mathscr{V}_{0,T}^s$ be the σ-algebra generated by $\mathbf{V}^{-\infty,0}(t,\mathbf{x})$, $0 \le t \le T$ and let us set

$$\mathscr{V}_{-\infty,0}^0 = \tau_{T,0}(\mathscr{V}_{0,T}^s) \subseteq \mathscr{V}_{-\infty,0}.$$

For any $f \in L^1(\Omega, \mathscr{V}_{-\infty,0}, P)$, such that $f \ge 0$, $\int_\Omega f\,dP = 1$, we define

$$[Qf][A] = \int_\Omega J^T(A,\omega)f(\omega)P(d\omega)$$

which is a probability measure on $(\Omega, \mathscr{V}_{-\infty,0}^{\perp})$.

LEMMA 6. (i) *For any f as described above $[Qf]$ is an absolutely continuous measure with respect to P.*

(ii) *The Radon–Nikodym derivative of* $[Qf]$ *with respect to* P, $d[Qf]/dP$, *is* $\mathscr{V}^s_{0,T}$*-measurable.*

(iii) $[Q1] = P$, *where* $\mathbf{1} = \mathbf{1}_\Omega$.

PROOF. (i) Suppose that $P[A] = 0$, for some $A \in \mathscr{V}^{\perp}_{-\infty,0}$. Then

$$[Qf][A] = \int_\Omega J^T(A, \omega) f(\omega) P(d\omega)$$
$$= \int_\Omega f(\omega) P(d\omega) \int_\Omega \mathbf{1}_A\big(\tau_{0,\tilde{\mathbf{X}}(T;\,\omega,\,\omega')}(\omega')\big) P(d\omega').$$

By Lemma 4 we get that the last expression is equal to

$$\tag{29} \begin{aligned} &\int_\Omega f(\omega) P(d\omega) \mathbf{E}\Big\{ 1_A\big(\tau_{0,\mathbf{X}(T;\,\omega)}(\omega)\big) \Big| \mathscr{V}_{-\infty,0} \Big\} \\ &\quad = \int_\Omega f(\omega) \mathbf{1}_A\big(\tau_{0,\mathbf{X}(T;\,\omega)}(\omega)\big) P(d\omega) = 0, \end{aligned}$$

since $\mathbf{E1}_A(\tau_{0,\mathbf{X}(T;\,\omega)}(\omega)) = \mathbf{E1}_A = P[A] = 0$ by virtue of Lemma 2.

(ii) Let

$$\theta_t(\omega) = \tau_{t,\mathbf{X}(t;\,\omega)}(\omega), \qquad t \in R.$$

Then the process

$$\Big\{ f(\theta_t(\omega)) \mathbf{1}_A\big(\tau_{0,\mathbf{X}(T;\,\theta_t(\omega))}(\theta_t(\omega))\big) \Big\}_{t\in R}$$

is stationary by virtue of Lemma 2. The right-hand side of (29) is hence equal to

$$\tag{30} \int_\Omega f(\theta_{-T}(\omega)) \mathbf{1}_A(\tau_{-T,\mathbf{0}}(\omega)) P(d\omega),$$

since by (18)

$$\mathbf{X}(-T;\omega) + \mathbf{X}(T;\theta_{-T}(\omega)) = \mathbf{X}(0;\omega) = \mathbf{0}.$$

Using the invariance of P with respect to $\tau_{-T,\mathbf{0}}(\omega)$, we get that expression (30) is equal to

$$\int_\Omega f\big(\tau_{0,\mathbf{X}(-T;\,\tau_{T,\mathbf{0}}(\omega))}(\omega)\big) \mathbf{1}_A(\omega) P(d\omega),$$

which by Lemma 4 equals

$$\int_\Omega \int_\Omega f\big(\tau_{0,\tilde{\mathbf{X}}^{0,\mathbf{0}}_T(-T;\,\omega,\,\omega')}(\omega)\big) \mathbf{1}_A(\omega') P(d\omega) P(d\omega')$$
$$= \int_A P(d\omega') \int_\Omega f\big(\tau_{0,\tilde{\mathbf{X}}^{0,\mathbf{0}}_T(-T;\,\omega,\,\omega')}(\omega)\big) P(d\omega),$$

where $\tilde{\mathbf{X}}^{0,\mathbf{0}}_T$ is defined by formula (19) for $a = -\infty$, $b = 0$. Since

$$\mathbf{V}^{-\infty,0}(t+T,\mathbf{x};\omega'), \mathbf{x} \in R^d, \ -T \le t \le 0$$

are $\mathscr{V}^s_{0,T}$ measurable, $\tilde{\mathbf{X}}^{0,\mathbf{0}}_T(-T;\omega,\omega')$ is jointly $\mathscr{V}_{-\infty,0} \otimes \mathscr{V}^s_{0,T}$-measurable.

Thus

$$\int_\Omega f\big(\tau_{0,\tilde{\mathbf{X}}_T^{0,\mathbf{0}}(-T;\,\omega,\,\psi')}(\omega)\big)P(d\omega)$$

is $\mathscr{V}^s_{0,T}$-measurable, which proves (ii) of the lemma.

(iii) Observe that

$$[Q\mathbf{1}][A]=\int_\Omega J^T(A,\omega)P(d\omega)=\mathbf{E}\big\{\mathbf{1}_A\big(\tau_{0,\mathbf{X}(T;\,\omega)}(\omega)\big)\big\}=\mathbf{E}\mathbf{1}_A=P[A].$$

The next to last equality holds by virtue of Lemma 2. □

For an arbitrary f we define Qf by the following formula

$$Qf=\frac{d[Qf]}{dP}\circ\tau_{-T,\mathbf{0}}.$$

Here $d[Qf]/dP$ is the Radon–Nikodym derivative of $[Qf]$ with respect to P on $(\mathscr{V}^{\perp}_{-\infty,0},P)$. By Lemma 6(ii),

$$Qf\in L^1\big(\Omega,\mathscr{V}^0_{-\infty,0},P\big).$$

The following lemma contains several useful properties of Q.

LEMMA 7. (i) Q: $L^1(\Omega,\mathscr{V}_{-\infty,0},P)\to L^1(\Omega,\mathscr{V}_{-\infty,0},P)$ *is a linear operator*.

(ii) $Qf\ge 0$ *for* $f\ge 0$.

(iii) $\int_\Omega Qf\,dP=\int_\Omega f\,dP$.

(iv) $Q\mathbf{1}=\mathbf{1}$.

(v) $\|Qf\|_{L^p}\le\|f\|_{L^p}$, *for any* $f\in L^p(\Omega,\mathscr{V}_{-\infty,0},P)$, $1\le p\le+\infty$.

PROOF. Parts (i) and (iv) follow directly from Lemma 6 and (ii) is obvious.

(iii) Note that from (29) we get

$$\int_{GU}Qf\,dP=[Qf][\Omega]=\int_\Omega f(\omega)\mathbf{1}_\Omega\big(\tau_{0,\mathbf{X}(T;\,\omega)}(\omega)\big)P(d\omega)=\int_\Omega f\,dP.$$

(v) Together (ii) and (iv) imply that Q is a contraction in $L^\infty(\Omega,\mathscr{V}_{-\infty,0},P)$ space (cf. [14], page 75). Therefore, by the Riesz–Torin convexity theorem (see [10], page 525) Q is a contraction in any $L^p(\Omega,\mathscr{V}^0_{-\infty,0},P)$ space. □

The next lemma will be of great use in establishing the Kolomogorov type estimates leading to the proof of weak compactness.

LEMMA 8. *Let* $p\ge 0$, N,k *be positive integers and* $s_{k+2}\ge\cdots\ge s_1\ge NT$, $1\le i\le d$. *Assume that* $Y\in L^1(\Omega,\mathscr{V}_{-\infty,0},P)$

Then

$$\mathbf{E}\left[\left|\int_{s_{k+1}}^{s_{k+2}}\mathbf{V}(\varrho)\,d\varrho\right|^p V_{i,\dots,i}(s_1,\dots,s_k)Y\right]$$

$$=\mathbf{E}\left[\left|\int_{s_{k+1}-NT}^{s_{k+2}-NT}\mathbf{V}(\varrho)\,d\varrho\right|^p V_{i,\dots,i}(s_1-NT,\dots,s_k-NT)Q^NY\right].$$

PROOF. Let us observe that

$$\mathbf{E}\left[\left|\int_{s_{k+1}}^{s_{k+2}}\mathbf{V}(\varrho)\,d\varrho\right|^{p}V_{i,\dots,i}(s_1,\dots,s_k)Y\right] \tag{31}$$
$$=\mathbf{E}\left\{\mathbf{E}\left[\left|\int_{s_{k+1}}^{s_{k+2}}\mathbf{V}(\varrho)\,d\varrho\right|^{p}V_{i,\dots,i}(s_1,\dots,s_k)\middle|\mathscr{V}_{-\infty,0}\right]Y\right\}.$$

Let us define

$$f(\omega)=\left|\int_{s_{k+1}}^{s_{k+2}}\mathbf{V}(\varrho,T,\mathbf{0};\omega)\,d\varrho\right|^{p}V_{i,\dots,i}(s_1,\dots,s_k,T,\mathbf{0};\omega)$$

and

$$\tilde{f}(\omega,\omega')=f(\omega').$$

Using the fact that

$$\mathbf{X}(t;\omega)=\mathbf{X}^{T,\mathbf{0}}\big(t;\tau_{0,\mathbf{X}(T;\omega)}(\omega)\big)+\mathbf{X}(T;\omega) \tag{32}$$

we can write that the right-hand side of (31) equals

$$\mathbf{E}\left\{\mathbf{E}\left[\left|\int_{s_{k+1}}^{s_{k+2}}\mathbf{V}\big(\varrho,T,\mathbf{0};\tau_{0,\mathbf{X}(T;\omega)}(\omega)\big)\,d\varrho\right|^{P}\right.\right. \tag{33}$$
$$\left.\left.\times V_{i,\dots,i}\big(s_1,\dots,s_k,T,\mathbf{0};\tau_{0,\mathbf{X}(T;\omega)}(\omega)\big)\middle|\mathscr{V}_{-\infty,0}\right]Y\right\}.$$

Applying Lemma 4 with $t=T$, $b,c=0$ and $a=-\infty$, we obtain that (33) equals

$$\int_\Omega Y(\omega)P(d\omega)\int_\Omega\left|\int_{s_{k+1}}^{s_{k+2}}\mathbf{V}\big(\varrho,T,\mathbf{0};\tau_{0,\tilde{\mathbf{X}}(T;\omega,\omega')}(\omega')\big)\,d\varrho\right|^{p} \tag{34}$$
$$\times V_{i,\dots,i}\big(s_1,\dots,s_k,T,\mathbf{0};\tau_{0,\tilde{\mathbf{X}}(T;\omega,\omega')}(\omega')\big)P(d\omega').$$

By the definition of $J^T(d\omega',\omega)$ given at the beginning of this section, we can conclude that the last expression in (34) is equal to

$$\int_\Omega Y(\omega)P(d\omega)\int_\Omega\left|\int_{s_{k+1}}^{s_{k+2}}\mathbf{V}(\varrho,T,\mathbf{0};\omega')\,d\varrho\right|^{p} \tag{35}$$
$$\times V_{i,\dots,i}(s_1,\dots,s_k,T,\mathbf{0};\omega')J^T(d\omega',\omega),$$

By virtue of Lemma 6(ii) and the definition of the operator Q given after the lemma, the expression in (35) is equal to

$$\int_\Omega\left|\int_{s_{k+1}}^{s_{k+2}}\tilde{\mathbf{V}}\big(\tau_{\varrho-T,\mathbf{X}^{T,0}(\varrho;\tau_{-T,0}(\omega'))}(\omega')\big)\,d\varrho\right|^{p}\tilde{V}_i\big(\tau_{s_k-T,\mathbf{X}^{T,0}(s_k;\tau_{-T,0}(\omega'))}(\omega')\big) \tag{36}$$
$$\times\cdots\times\tilde{V}_i\big(\tau_{s_1-T,\mathbf{X}^{T,0}(s_1;\tau_{-T,0}(\omega'))}(\omega')\big)QYP(d\omega'),$$

where QY is $\mathscr{V}_{-\infty,0}^{0}$-measurable according to Lemmas 6 and 7. Using the fact that

$$\mathbf{X}^{T,0}(\varrho;\tau_{-T,0}(\omega')) = \mathbf{X}(\varrho - T;\omega')$$

we get that (36) equals to

$$\int_\Omega \left| \int_{s_{k+1}-T}^{s_{k+2}-T} \mathbf{V}(\varrho;\omega')\,d\varrho \right|^p V_{i,\dots,i}(s_1 - T,\dots,s_k - T;\omega') QY(\omega') P(d\omega').$$

Repeating the above procedure N times we obtain the desired result. □

REMARK. There is a possible interpretation of what we have done so far in terms of a certain Markov chain defined on the abstract space of random environments Ω. Let us consider a random family of maps $\mathscr{Z}(\cdot;\omega)$: $\Omega \to \Omega$, $\omega \in \Omega$, defined by

$$\mathscr{Z}(\omega;\omega') = \mathscr{Z}_\omega^T(\tau_{-T,\mathbf{0}}(\omega')).$$

Using the argument employed to prove Lemma 6(ii), we note that the map $\mathscr{Z}$: $\Omega \times \Omega \to \Omega$ is $\mathscr{V}_{-\infty,0} \otimes \mathscr{V}_{-\infty,0}^{\perp}$ to $\mathscr{V}_{-\infty,0}$ measurable and therefore $\{\Xi_n^\omega\}_{n\ge 0}$ given by

$$\Xi_0^\omega(\omega') = \omega$$

$$\Xi_{n+1}^\omega(\omega') = \mathscr{Z}(\Xi_n^\omega(\omega');\omega'), \qquad n \ge 0$$

is a Markov family with values in an abstract state space of random environments $(\Omega, \mathscr{V}_{-\infty,0})$ defined over the probability space $(\Omega, \mathscr{V}_{-\infty,0}^{\perp}, P)$ where Q is its transition operator. Lemma 8 can be now formulated as follows.

LEMMA 9. *Let $p \ge 0$, N, k be positive integers and $s_{k+2} \ge \cdots \ge s_1 \ge NT$, $1 \le i \le d$, Assume that $Y \in L^1(\Omega, \mathscr{V}_{-\infty,0}, P)$. Then*

$$\mathbf{E}\left[\left| \int_{s_{k+1}}^{s_{k+2}} \mathbf{V}(\varrho)\,d\varrho \right|^p V_{i,\dots,i}(s_1,\dots,s_k) Y \right]$$

$$= \mathbf{E}_\omega \mathbf{E}_{\omega'} \left[\left| \int_{s_{k+1}-NT}^{s_{k+2}-NT} \mathbf{V}(\varrho;\Xi_N^\omega(\omega'))\,d\varrho \right|^p \right.$$

$$\left. \times V_{i,\dots,i}(s_1 - NT,\dots,s_k - NT;\Xi_N^\omega(\omega')) Y(\omega) \right].$$

5. Rate of convergence of $\{Q^n Y\}_{n\ge 0}$. Our main objective in this section is to prove the following lemma.

LEMMA 10. *Let Y be a random variable belonging to $L^2(\Omega, \mathscr{V}_{-\infty,0}, P)$ such that $\mathbf{E}Y = 0$.*

Then for any $s > 0$ there is a constant C depending only on s and $\|Y\|_{L^2}$ such that

$$\|Q^n Y\|_{L^1} \le \frac{C}{n^s} \quad \textit{for all } n.$$

PROOF. Note that, according to the definition of $[Qf]$, for any $f \geq 0$ belonging to $L^1(\Omega, \mathscr{V}_{-\infty,0}, P)$ and $A \in \mathscr{V}^{\perp}_{-\infty,0}$

$$[Qf][A] = \int_A \frac{d[Qf]}{dP}\, dP = \int_\Omega J^T(A,\omega) f(\omega) P(d\omega). \tag{37}$$

According to [15], page 149, Section 5, the absolutely continuous with respect to P part of $J^T(d\omega', \omega)$ has a density given by the formula

$$\int_{R^d} \frac{F\nu_0^T(d\mathbf{x}; \omega, \omega')}{G_T(\mathbf{0}; \omega, \tau_{0,\mathbf{x}}(\omega'))}. \tag{38}$$

Here $\nu_{\mathbf{x}}^T(U; \omega, \omega')$ stands for the cardinality of those $\mathbf{y} \in U$ for which

$$\psi_t(\mathbf{y}; \omega, \omega') = \mathbf{x},$$

where

$$\psi_t(\mathbf{x}; \omega, \omega') = \mathbf{x} + \tilde{\mathbf{X}}(t; \omega, \tau_{0,\mathbf{x}}(\omega')),$$
$$G_T(\mathbf{x}; \omega, \psi') = \det \nabla\psi_t(\mathbf{x}; \omega, \omega').$$

Note that ψ_t is at least of C^1-class because

$$\frac{d\psi_t(\mathbf{x})}{dt} = \mathbf{V}_{-\infty,0}(t, \psi_t(\mathbf{x}) - \mathbf{x}; \omega) + \mathbf{V}^{-\infty,0}(t, \psi_t(\mathbf{x}); \omega'), \qquad \psi_0(\mathbf{x}) = \mathbf{x} \tag{39}$$

and both $\mathbf{V}_{-\infty,0}, \mathbf{V}^{-\infty,0}$ are C^1 smooth in $\mathbf{x}$. As a result we know that $\nabla\psi_t(\mathbf{x})$ exists and satisfies

$$\nabla\psi_t(\mathbf{x}; \omega, \omega') = \nabla\psi_t(\mathbf{0}; \omega, \tau_{0,\mathbf{x}}(\omega')) = \mathbf{I} + \nabla\tilde{\mathbf{X}}(t; \omega, \tau_{0,\mathbf{x}}(\omega')).$$

We also have that

$$\frac{d}{dt}\nabla\psi_t(\mathbf{0}; \omega, \omega') = \nabla_{\mathbf{x}}\mathbf{V}_{-\infty,0}(t, \tilde{\mathbf{X}}(t); \omega)[\nabla\psi_t(\mathbf{0}; \omega, \omega') - \mathbf{I}] + \nabla_{\mathbf{x}}\mathbf{V}^{-\infty,0}(t, \tilde{\mathbf{X}}(t); \omega')\nabla\psi_t(\mathbf{0}; \omega, \omega') \tag{40}$$

and

$$\nabla\psi_0(\mathbf{0}; \omega, \omega') = \mathbf{I}.$$

The expression (38) is $P(d\omega')$ a.s. finite according to [15] [see formula (13) in that article]. We have therefore that

$$J^T(A, \omega) \geq \int_A P(d\omega') \int_{R^d} \frac{\nu_0^T(d\mathbf{x}; \omega, \omega')}{G_T(\mathbf{0}; \omega, \tau_{0,\mathbf{x}}(\omega'))}. \tag{41}$$

Let $\alpha(\varrho)$ be an increasing, for $\varrho \geq 0$, smooth function such that there exists the following:

(i) $\alpha(-\varrho) = \alpha(\varrho)$;
(ii) $\alpha(0) = 0$;
(iii) $\alpha'(\varrho) > 0$, for $\varrho > 0$;
(iv) $\alpha(\varrho) = \sqrt{\varrho}$, for $\varrho \geq 1$.

Set $\varphi(\mathbf{x}) = \alpha(|\mathbf{x}|)$. Fix $\gamma \in (1/2, 1)$. For any $\lambda > 0$, let

$$K_n(\lambda) = \left[\omega\colon \sup_{0 \le t \le T} \left[|\mathbf{V}_{-\infty,0}(t,\mathbf{x})| + |\nabla_{\mathbf{x}} \mathbf{V}_{-\infty,0}(t,\mathbf{x})| \right] \le \lambda(\varphi(\mathbf{x}) + \log^\gamma n)\right].$$

Let $\mathbf{V}_{-\infty,0}(t,\mathbf{x}) = (V^{(1)}_{-\infty,0}(t,\mathbf{x}), \dots, V^{(d)}_{-\infty,0}(t,\mathbf{x}))$. Consider the processes

$$W_n^{(i)}(t,\mathbf{x}) = \frac{V^{(i)}_{-\infty,0}(t,\mathbf{x})}{\varphi(\mathbf{x}) + \log^\gamma n} \quad \text{for } 0 \le t \le T, \mathbf{x} \in R^d, i = 1, \dots, d.$$

It is easy to see that for any $W_n^{(i)}$ the minimal number $N_n^{(i)}$ of d-balls with radius $\varepsilon > 0$ needed to cover R^d satisfies

$$N_n^{(i)} \le K\varepsilon^{-4d},$$

where K does not depend on ε, n and i. The Borell–Fernique–Talagrand type of estimates of the tail probabilities for Gaussian fields (see, e.g., [2], page 120, Theorem 5.2) imply that there exist constants $C, \Lambda > 0$ independent of n so that for $\lambda \ge \Lambda$ and all n we have

$$P\left[K_n(\lambda)^c\right] \le C\lambda^{4d+1} \exp\left\{-\frac{\lambda^2}{8\sigma_n^2}\right\},$$

where

$$K_n(\lambda)^c = \Omega \setminus K_n(\lambda),$$

$$\sigma_n^2 = \sup_{0 \le t \le T,\, \mathbf{x} \in R^d} \mathbf{E}\left[\frac{|\mathbf{V}_{-\infty,0}(t,\mathbf{x})|^2 + |\nabla_{\mathbf{x}} \mathbf{V}_{-\infty,0}(t,\mathbf{x})|^2}{(\varphi(\mathbf{x}) + \log^\gamma n)^2}\right]$$

$$= \frac{1}{\log^{2\gamma} n} \sup_{0 \le t \le T} \mathbf{E}\left[|\mathbf{V}_{-\infty,0}(t,\mathbf{0})|^2 + |\nabla_{\mathbf{x}} \mathbf{V}_{-\infty,0}(t,\mathbf{0})|^2\right] = \frac{C}{\log^{2\gamma} n},$$

for some constant C. Hence

$$P\left[K_n(\lambda)^c\right] \le C_{01}\lambda^{4d+1} \exp\left\{-C_{02}\lambda^2 \log^{2\gamma} n\right\}, \tag{42}$$

for $\lambda \ge \Lambda$ and certain constants C_{01}, C_{02}. Let $0 < \nu < 1$ be such that

$$\frac{\Lambda}{\Lambda + 1} e^{\nu(1+\Lambda)T} < 1$$

and $\nu\Lambda < 1$.

In the sequel we will denote $K_n(\Lambda)$ by simply writing K_n, $n \ge 1$. Define

$$L_m = \left[\omega \in \Omega\colon \sup_{0 \le t \le T} \left[|\mathbf{V}^{-\infty,0}(t,\mathbf{x})| + |\nabla_{\mathbf{x}} \mathbf{V}^{-\infty,0}(t,\mathbf{x})| \right] \le \nu|\mathbf{x}| + \log^\gamma m\right].$$

The fact that $\lim_{m \to +\infty} P[L_m] = 1$ can be proven precisely in the same way as in the argument used in Remark 3 in Section 2. There exists a constant C_ν depending on ν only such that $\varphi(\mathbf{x}) \le \nu|\mathbf{x}| + C_\nu$. For $\omega \in K_n$, $\omega' \in L_m$ we get

$$\left|\frac{d\tilde{\mathbf{X}}(t)}{dt}\right| \le \nu(1 + \Lambda)|\tilde{\mathbf{X}}(t)| + C_\nu\Lambda + \Lambda \log^\gamma n + \log^\gamma m,$$

$$\tilde{\mathbf{X}}(0) = 0.$$

Hence by the Gronwall inequality,

$$\sup_{0\le t\le T}\left|\tilde{\mathbf{X}}(t)\right| \le \frac{e^{\nu(1+\Lambda)T}-1}{\nu(1+\Lambda)}(C_1+\Lambda\log^\gamma n+\log^\gamma m). \tag{43}$$

Here $C_1 = C_\nu\Lambda$. Using (40) we get that $|\nabla\psi_0(\mathbf{0})| = 1$ and

$$\begin{aligned}\frac{d}{dt}|\nabla\psi_t(\mathbf{0})| &\le \sup_{0\le t\le T}\left|\nabla_{\mathbf{x}}\mathbf{V}_{-\infty,0}\big(t,\tilde{\mathbf{X}}(t)\big)\right|\big[|\nabla\psi_t(\mathbf{0})|+1\big]\\ &\quad+\sup_{0\le t\le T}\left|\nabla_{\mathbf{x}}\mathbf{V}^{-\infty,0}\big(t,\tilde{\mathbf{X}}(t)\big)\right|\left|\nabla\psi_t(\mathbf{0})\right|\\ &\le\Big(\Lambda\nu\sup_{0\le t\le T}\left|\tilde{\mathbf{X}}(t)\right|+C_1+\Lambda\log^\gamma n\Big)\big[|\nabla\psi_t(\mathbf{0})|+1\big]\\ &\quad+\Big(\nu\sup_{0\le t\le T}\left|\tilde{\mathbf{X}}(t)\right|+\log^\gamma m\Big)|\nabla\psi_t(\mathbf{0})|\\ &\le e^{\nu(1+\Lambda)T}(C_1+\Lambda\log^\gamma n+\log^\gamma m)|\nabla\psi_t(\mathbf{0})|\\ &\quad+\frac{\Lambda}{\Lambda+1}(e^{\nu(1+\Lambda)T}-1)(C_1+\Lambda\log^\gamma n+\log^\gamma m)\\ &\quad+C_1+\Lambda\log^\gamma n,\end{aligned}\tag{44}$$

for $0\le t\le T$.

The last inequality in (44) follows from estimate (43) of $\sup_{0\le t\le T}|\tilde{\mathbf{X}}(t)|$. Hence by the Gronwall inequality

$$\begin{aligned}\sup_{0\le t\le T}|\nabla\psi_t(\mathbf{0})| &\le 2\Lambda\exp\{e^{\nu(1+\Lambda)T}(C+\Lambda\log^\gamma n+\log^\gamma m)T\}\\ &\le C_{21}\exp(C_{22}\log^\gamma n)\exp(C_{23}\log^\gamma m)\\ &\qquad\qquad\text{for } \omega'\in L_m,\ \omega\in K_n.\end{aligned}\tag{45}$$

The next lemma establishes how large the random set $[\mathbf{x}\colon \psi_T(\mathbf{x})=\mathbf{0}]$ is for $\omega'\in L_m$, $\omega\in K_n$. Before the formulation of the lemma, let us introduce the following notation: $B_\varrho(\mathbf{0})$ denotes a ball of radius ϱ with a center at $\mathbf{0}\in R^d$.

LEMMA 11. *For $\omega'\in L_m$, $\omega\in K_n$ the set $[\mathbf{x}\colon \psi_T(\mathbf{x};\omega,\omega')=\mathbf{0}]$ is nonempty. There exists a constant C_{31} independent of n, m such that*

$$\big[\mathbf{x}\colon \psi_T(\mathbf{x})=\mathbf{0}\big]\subseteq B_{C_{31}(\log^\gamma n+\log^\gamma m)}(\mathbf{0}).$$

Assuming the above lemma, observe that

$$\bigcup_{|x|\le C_{31}(\log^\gamma n+\log^\gamma m)}\tau_{0,\mathbf{x}}(L_m)\subseteq L_{[M]+1},$$

where

$$\log^\gamma M = C_{31}(\log^\gamma n+\log^\gamma m). \tag{46}$$

Indeed if $\omega' \in L_m$, $\mathbf{x}_1 \in B_{C_{31}(\log^\gamma n + \log^\gamma m)}(\mathbf{0})$ then

$$\begin{aligned}\left|\mathbf{V}^{-\infty,0}\left(t,\mathbf{x};\tau_{0,\mathbf{x}_1}(\omega')\right)\right| &\le \nu|\mathbf{x}+\mathbf{x}_1| + \log^\gamma m\\ &\le \nu|\mathbf{x}| + C_{31}\nu(\log^\gamma n + \log^\gamma m) + \log^\gamma m\\ &\le \nu|\mathbf{x}| + (1+C_{31}\nu)(\log^\gamma n + \log m)\\ &\le \nu|\mathbf{x}| + C_{31}(\log^\gamma n + \log^\gamma m)\end{aligned}$$

provided that C_{31} in Lemma 11 has been chosen sufficiently large. Hence $\tau_{0,\mathbf{x}_1}(\omega') \in L_{[M]+1}$ for the choice of M according to (46). Therefore we can write that

$$\begin{aligned}&\int_{R^d} \frac{\nu_0^T(d\mathbf{x};\omega,\omega')}{G_T(\mathbf{0};\omega,\tau_{0,x}(\omega'))}\\ &\quad\ge C_{41}\int_{R^d}\frac{\nu_0^T(d\mathbf{x};\omega,\omega')}{|\nabla\psi_T(\mathbf{0};\omega,\tau_{0,x}(\omega'))|}\\ &\quad\ge C_{42}\frac{1}{\exp(C_{23}\log^\gamma([M]+1))}\frac{1}{\exp(C_{22}\log^\gamma n)}\\ &\quad\ge C_{43}\frac{1}{\exp(C_{44}\log^\gamma m)}\frac{1}{\exp(C_{45}\log^\gamma n)}.\end{aligned}\tag{47}$$

The last but one inequality follows from Lemma 11 and estimate (45). Taking into account (41) we see that the left-hand side of (37) is greater than or equal to

$$\begin{aligned}&\int_\Omega f(\omega)P(d\omega)\int_A P(d\omega')\int_{R^d}\frac{\nu_0^T(d\mathbf{x};\omega,\omega')}{G_T(\mathbf{0};\omega,\tau_{0,\mathbf{x}}(\omega'))}\\ &\quad=\sum_{m,n=0}^{+\infty}\int_\Omega\int_A f(\omega)\mathbf{1}_{K_{n+1}\setminus K_n(\omega)\mathbf{1}L_{m+1}\setminus L_m}(\omega')P(d\omega)P(d\omega')\\ &\qquad\times\int_{R^d}\frac{\nu_0^T(d\mathbf{x};\omega,\omega')}{G_T(\mathbf{0};\omega,\tau_{0,\mathbf{x}}(\omega'))}.\end{aligned}\tag{48}$$

Here $K_0 = L_0 = \phi$. Using (47) we can estimate the right-hand side of (48) from below by

$$\begin{aligned}&C_{51}\sum_{m,n=0}^{+\infty}\int_\Omega\int_A f(\omega)\mathbf{1}_{K_{n+1}\setminus K_n}(\omega)\mathbf{1}_{L_{m+1}\setminus L_m}(\omega')\frac{1}{\exp(C_{44}\log^\gamma n)}\\ &\qquad\times\frac{1}{\exp(C_{45}\log^\gamma m)}P(d\omega)P(d\omega')\\ &\quad=\int_A\Gamma(\omega')P(d\omega')\int_\Omega f(\omega)\,\Delta(\omega)\,P(d\omega),\end{aligned}\tag{49}$$

where

$$\Gamma(\omega') = C_{51}\frac{1}{\exp(C_{45}\log^{\gamma} m)} \quad \text{on } L_{m+1}\setminus L_m,$$

$$\Delta(\omega) = \frac{1}{\exp(C_{44}\log^{\gamma} n)} \quad \text{on } K_{n+1}\setminus K_n.$$

Note that each $L_m \in \mathscr{V}^{s}_{-\infty,0} \subseteq \mathscr{V}^{\perp}_{-\infty,0}$ so Γ is $\mathscr{V}^{\perp}_{-\infty,0}$ measurable. Combining this with (49) and (37) we get that

$$\frac{d[Qf]}{dP}(\omega') \ge \Gamma(\omega')\int_{\Omega} f\Delta\, dP \qquad P(d\omega') \text{ a.s.} \tag{50}$$

Hence

$$Qf \ge \Gamma\tau_{-T\,0}\int_{\Omega} f\Delta\, dP$$

P a.s. Now denote $Y_n = Q^n Y$. Choose the minimum of $\int_{\Omega} Y_n^{+}\Delta\, dP$ and $\int_{\Omega} Y_n^{-}\Delta\, dP$; say it is the first one. We have then by (50),

$$\begin{aligned}\|Y_{n+1}\|_{L^1} &\le \|QY_n^{+}\|_{L^1} + \|QY_n^{-}\|_{L^1} - \int_{\Omega} Y_n^{+}\Delta\, dP\int_{\Omega}\Gamma\, dP\\ &\le \|Y_n\|_{L^1} - \int_{K_n} Y_n^{+}\Delta\, dP\int_{\Omega}\Gamma\, dP\\ &\le \|Y_n\|_{L^1} - \exp(-C_{44}\log^{\gamma} n)\int_{K_n} Y_n^{+}\, dP\int_{\Omega}\Gamma\, dP\\ &= \|Y_n\|_{L^1} + \exp(-C_{44}\log^{\gamma} n)\int_{\Omega}\Gamma\, dP\left(\int_{K_n^c} Y_n^{+}\, dP - \int_{\Omega} Y_n^{+}\, dP\right).\end{aligned} \tag{51}$$

Since $\int_{\Omega} Y_n\, dP = 0$ we get $\int_{\Omega} Y_n^{+}\, dP = 1/2\|Y_n\|_{L^1}$. The Schwarz inequality guarantees that

$$\int_{K_n^c} Y_n^{+}\, dP \le \sqrt{P[K_n^c]}\,\|Y_n^{+}\|_{L^2}$$

The right-hand side of the above inequality is, by (42) and Lemma 7(v), less than or equal to

$$C_{61}\exp\big(-C_{62}\log^{2\gamma} n\big)\|Y\|_{L^2}.$$

Finally we can conclude that

$$\|Y_{n+1}\|_{L^1} \le \|Y_n\|_{L^1}\big(1 - C_{70}\exp(-C_{72}\log^{\gamma} n)\big) + C_{71}\exp\big(-C_{73}\log^{2\gamma} n\big).$$

The positive constants C_{70}, C_{71} depend only on $\|Y\|_{L^2}$. We can observe that

$$\begin{aligned}\|Y_n\|_{L^1} &\le C_{71}\sum_{k=1}^{n}\exp\big(-C_{73}\log^{2\gamma} k\big)\prod_{p=k+1}^{n}\big(1 - C_{70}\exp(-C_{72}\log^{\gamma} p)\big)\\ &\quad + \|Y\|_{L^1}\prod_{p=1}^{n}\big(1 - C_{70}\exp(-C_{72}\log^{\gamma} p)\big).\end{aligned} \tag{52}$$

Now the kth term of the sum on the right-hand side of (52) can be estimated by

$$C_{71}\exp(-C_{73}\log^{2\gamma}k)\exp\left\{-C_{70}\sum_{p=k+1}^{n}\exp(-C_{72}\log^{\gamma}p)\right\}. \tag{53}$$

For $k > [n/2]$ we get therefore an estimate of (53) by the first factor, that is, by

$$\frac{C_{81}}{n^{C_{82}\log^{2\gamma-1}(n/2)}}.$$

For $k < [n/2]$,

$$\sum_{p=k+1}^{n}\exp(-C_{72}\log^{\gamma}p) = \sum_{p=k+1}^{n}\frac{1}{p^{C_{72}/\log^{1-\gamma}p}} \geq n^{1-r}$$

for $0 < r < 1$ and n sufficiently large. Therefore (53) can be estimated by $C_{91}\exp(-C_{71}n^{1-r})$. Hence Lemma 10 has been completely proven, provided that we prove Lemma 11. □

Proof of Lemma 11. For $\omega \in K_n$, $\omega' \in L_m$, consider the mapping F: $R^d \to R^d$ constructed in the following way. For any $\mathbf{x} \in R^d$, solve for $0 \leq t \leq T$ the system of O.D.E.'s

$$\begin{aligned} \frac{d\mu_t(\mathbf{x})}{dt} &= \mathbf{V}_{-\infty,0}(t, \mu_t(\mathbf{x}) - \mathbf{x}; \omega) + \mathbf{V}^{-\infty,0}(t, \mu_t(\mathbf{x}); \omega'), \\ \mu_T(\mathbf{x}) &= 0. \end{aligned} \tag{54}$$

Let us set

$$F(\mathbf{x}) = \mu_0(\mathbf{x}).$$

Note that

$$\mu_t(\mathbf{x}) = \phi_t(\mathbf{x}) \quad \text{for all } 0 \leq t \leq T$$

iff

$$F(\mathbf{x}) = \mathbf{x}.$$

Observe that then $\mathbf{x}$ belongs to the set

$$[\mathbf{x}: \phi_T(\mathbf{x}) = \mathbf{0}].$$

We can write therefore that

$$\begin{aligned} \left|\frac{d\mu_t(\mathbf{x})}{dt}\right| &\leq \Lambda\nu|\mu_t(\mathbf{x}) - \mathbf{x}| + C_1 + \Lambda\log^{\gamma}n + \nu|\mu_t(\mathbf{x})| + \log^{\gamma}m \\ &\leq (\Lambda+1)\nu|\mu_t(\mathbf{x})| + \Lambda\nu|\mathbf{x}| + C_1 + \Lambda\log^{\gamma}n + \log^{\gamma}m \end{aligned}$$

and $\mu_T(\mathbf{0}) = \mathbf{0}$. By the Gronwall inequality,

$$
\begin{aligned}
|\mu_t(\mathbf{x})| &\le \frac{e^{(\Lambda+1)\nu T}-1}{(\Lambda+1)\nu}(\Lambda\nu|\mathbf{x}| + C_1 + \Lambda\log^\gamma n + \log^\gamma m) \\
&= \frac{\Lambda}{\Lambda+1}(e^{(\Lambda+1)\nu T}-1)|\mathbf{x}| \\
&\quad + \frac{e^{(\Lambda+1)\nu T}-1}{(\Lambda+1)\nu}(C_1 + \Lambda\log^\gamma n + \log^\gamma m).
\end{aligned} \tag{55}
$$

For

$$
R \ge \frac{(e^{(\Lambda+1)\nu T}-1)/((\Lambda+1)\nu)(C_1 + \Lambda\log^\gamma n + \log^\gamma m)}{1-(\Lambda/\Lambda+1)(e^{(\Lambda+1)\nu T}-1)}
$$

we can observe that $|\mathbf{x}| \le R$ implies $|F(\mathbf{x})| \le R$. Choose therefore $R = C(\log^\gamma n + \log^\gamma m)$ for C sufficiently large independent of n, m. By Brouwer's theorem there exists an $\mathbf{x}$ such that $F(\mathbf{x}) = \mathbf{x}$. Then, as we have noted $[\mathbf{x}: \psi_T(\mathbf{x}) = \mathbf{0}]$ is not empty. The second part of the lemma follows from the a priori estimate (55). □

6. Tightness and the limit identification. The first step toward establishing tightness of the family $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, is to check that for any $L > 0$ there is $C > 0$ so that for all $0 \le s \le t \le L$, $\varepsilon > 0$,

$$
\mathbf{E}|\mathbf{X}_\varepsilon(t) - \mathbf{X}_\varepsilon(s)|^2 \le C(t-s). \tag{56}
$$

The left-hand side of (56) equals

$$
\begin{aligned}
&\varepsilon^2\mathbf{E}\left|\mathbf{X}\left(\frac{t}{\varepsilon^2}\right) - \mathbf{X}\left(\frac{s}{\varepsilon^2}\right)\right|^2 \\
&\quad = 2\varepsilon^2\sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} ds_1 \int_0^{s_1} \mathbf{E}[V_{i,i}(s_1,\varrho)]\, d\varrho \\
&\quad = 2\varepsilon^2\sum_{i=1}^d \int_{s/\varepsilon^2}^{t/\varepsilon^2} ds_1 \int_{s/\varepsilon^2}^{s_1} \mathbf{E}\big[V_i(s_1-\varrho)\tilde{V}_i\big]\, d\varrho \\
&\quad = 2\varepsilon^2\sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} ds_1 \int_0^{s_1} \mathbf{E}\big[V_i(s_1-\varrho)\tilde{V}_i\big]\, d\varrho \\
&\quad = 2\varepsilon^2\sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} ds_1 \sum_{k=1}^{[s_1/T]} \int_{s_1-kG}^{s_1-k-1)T} \mathbf{E}\big[V_i(s_1-\varrho)\tilde{V}_i\big]\, d\varrho \\
&\qquad + 2\varepsilon^2\sum_{i=1}^d \int_0^{(t-s)/\varepsilon^2} ds_1 \int_0^{s_1-[s_1/T]T} \mathbf{E}\big[V_i(s_1-\varrho)\tilde{V}_i\big]\, d\varrho.
\end{aligned} \tag{57}
$$

Let us notice that the stationarity of $\{V_i^2(t,\mathbf{X}(t))\}_{t\ge 0}$ implies that

$$
\left|\mathbf{E}\big[V_i(s_1-\varrho)\tilde{V}_i\big]\right| \le \big[\mathbf{E}V_i^2(s_1-\varrho)\big]^{1/2}\|\tilde{V}_i\|_{L^2} = \|\tilde{V}_i\|_{L^2}^2.
$$

Therefore the last term on the right-hand side of (57) is of order $(t - s)$. We will be concerned with the first term only. By Lemma 8 it equals

$$2\varepsilon^2 \sum_{i=1}^{d} \int_0^{(t-s)/\varepsilon^2} ds_1 \sum_{k=1}^{[s_1/T]} \int_{s_1-kT}^{s_1-(k-1)T} \mathbf{E}\big[V_i(s_1 - \varrho - (k-1)T)Q^{k-1}\tilde{V}_i\big]\, d\varrho. \tag{58}$$

Let

$$W = V_i(s_1 - \varrho - (k-1)T).$$

Because the one-dimensional distributions of W are Gaussian, $\mathbf{E}W = 0$ and $\mathbf{E}W^2 = \mathbf{E}\tilde{V}_i^2$, we get from that and Lemma 10,

$$\begin{aligned}\big|\mathbf{E}\big(WQ^{k-1}\tilde{\mathbf{V}}_i\big)\big| &\le \int_{|W|\ge k} |WQ^{k-1}\tilde{V}_i|\, dP + \int_{|W|\le k} |WQ^{k-1}\tilde{V}_i|\, dP \\ &\le \left(\int_{|W|\ge k} W^2\, dP\right)^{1/2} \|Q^{k-1}\tilde{V}_i\|_{L^2} + k\|Q^{k-1}\tilde{V}_i\|_{L^1} \le \frac{C}{k^2},\end{aligned}$$

where C is some constant. Hence (58) can be estimated by

$$C\varepsilon^2 \sum_{i=1}^{d} \int_0^{(t-s)/\varepsilon^2} ds_1 \sum_{k=1}^{[s_1/T]} \frac{1}{k^2} \le C'(t-s).$$

COROLLARY 1. *The integrals*

$$\int_0^{+\infty} \mathbf{E}\big[V_i(\varrho, \mathbf{X}(\varrho))V_i(0, \mathbf{0})\big]\, d\varrho,$$

are convergent for $i, j = 1, \dots, d$.

Let us estimate

$$\varepsilon^2 \mathbf{E}\left[|\mathbf{X}_\varepsilon(u) - \mathbf{X}_\varepsilon(t)|^p \left(X_i\left(\frac{t}{\varepsilon^2}\right) - X_i\left(\frac{s}{\varepsilon^2}\right)\right)^2\right], \tag{59}$$

for $0 \le s \le t \le u \le L$, $0 < p < 2$. The expression in (59) equals

$$\begin{aligned}&2\varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} ds_1 \int_{s/\varepsilon^2}^{s_1} \mathbf{E}\big[|\mathbf{X}_\varepsilon(u) - \mathbf{X}_\varepsilon(t)|^p V_{i,i}(s_1, \varrho)\big]\, d\varrho \\ &\quad = 2\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} ds_1 \int_0^{s_1} \mathbf{E}\big[|\mathbf{X}_\varepsilon(u-s) - \mathbf{X}_\varepsilon(t-s)|^p\, V_{i,i}(s_1, \varrho)\big]\, d\varrho \\ &\quad = 2\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} ds_1 \int_0^{s_1} \mathbf{E}\big[|\mathbf{X}_\varepsilon(u - s - \varepsilon^2\varrho) - \mathbf{X}_\varepsilon(t - s - \varepsilon^2\varrho)|^p \\ &\qquad\qquad \times V_i(s_1 - \varrho)\tilde{V}_i\big]\, d\varrho.\end{aligned}$$

The last two equalities follow from the stationarity of the process

$$\left\{\left|\varepsilon\int_{(t-s-\varepsilon^2\varrho)/\varepsilon^2}^{(u-s-\varepsilon^2\varrho)/\varepsilon^2}\mathbf{V}(\varrho'+r)\,d\varrho'\right|^p V_{i,i}(s_1+r,\varrho+r)\right\}_{r\geq 0}$$

(cf. Lemma 1). The left-hand side of (59) is equal to

$$2\varepsilon^2\int_0^{(t-s)/\varepsilon^2}ds_1\sum_{k=1}^{d}\int_{s_1-kT}^{s_1-(k-1)T}\mathbf{E}\Big\{\big|\mathbf{X}_\varepsilon(u-s-\varepsilon^2\varrho)$$
$$-\mathbf{X}_\varepsilon(t-s-\varepsilon^2\varrho)\big|^p V_i(s_1-\varrho)\tilde{V}_i\Big\}\,d\varrho$$

$$(60)\quad =2\varepsilon^2\int_0^{(t-s)/\varepsilon^2}ds_1\sum_{k=1}^{d}\int_{s_1-kT}^{s_1-(k-1)T}\mathbf{E}\Big\{\big|\mathbf{X}_\varepsilon(u-s-\varepsilon^2\varrho-(k-1)\varepsilon^2T)$$
$$-\mathbf{X}_\varepsilon(t-s-\varepsilon^2\varrho-(k-1)\varepsilon^2T)\big|^p$$
$$\times V_i(s_1-\varrho-(k-1)T)Q^{k-1}\tilde{V}_i\Big\}\,d\varrho.$$

Let

$$\Gamma_1=\big|\mathbf{X}_\varepsilon(u-s-\varepsilon^2\varrho-(k-1)\varepsilon^2T)-\mathbf{X}_\varepsilon(t-s-\varepsilon^2\varrho-(k-1)\varepsilon^2T)\big|^p$$

and

$$\Gamma_2=V_i(s_1-\varrho-(k-1)T).$$

The expectation on the right-hand side of (60) can be estimated as follows:

$$(61)\quad \mathbf{E}\Gamma_1\Gamma_2Q^{k-1}\tilde{V}_i=\int_{\Gamma_1\leq k^\nu(u-t)^\alpha}\Gamma_1\Gamma_2Q^{k-1}\tilde{V}_i\,dP+\int_{\Gamma_1>k^\nu(u-t)^\alpha}\Gamma_1\Gamma_2Q^{k-1}\tilde{V}_i\,dP,$$

where $\alpha,\nu>0$ are to be determined later. Note that

$$\left|\int_{\Gamma_1\leq k^\nu(u-t)^\alpha}\Gamma_1\Gamma_2Q^{k-1}\tilde{V}_i\,dP\right|$$
$$\leq k^\nu(u-t)^\alpha\int_\Omega|\Gamma_2Q^{k-1}\tilde{V}_i|\,dP$$
$$(62)\quad =k^\nu(u-t)^\alpha\left\{\int_{|\Gamma_2|>k}|\Gamma_2Q^{k-1}\tilde{V}_i|\,dP+\int_{|\Gamma_2|\leq k}|\Gamma_2Q^{k-1}\tilde{V}_i|\,dP\right\}$$
$$\leq k^\nu(u-t)^\alpha\left[\left(\int_{|\Gamma_2|\Gamma k}\Gamma_2^2\,dP\right)^{1/2}\|Q^{k-1}\tilde{V}_i\|_{L^2}+k\|Q^{k-1}\tilde{V}_i\|_{L^1}\right].$$

Using again the fact that Γ_2 is a Gaussian random variable with zero mean and variance $\mathbf{E}\tilde{V}_i^2$ in combination with Lemma 10 and Lemma 7(v), we get that (62) can be estimated from above by $(C/k^2)(u-t)^\alpha$. The second term on the right-hand side of (61) can be estimated from above as follows:

$$\left|\int_{\Lambda_1 > k^{\nu}(u-t)^{\alpha}} \Gamma_1\Gamma_2 Q^{k-1}\tilde{V}_i\, dP\right| \le \left(P\left[\Gamma_1 > k^{\nu}(u-t)^{\alpha}\right]\right)^{\beta_1}\left(\mathbf{E}\Gamma_1^{2/p}\right)^{p/2}\left(\mathbf{E}\Gamma_2^{1/\beta_2}\right)^{\beta_2}\|Q^{k-1}\tilde{V}_i\|_{1/L^{\beta_3}}, \tag{63}$$

for some $\beta_1, \beta_2, \beta_3 > 0$. Notice that

$$P\left[\Gamma_1 > k^{\nu}(u-t)^{\alpha}\right] \le \frac{1}{k^{2\nu/p}(u-t)^{2\alpha/p}}\mathbf{E}\Gamma_1^{2/p} \le \frac{C}{k^{2\nu/p}}(u-t)^{1-2\alpha/p}.$$

Here we use the fact that, according to (56),

$$\left(\mathbf{E}\Gamma_1^{2/p}\right)^{p/2} \le C(u-t)^{p/2}$$

for some constant C. Choosing α such that

$$1 - 2\alpha/p > 0$$

and ν such that

$$\frac{\nu\beta_1}{p} \ge 1,$$

we get an estimate of (63) from above by $(C(u-t)^{\beta}/k^2$, where $\beta > 0$. Combining this with the bound for the first term of (60) obtained before we get an estimate of the expression on the left-hand side of (60) from above by

$$C\varepsilon^2\int_0^{(t-s)/\varepsilon^2} ds_1 \sum_{k=1}^{[s_1/T]} \frac{(u-t)^{\beta}}{k^2} \le C'(t-s)(u-t)^{\beta}.$$

This proves tightness of $\{\mathbf{X}_{\varepsilon}(t)\}_{t\ge 0}$, $\varepsilon > 0$, $0 \le t \le L$ in light of the following lemma.

LEMMA 12. *Suppose that* $[Y_{\varepsilon}(t)\}_{t\ge 0}$, $\varepsilon > 0$ *is a family of processes with trajectories in* $C[0, +\infty)$ *such that for any* $L > 0$ *there exist* $p, C, \nu > 0$ *so that for* $0 \le s \le t \le u \le L$,

$$\mathbf{E}\left|Y_{\varepsilon}(t) - Y_{\varepsilon}(s)\right|^2\left|Y_{\varepsilon}(u) - Y_{\varepsilon}(t)\right|^p \le C(u-s)^{1+\nu}$$

and $Y_{\varepsilon}(0) = 0$, $\varepsilon > 0$. *Then* $\{Y_{\varepsilon}(t)\}_{t\ge 0}$, $\varepsilon > 0$ *is tight.*

PROOF. It is enough to prove tightness of the above family on $C[0, L]$, for any $L > 0$ (see [32]). From the proof of Theorem 15.6, page 128 of [5], we obtain that for any $\varepsilon, \eta > 0$ there is $\delta > 0$ such that $P[w''_{Y_{\varepsilon}}(\delta) > \varepsilon] < \eta$. Here

$$w''_{Y_{\varepsilon}}(\delta) = \sup_{0\le u-s\le\delta}\ \sup_{s\le t\le u} \min\left[\left|Y_{\varepsilon}(u) - Y_{\varepsilon}(t)\right|, \left|Y_{\varepsilon}(t) - Y_{\varepsilon}(s)\right|\right].$$

One can prove that because $t \mapsto Y_{\varepsilon}(t)$ is continuous $w''_{Y_{\varepsilon}}(\delta) < w_{Y_{\varepsilon}}(\delta) < 4w''_{Y_{\varepsilon}}(\delta)$, where $w_{Y_{\varepsilon}}(\delta) = \sup_{0\le u-s\le\delta}|Y_{\varepsilon}(u) - Y_{\varepsilon}(t)|$. By Theorem 8.2, page 55 of [5], the family $\{Y_{\varepsilon}(t)\}_{L\ge t\ge 0}$, $\varepsilon > 0$ is tight over $C[0, L]$. □

Now after we have established that $\{\mathbf{X}_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$ is weakly compact, we have to prove that there is only one process whose law can be a weak limit of the laws of the processes from the family. The first step in that direction is the following lemma. First let us denote by $(X_\varepsilon^1(t), \dots, X_\varepsilon^d(t))$ the components of the process $\{\mathbf{X}_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$.

LEMMA 13. *For any* $\gamma \in (0,1)$, M *a positive integer*, $\psi\colon (R^d)^M \to R_+$ *continuous*, $0 \leq s_1 \leq \cdots \leq s_M \leq s$ *and* $i = 1, \dots, d$, *there is* $C > 0$ *such that for all* $\varepsilon > 0$ *and* $0 \leq s \leq t \leq L$ *we have*

$$\left|\mathbf{E}\left\{\left(X_\varepsilon^i(t+\varepsilon^\gamma) - X_\varepsilon^i(t)\right)\psi(\mathbf{X}_\varepsilon(s_1), \dots, \mathbf{X}_\varepsilon(s_N))\right\}\right| \leq C\varepsilon. \tag{64}$$

PROOF. The expression under the absolute value in (64) is equal to

$$\mathbf{E}\left\{\varepsilon\left[\int_{t/\varepsilon^2}^{(t/\varepsilon^2)+\varepsilon^{\gamma-2}} V_i(\varrho)\,d\varrho\right]\psi\left(\varepsilon\int_0^{s_1/\varepsilon^2}\mathbf{V}(\varrho)\,d\varrho, \dots, \varepsilon\int_0^{s_M/\varepsilon^2}\mathbf{V}(\varrho)\,d\varrho\right)\right\}. \tag{65}$$

Using Lemma, 1 we get that (65) equals to

$$\mathbf{E}\left[\varepsilon\int_0^{\varepsilon^{\gamma-2}} V_i(\varrho)\Phi\,d\varrho\right], \tag{66}$$

where

$$\Psi = \psi\left(\varepsilon\int_{-t/\varepsilon^2}^{(s_1-t)/\varepsilon^2}\mathbf{V}(\varrho)\,d\varrho, \dots, \varepsilon\int_{-t/\varepsilon^2}^{(s_M-t)/\varepsilon^2}\mathbf{V}(\varrho)\,d\varrho\right)$$

is $\mathscr{V}_{-\infty,0}$-measurable. But (66) is equal to

$$\sum_{k=1}^{[\varepsilon^{\gamma-2}/T]} \varepsilon\int_0^{\varepsilon^{\gamma-2}} \mathbf{E}[V_i(\varrho)\Psi]\,d\varrho + O(\varepsilon). \tag{67}$$

Using Lemma 8, the fact that $Q\mathbf{1} = \mathbf{1}$ in combination with Lemma 2 and Lemma 4, we get that (67) equals to

$$\sum_{k=1}^{[\varepsilon^{\gamma-2}/T]} \varepsilon\int_{\varepsilon^{\gamma-2}-kT}^{\varepsilon^{\gamma-2}-(k-1)T} \mathbf{E}\left[V_i(\varrho-(k-1)T)Q^{k-1}(\Psi - \mathbf{E}\Psi)\right]d\varrho + O(\varepsilon).$$

By Lemma 10 and estimates identical with those applied to prove (57), we get the asserted estimate. □

LEMMA 14. *Under the assumptions of Lemma* 13, *the following estimates hold. There exists* $C > 0$ *such that for all* $\varepsilon > 0$,

$$\left|\mathbf{E}\left\{\left[\left(X_\varepsilon^i(t+\varepsilon^\gamma) - X_\varepsilon^i(t)\right)\left(X_\varepsilon^j(t+\varepsilon^\gamma) - X_\varepsilon^j(t)\right) - \kappa_{ij}\right]\right.\right.$$
$$\left.\left.\times\,\psi(\mathbf{X}_\varepsilon(s_1), \dots, \mathbf{X}_\varepsilon(s_N))\right\}\right| \leq C\varepsilon,$$

for $i, j = 1, \dots, d$. *Here* κ_{ij} *is given by* (11).

PROOF. Note that

$$\begin{aligned}
&\mathbf{E}\{(X_\varepsilon^i(t+\varepsilon^\gamma)-X_\varepsilon^i(t))(\mathbf{X}_\varepsilon^j(t+\varepsilon^\gamma)-X_\varepsilon^j(t))\\
&\qquad\qquad\times\psi(\mathbf{X}_\varepsilon(s_1),\dots,\mathbf{X}_\varepsilon(s_N))\}\,d\varrho'\\
&\quad=\varepsilon^2\mathbf{E}\left[\int_0^{\varepsilon^{\gamma-2}}d\varrho\int_0^{\varrho}V_i(\varrho)V_j(\varrho')\Psi\,d\varrho'\right]\\
&\qquad+\varepsilon^2\mathbf{E}\left[\int_0^{\varepsilon^{\gamma-2}}d\varrho\int_0^{\varrho}V_j(\varrho)V_i(\varrho')\,d\varrho'\right],
\end{aligned}\tag{68}$$

where Ψ is as in the previous lemma. Consider only the first term. We can rewrite it as being equal to

$$\begin{aligned}
&\varepsilon^2\int_0^{\varepsilon^{\gamma-2}}d\varrho\int_0^{\varrho}\mathbf{E}[V_i(\varrho)V_j(\varrho')]\,d\varrho'\,\mathbf{E}\Psi\\
&\qquad+\varepsilon^2\int_0^{\varepsilon^{\gamma-2}}d\varrho\int_0^{\varrho}\mathbf{E}[V_i(\varrho)V_j(\varrho')(\Psi-\mathbf{E}\Psi)]\,d\varrho'.
\end{aligned}\tag{69}$$

Using Lemma 1 and the change of variables $\varepsilon^2\varrho\mapsto\varrho$, the first term of (69) can be written as

$$\int_0^{\varepsilon^\gamma}d\varrho\int_0^{\varrho/\varepsilon^2}\mathbf{E}[V_i(\varrho')\tilde{V}_j]\,d\varrho'\,\mathbf{E}\Psi.\tag{70}$$

Using a computation identical with that to obtain (58), we get that the difference between (70) and

$$\varepsilon^\gamma\int_0^{+\infty}\mathbf{E}[V_i(\varrho,\mathbf{X}(\varrho))\tilde{V}_j]\,d\varrho$$

is less than or equal to

$$\begin{aligned}
&\int_0^{\varepsilon^\gamma}d\varrho\left|\int_{\varrho/\varepsilon^2}^{+\infty}\mathbf{E}[V_i(\varrho')\tilde{V}_j]\,d\varrho'\right|\\
&\quad\le\int_0^{\varepsilon^\gamma}d\varrho\sum_{k=[\varrho/\varepsilon^2T]}^{+\infty}\int_{(\varrho/\varepsilon^2)-kT}^{(\varrho/\varepsilon^2)-(k-1)T}\left|\mathbf{E}[V_i(\varrho'-(k-1)T)Q^{k-1}\tilde{V}_j]\right|d\varrho'\\
&\quad\le C\int_0^{\varepsilon^\gamma}\sum_{k=[\varrho/\varepsilon^2T]}^{+\infty}\frac{1}{k^2}\,d\varrho\le C\varepsilon.
\end{aligned}$$

The second term of (69) is estimated essentially with the help of the same argument as the one used in the proof of Lemma 13. We consider two cases: first, when $\varrho'<\varepsilon^{1-\gamma}$, then by Lemma 10 we get the desired estimate; second, when $\varrho'<\varepsilon^{1-\gamma}$, we use the Schwarz inequality and reach the estimate as claimed in the lemma. □

LEMMA 15. *For any $\gamma\in(0,1)$ and $0<\gamma'<\gamma$, there is a constant C so that*

$$\mathbf{E}|\mathbf{X}_\varepsilon(t+\varepsilon^\gamma)-\mathbf{X}_\varepsilon(t)|^4\le C\varepsilon^{2\gamma'}.\tag{71}$$

PROOF. The proof of (71) reduces to an estimation of

$$\varepsilon^4 \int_0^{\varepsilon^{\gamma-2}} d\varrho_1 \int_0^{\varrho_1} d\varrho_2 \int_0^{\varrho_2} d\varrho_3 \int_0^{\varrho_3} \mathbf{E}\big[V_{i,i,i}(\varrho_2, \varrho_3, \varrho_4)\tilde{V}_i\big]\, d\varrho_4. \tag{72}$$

If $\varrho_4 < \varepsilon^{-\mu}$, $\mu > 0$ we can use Lemma 10 and estimate the same way as we have done to obtain estimate (58).

For $\varrho_4 \le \varepsilon^{-\mu}$ we get that (72) is equal to

$$\varepsilon^4 \int_0^{\varepsilon^{\gamma-2}} d\varrho_1 \int_0^{\varrho_1} d\varrho_2 \int_0^{\varrho_2} d\varrho_3 \int_0^{\varrho_3 \wedge \varepsilon^{-\mu}} \mathbf{E}[V_{i,i}(\varrho_2, \varrho_3)W]\, d\varrho_4$$

$$+ \varepsilon^4 \int_0^{\varepsilon^{\gamma-2}} d\varrho_1 \int_0^{\varrho_1} d\varrho_2 \int_0^{\varrho_2} d\varrho_3 \int_0^{\varrho_3 \wedge \varepsilon^{-\mu}} \mathbf{E}[V_{i,i}(\varrho_2, \varrho_3)]\mathbf{E}\big[V_i(\varrho_4)\tilde{V}_i\big]\, d\varrho_4,$$

where

$$W = V_i(\varrho_4)\tilde{V}_i - \mathbf{E}\big[V_i(\varrho_4)\tilde{V}_i\big].$$

The second term above is estimated easily with the help of Lemma 10 as we have done in the proof of (56). The estimation of the first term can be done separately for $\varrho_3 - \varrho_4 \ge \varepsilon^{-\mu}$ and then for $\varrho_3 - \varrho_4 \le \varepsilon^{-\mu}$. In the first case we estimate with the help of Lemmas 8 and 10 precisely as we have done before. For $\varrho_3 - \varrho_4 \le \varepsilon^{-\mu}$, we can estimate the corresponding integral by

$$C\varepsilon^4 \varepsilon^{2(\gamma-2)}\varepsilon^{-2\mu} = C\varepsilon^{2(\gamma-\mu)}. \qquad \square$$

Let $f \in C_0^\infty(R^d)$. Consider ψ as in Lemma 14. Let $t_i = s + i\varepsilon^\gamma$. Then using Taylor's formula, we get

$$\begin{aligned}
&\mathbf{E}[f(\mathbf{X}_\varepsilon(t)) - f(\mathbf{X}_\varepsilon(s))]\Psi \\
&\quad = \sum_{i:\, s<t_i<t} \mathbf{E}\{([\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)], (\nabla f)(\mathbf{X}_\varepsilon(t_i)))\Psi\} \\
&\qquad + \tfrac{1}{2} \sum_{i:\, s<t_i<t} \mathbf{E}\{([\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)] \\
&\qquad\qquad \otimes [\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)], (\nabla \otimes \nabla f)(\mathbf{X}_\varepsilon(t_i)))\Psi\} \\
&\qquad + \tfrac{1}{6} \sum_{i:\, s<t_i<t} \mathbf{E}\{([\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)] \\
&\qquad\qquad \otimes [\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}\varepsilon(t_i)] \otimes [\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)], \\
&\qquad\qquad\qquad (\nabla \otimes \nabla \otimes \nabla f)(\theta_i))\Psi\}.
\end{aligned} \tag{73}$$

Here θ_i is a point on the segment $[\mathbf{X}_\varepsilon(t_i), \mathbf{X}_\varepsilon(t_{i+1})]$. The last term of (73) can be estimated by

$$\frac{C}{\varepsilon^\gamma}\Big[\mathbf{E}|\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)|^4\Big]^{3/4}$$

with the help of the Schwarz lemma. Using Lemma 15, we get that this term is of order of magnitude $C\varepsilon^{3\gamma'/2-\gamma} = o(1)$. The first term is estimated with the help of Lemma 13 by $C\varepsilon^{1-\gamma}$. The second term of (73) can be rewritten as being equal to

$$\frac{1}{2} \sum_{i:\, s<t_i<t} \mathbf{E}\Big\{\big([\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)] \otimes [\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)] - \kappa^\varepsilon_{i,j}, (\nabla \otimes \nabla f)(\mathbf{X}_\varepsilon(t_i))\big)\Psi\Big\}$$
$$+ \frac{1}{2} \sum_{i:\, s<t_i<t} \mathbf{E}\Big\{\big(\kappa^\varepsilon_{i,j}, (\nabla \otimes \nabla f)(\mathbf{X}_\varepsilon(t_i))\big)\Psi\Big\}.$$

where

$$\kappa^\varepsilon_{i,j} = \mathbf{E}\{[\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)] \otimes [\mathbf{X}_\varepsilon(t_{i+1}) - \mathbf{X}_\varepsilon(t_i)]\}.$$

The first term of the above sum tends to zero faster than ε^γ. The latter is equal to

$$\kappa_{ij}\varepsilon^\gamma \mathbf{E}\Psi$$

up to a term of magnitude $o(\varepsilon^\gamma)$.

Summarizing, any weak limit of $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, must have a law in $C([0, +\infty); R^d)$ such that for any $f \in C_0^\infty(R^d)$,

$$f(\mathbf{x}(t)) - \frac{1}{2} \sum_{i,j=1,\dots,d} \kappa_{ij} \int_0^t \partial^2_{ij} f(\mathbf{x}(\varrho))\, d\varrho, \qquad t \ge 0$$

is a martingale. This fact, according to [33], identifies the measure uniquely and thus concludes the proof of Theorem 1. □

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