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EXPONENTIAL DECAY AND ERGODICITY OF COMPLETELY ASYMMETRIC LÉVY PROCESSES IN A FINITE INTERVAL

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Consider a completely asymmetric Lévy process which has absolutely continuous transition probabilities. We determine the exponential decay parameter ρ and the quasistationary distribution for the transition probabilities of the Lévy process killed as it exits from a finite interval, prove that the killed process is ρ-positive and specify the ρ-invariant function and measure.

1. Introduction. A completely asymmetric Lévy process is a real-valued random process with stationary and independent increments, which has either no positive or no negative jumps. Such processes have been considered frequently in applied probability, in connection with theories of dams, queues, insurance risks, continuous branching processes and so on (see [3], [4], [5], [12] and [18], and references therein). A further motivation for their study has been provided by recently by Le Gall and Le Jan [11], who discovered a remarkable link with random trees and superprocesses.

One of the most interesting aspects of the theory concerns the so-called two-sided exit problem, which consists of specifying the distribution of certain variables related to the first exit-time from a finite interval (see, in particular, [18], [8], [14] and [16]). The purpose of this paper is to investigate ergodic properties of a completely asymmetric Lévy process killed as it exits from some finite interval. Typically, provided that the one-dimensional distributions of X are absolutely continuous, we determine the (exponential) decay parameter ρ of the semigroup and the quasistationary distribution, prove that the process is ρ-positive in the classification of Tuominen and Tweedie [20] and specify the ρ-invariant function and measure. Section 6 contains the main results in that direction.

Our approach relies on special properties of fluctuation theory for completely asymmetric Lévy processes, elementary features of entire functions, Tauberian theorems and, finally, the R-theory developed by Tuominen and Tweedie [20] for a general irreducible Markov process. It should be clear that the method also applies in discrete time to study upward-skip-free (or downward-skip-free) random walks in a finite interval, now using the Perron–Frobenius theorem; we refer to [15], Chapter V, and the references therein for more on this topic, in particular, the connection with Toeplitz matrices. The method does not seem to apply to general Lévy processes, partly due to the lack of explicit formulas for the resolvent densities (to this end, compare,

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e.g., with the asymptotic results of Pruitt and Taylor [13] in the case of an asymmetric Cauchy process).

2. Preliminaries. This section introduces the notation and reviews standard material on completely asymmetric Lévy processes that can be found for instance in [3] or in [1], Chapter VII.

2.1. *Notation.* For the sake of simplicity, we will focus on a Lévy process $X = (X_t,\ t \geq 0)$ with no positive jumps (one also says that X is spectrally negative). The case when X is the negative of a subordinator is degenerate for our purpose and will be implicitly excluded in the sequel. The law of the Lévy process started at $x \in \mathbb{R}$ will be denoted by $\mathbb{P}_x$; its Laplace transform is given by

$$\mathbb{E}_0(\exp\{\lambda X_t\}) = \exp t\psi(\lambda), \qquad \lambda, t \geq 0.$$

The function ψ: $[0, \infty) \to (-\infty, \infty)$ is convex with $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$. We denote its right-inverse function by Φ: $[0, \infty) \to [0, \infty)$, so that

$$\psi(\Phi(\lambda)) = \lambda \quad \text{for all } \lambda \geq 0.$$

Let I stand for the infimum process,

$$I_t = \inf\{X_s,\ s \leq t\},$$

and let $\tau(q)$ stand for an exponential time with parameter $q > 0$ which is independent of X. It is well known that

$$X_{\tau(q)} - I_{\tau(q)} \text{ and } I_{\tau(q)} \text{ are independent,} \tag{1}$$

that $X_{\tau(q)} - I_{\tau(q)}$ has an exponential distribution,

$$\mathbb{P}_0(X_{\tau(q)} - I_{\tau(q)} \in dx) = \Phi(q)\exp\{-\Phi(q)x\}\, dx, \qquad x \geq 0, \tag{2}$$

and that the Laplace transform of $I_{\tau(q)}$ is given by

$$\mathbb{E}_0(\exp\{\lambda I_{\tau(q)}\}) = \frac{q(\Phi(q) - \lambda)}{\Phi(q)(q - \psi(\lambda))}, \qquad \lambda \geq 0. \tag{3}$$

Writing $X_{\tau(q)} = (X - I)_{\tau(q)} + I_{\tau(q)}$, one observes from (1) and (2) that $X_{\tau(q)}$ has an absolutely continuous distribution. That is to say that the resolvent kernels of the Lévy process are absolutely continuous.

2.2. *Exit from a finite interval and the scale function.* We now turn our attention to the two-sided exit problem. Fix $a > 0$ and denote the first exit time from $(0, a)$ by

$$T = \inf\{t\colon X_t \notin (0, a)\}.$$

According to a fundamental result essentially due to Takács, the probability that the process started at $x \in (0, a)$ exits from $(0, a)$ at the upper boundary point can be expressed as follows (see [1], Theorem VII.8, or [14]). There exists a unique continuous function W: $[0,\infty) \to [0,\infty)$ with Laplace transform

$$\int_0^\infty e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)}, \qquad \lambda > \Phi(0), \tag{4}$$

such that, for every $x \in (0,a)$,

$$\mathbb{P}_x(X_T = a) = W(x)/W(a). \tag{5}$$

The function W is strictly increasing, it will be referred to as the scale function. More precisely, one can refine (5) and calculate the Laplace transform of the exit time T on the event that the exit occurs at the upper boundary point:

$$\mathbb{E}_x(\exp\{-qT\}, X_T = a) = W^{(q)}(x)/W^{(q)}(a), \qquad q > 0, \tag{6}$$

where $W^{(q)}$: $[0,\infty) \to [0,\infty)$ is the continuous function with Laplace transform

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)\, dx = \frac{1}{\psi(\lambda) - q}, \qquad \lambda > \Phi(q). \tag{7}$$

Formally, (6) reduces to (5) applied to the Lévy process killed at rate q, and one can easily make this informal argument rigorous. See also Theorem 3 in [16] for an alternative approach.

The function $W^{(q)}$ is strictly increasing. The corresponding Stieltjes measure on $[0,\infty)$ is denoted by $W^{(q)}(dx)$; in particular, its mass at 0 is $W^{(q)}(0)$. Observe that, by (3) and Laplace inversion, the distribution of $-I_{\tau(q)}$ is given in terms of $W^{(q)}$ by

$$\mathbb{P}_0(-I_{\tau(q)} \in dx) = \frac{q}{\Phi(q)} W^{(q)}(dx) - qW^{(q)}(x)\, dx, \qquad x \geq 0. \tag{8}$$

The simple identity

$$\frac{1}{\psi(\lambda) - q} = \sum_{k=0}^{\infty} q^k \psi(\lambda)^{-k-1}, \qquad \lambda > \Phi(q),$$

together with (4) and (7) yields the following expression for $W^{(q)}(x)$ as a power series:

$$W^{(q)}(x) = \sum_{k=0}^{\infty} q^k W^{*k+1}(x), \tag{9}$$

where $W^{*n} = W * \cdots * W$ denotes the nth convolution power of the function W. More precisely, the fact that the scale function increases entails, by induction,

$$W^{*k+1}(x) \leq \frac{x^k W(x)^{k+1}}{k!}, \qquad x \geq 0,\ k \in \mathbb{N}, \tag{10}$$

and this justifies (9). Equation (9) also appears as (6) in [10].

3. Resolvent density. The Lévy process killed when it exits from $(0, a)$ has the strong Markov property. We denote its transition probabilities by $(P^t,\ t \geq 0)$, that is,

$$P^t(x, A) = \mathbb{P}_x(X_t \in A,\ t < T) \quad \text{for } x \in (0, a)$$

[where $A \subseteq (0, a)$ stands for a generic Borel set], and its q-resolvent kernel by

$$U^q(x, A) = \int_0^\infty P^t(x, A)e^{-qt}\, dt = \mathbb{E}_x\left(\int_0^T e^{-qt}\mathbf{1}_{\{X_t \in A\}}\, dt\right), \qquad q \geq 0.$$

Since the Lévy process has absolutely continuous resolvent kernels, it follows immediately from the Radon–Nikodym theorem and the foregoing equality that

$$U^q(x, A) = \int_A u^q(x, y)\, dy,$$

where u^q is known as the q-resolvent density for the killed process. The purpose of this section is to express these densities in terms of our data. The result is due to Suprun (see [16], Theorem 2; a less simple expression was given previously by Emery [8]).

THEOREM 1 [16]. *For every* $x, y \in (0, a)$, *set*

$$u^q(x, y) = \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - \mathbf{1}_{\{x \geq y\}}W^{(q)}(x-y).$$

Then u^q *is a version of the* q-*resolvent kernel* U^q.

We stress that Theorem 1 holds in particular for $q = 0$, where $U^0 = U$ is the potential kernel and $W^{(0)} = W$ the scale function.

Suprun's proof relies heavily on analytic arguments; we present here a probabilistic proof based on the following formula for the resolvent of X killed as it enters the nonpositive half-line, which is due again to Suprun (see [16], Theorem 1). Recall that $\tau(q)$ is an independent exponential time with parameter $q > 0$.

LEMMA 1 [16]. *We have, for every* $x, y > 0$ *and* $q > 0$,

$$q^{-1}\mathbb{P}_x\big(X_{\tau(q)} \in dy,\ I_{\tau(q)} > 0\big)/dy = \exp\{-\Phi(q)y\}W^{(q)}(x) - \mathbf{1}_{\{x \geq y\}}W^{(q)}(x-y).$$

PROOF. Using (1), (2) and (8), we obtain

$$\begin{aligned} &q^{-1}\mathbb{P}_x(X_{\tau(q)} \in dy,\ I_{\tau(q)} > 0)\\ &\quad = \left(\int_{(x-y,x]} \Phi(q)\exp\{-\Phi(q)(y+t-x)\}\left(\frac{1}{\Phi(q)}W^{(q)}(dt) - W^{(q)}(t)\, dt\right)\right) dy, \end{aligned}$$

where we agree that $W^{(q)} \equiv 0$ on $(-\infty, 0)$.

By an integration by parts,

$$\int_{x-y}^{x} \Phi(q)\exp\{-\Phi(q)(y+t-x)\}W^{(q)}(t)\,dt$$
$$= \int_{(x-y,\,x]} \exp\{-\Phi(q)(y+t-x)\}W^{(q)}(dt)$$
$$- \big[\exp\{-\Phi(q)(y+t-x)\}W^{(q)}(t)\big]_{t=x-y}^{x}.$$

The statement follows. □

PROOF OF THEOREM 1. Applying the strong Markov property at time T and the lack-of-memory of the exponential law, we get

$$\begin{aligned} qu^q(x, y)\,dy &= \mathbb{P}_x\big(X_{\tau(q)} \in dy,\ \tau(q) < T\big) \\ &= \mathbb{P}_x\big(X_{\tau(q)} \in dy,\ I_{\tau(q)} > 0\big) \\ &\quad - \mathbb{P}_x\big(X_T = a,\ T < \tau(q)\big)\mathbb{P}_a\big(X_{\tau(q)} \in dy,\ I_{\tau(q)} > 0\big) \\ &= \alpha - \beta\gamma. \end{aligned}$$

The quantities α and γ are given by Lemma 1, and β by (6). This entails the formula stated in Theorem 1 for $q > 0$. The limit case $q = 0$ follows by approximation, using (9), for example. □

Specializing Theorem 1 to the case $A = (0, a)$ yields the following formula for the Laplace transform of the first exit time T.

COROLLARY 1. *Denote the indefinite integral of* $W^{(q)}$ *by* $\overline{W}^{(q)}$; *that is,*

$$\overline{W}^{(q)}(x) = \int_0^x W^{(q)}(t)\,dt, \qquad x \ge 0.$$

For every $q > 0$, *we have*

$$\mathbb{E}_x\left(\exp\{-qT\}\right) = 1 + \overline{W}^{(q)}(x) - W^{(q)}(x)\overline{W}^{(q)}(a)/W^{(q)}(a).$$

Corollary 1 extends Theorem 1 in [2], which was proven in the stable case. We also point out that Theorem 1 enables us to determine the joint distribution of (T, X_{T-}, Δ_T), where $\Delta_T = X_T - X_{T-}$ stands for the (possible) jump at time T.

COROLLARY 2. *For every* $x, y \in (0, a)$ *and* $z \le -y$, *we have*

$$\mathbb{E}_x(\exp\{-qT\},\ X_{T-} \in dy,\ \Delta_T \in dz) = u^q(x, y)\,dy\Lambda(-y + dz),$$

where Λ *denotes the Lévy measure of* X.

PROOF. The jump process $\Delta = (\Delta_t,\ t \ge 0)$ of the Lévy process is a Poisson point process with characteristic measure Λ. The statement follows from the compensation formula for Poisson point processes. □

The quantity

$$\mathbb{E}_x(\exp\{-qT\},\ X_T = a,\ \Delta_T = 0) = \mathbb{E}_x(\exp\{-qT\},\ X_T = a)$$

is given by (6); the value of

$$\mathbb{E}_x(\exp\{-qT\},\ X_T = 0,\ \Delta_T = 0) = \mathbb{E}_x(\exp\{-qT\},\ X_T = 0)$$

then follows from Corollaries 1 and 2. This solves completely the two-sided exit problem (see [8], [14] and [17]).

4. Irreducibility. The purpose of this section is to investigate the irreducibility of the killed Lévy process. The necessary and sufficient condition stated in Proposition 1 below is intuitively obvious, although the rigorous proof is perhaps less simple than one might have expected.

It is well known that $\lim_{\lambda\to\infty} \lambda^{-2}\psi(\lambda)$ exists; we call this limit the Brownian coefficient of X. We also say that X has no jumps of absolute length less than a if the Lévy measure of X gives zero mass to $(-a, 0)$. This terminology should be clear.

PROPOSITION 1. *If the Brownian coefficient of X is zero and X has no jumps of absolute length less than a, then*

$$\mathbb{P}_x\big(X_t = y \text{ for some } t < T\big) = 0 \quad \text{for all } 0 < y < x < a.$$

Otherwise

$$\mathbb{P}_x\big(X_t = y \text{ for some } t < T\big) > 0 \quad \text{for all } x, y \in (0, a).$$

PROOF. The first assertion is obvious. If the Brownian coefficient of X is zero and X has no jumps of absolute length less than a, then X is a compound Poisson process with drift. The typical sample path of the Lévy process killed as it exits from $(0, a)$ is that of a uniform motion to the right in $(0, a)$, killed both at some deterministic rate [this corresponds to the first jump that necessarily takes X out of $(0, a)$] and when the motion reaches a. In particular, such a process started at x never visits any $y < x$.

Next, we observe that, since X has no positive jumps,

$$\text{(11)}\quad \mathbb{P}_x\big(X_t = y \text{ for some } t < T\big) = W(x)/W(y) > 0, \qquad 0 < x \le y < a$$

[by (5) and because W is strictly increasing]. Applying the strong Markov property, we see that the second assertion reduces to showing that

$$\text{(12)}\qquad \mathbb{P}_x\big(X_t < y \text{ for some } t < T\big) > 0, \qquad 0 < y < x < a.$$

Suppose first that X has a nonzero Brownian coefficient and no jumps of absolute length less than a. The process $(X_t,\ t < T)$ can then be thought of as a Brownian motion with a possible drift, killed both at some deterministic rate (which is zero iff X has no jumps at all) and as it exits from $(0, a)$. It is easy to check that (12) holds in that case.

Second, suppose that the support of the Lévy measure contains some $-\ell \in (-a, 0)$. For any arbitrarily small $\varepsilon > 0$, consider

$$\delta_\ell = \inf\{t\colon X_t - X_{t-} \in (-\ell - \varepsilon, -\ell + \varepsilon)\}$$

the instant of the first jump of absolute length between $\ell - \varepsilon$ and $\ell + \varepsilon$. It is well known that δ_ℓ has an exponential distribution with finite parameter and is independent of the process X^ℓ obtained from X by discarding the jumps of absolute length in $(\ell - \varepsilon, \ell + \varepsilon)$. Using the right-continuity of X^ℓ and the fact that $\mathbb{P}(\delta_\ell < \eta) > 0$ for every $\eta > 0$, one deduces readily that

$$\mathbb{P}_x\big(X_t < x - \ell + \varepsilon \text{ for some } t < T\big) > 0 \qquad \text{whenever } x > \ell. \tag{13}$$

Next, combining (11), the strong Markov property at the first passage time at $\ell + \varepsilon$ and (13), we see that (12) holds provided that $x < \ell$. Finally, the case when $x > \ell$ can be reduced to the preceding, applying n times the strong Markov property and (13), where $n = [x/\ell]$ is the integer part of x/ℓ. □

One says that the transition probabilities P^t are Lebesgue irreducible if, for every Borel set $A \subseteq (0, a)$ with positive Lebesgue measure, the potential $U(x, A)$ of A is positive for every $x \in (0, a)$. Recall that a simple version of the potential density $u(x, y)$ has been given in Theorem 1 in terms of the scale function.

COROLLARY 3. *Suppose that the Brownian coefficient is positive, or that X has jumps of absolute length less than a. Then $u(x, y) > 0$ for every $x, y \in (0, a)$, and as a consequence P^t is Lebesgue irreducible.*

PROOF. It follows from (5) that, for $0 < y \leq x < a$,

$$\mathbb{P}_x\big(X_t \leq y \text{ for some } t < T, X_t = a\big) \leq \mathbb{P}_x\big(X_t \leq y \text{ for some } t < T,\ X_T = a\big) = \frac{W(x)}{W(a)} - \frac{W(x-y)}{W(a-y)}$$

and we know from Proposition 1 and the Markov property that the left-hand side is positive. Then, by Theorem 1,

$$u(x, y) = \frac{W(x)W(a-y)}{W(a)} - \mathbf{1}_{\{x \geq y\}} W(x-y) > 0. \qquad \square$$

We next turn our attention to the continuity of the transition probabilities P^t in the space and time variables. In this direction, we say that absolute continuity (AC) holds if the one-dimensional distributions of the Lévy process are absolutely continuous, that is,

$$\mathbb{P}_0(X_t \in dx) \ll dx \quad \text{for every } t > 0. \tag{AC}$$

It is known that (AC) holds whenever the Brownian coefficient is positive, or when the mass of the absolutely continuous part of the Lévy measure is infinite (see [19]). It should also be clear that (AC) implies that the conditions of Corollary 3 are satisfied [otherwise X would be a compound Poisson process with drift and (AC) plainly fails in that case].

PROPOSITION 2. *Suppose that* (AC) *holds. Then the following hold*:

(i) *The mapping* $t \to P^t(x, A)$ *is continuous on* $(0, \infty)$ *for every* $x \in (0, a)$ *and Borel set* $A \subseteq (0, a)$.

(ii) *For every* $t > 0$, P^t *has the strong Feller property. That is, for every Borel bounded function* f, $P^t f$ *is a continuous function on* $(0, a)$.

Here we use the standard notation $P^t f(x) = \int_{(0,a)} f(y) P^t(x, dy)$.

The proof of Proposition 2 relies on two lemmas.

LEMMA 2. *If* (AC) *holds, then for every* $x \in (-\infty, \infty)$ *and Borel set* $A \subseteq \mathbb{R}$, *the mapping* $t \to \mathbb{P}_x(X_t \in A)$ *is continuous on* $(0, \infty)$.

PROOF. According to Hawkes ([9], Theorem 2.2), there is a version $(t, x) \to p_t(x)$ of the density of the one-dimensional distributions of X such that, for every Borel bounded function f and $x \in \mathbb{R}$,

$$\mathbb{E}_x(f(X_t)) = p_t * f(-x)$$

and $p_{t+s} = p_t * p_s$ for every $s, t > 0$. Moreover, $x \to p_t * f(-x)$ is a continuous function. Because the probability measure $p_\varepsilon(y)dy$ converges weakly to the Dirac point mass at 0 as $\varepsilon \to 0+$, we deduce that $p_{t+\varepsilon} * f = p_\varepsilon * (p_t * f)$ converges pointwise to $p_t * f$ as $\varepsilon \to 0+$. This establishes the right-continuity in the lemma.

To prove the left-continuity, we take $0 < \eta < \varepsilon < t$ and write

$$p_{t-\eta} * f = p_{\varepsilon-\eta} * (p_{t-\varepsilon} * f).$$

As $\eta \to 0+$, the probability measure $p_{\varepsilon-\eta}(y)\,dy$ converges weakly to $p_\varepsilon(y)dy$ (because the sample paths of a Lévy process are continuous at time ε, a.s.), and therefore $p_{t-\eta} * f$ converges pointwise to $p_\varepsilon * (p_{t-\varepsilon} * f) = p_t * f$. □

LEMMA 3. *Suppose that* (AC) *holds. The distribution of the exit time* T *has no atom, that is*,

$$\mathbb{P}_x(T = t) = 0 \quad \textit{for every } x \in (0, a) \textit{ and } t \geq 0.$$

PROOF. Because the sample paths of a Lévy process are continuous at each fixed time, a.s., we have

$$\mathbb{P}_x(T = t) \leq \mathbb{P}_x(X_t = 0 \text{ or } X_t = a)$$

and the right-hand-side is zero by (AC). □

We are now able to prove Proposition 2.

PROOF OF PROPOSITION 2. (i) We have, by an application of the strong Markov property at time T,

$$\mathbb{P}_x(X_t \in A) = P^t(x, A) + \int_0^t \int_{\mathbb{R}} \mathbb{P}_x(T \in ds,\ X_T \in dy)\mathbb{P}_y(X_{t-s} \in A).$$

The left-hand side is continuous in the variable $t > 0$ by Lemma 2. The same holds for the integral on the right-hand side, because the distribution of T has no atom.

(ii) A Lévy process has the Feller property. By Theorem 2.2 in [9], it also has the strong Feller property when (AC) holds. One says that it is a doubly Feller process. By a general result of Chung [7], a doubly Feller process killed upon hitting an open set remains doubly Feller. □

REMARK. The argument in the proof of Lemma 2 applies under the sole condition that, for each $t > 0$ and Borel bounded function f, the mapping

$$x \to \mathbb{E}_x(f(X_t)) \tag{14}$$

is both finely and cofinely continuous, and this condition seems weaker than (AC). However, the single points are not polar for the Lévy process, and 0 is regular for $(0, \infty)$. According to Bretagnolle [6], either X has bounded variation and then the fine and co-fine topologies are the right and left topologies, respectively, or X has unbounded variation and then the right and the left topologies are simply the usual Euclidean topology. In other words, a function that is both finely and cofinely continuous is in fact continuous in the Euclidean sense in any case. However, according to Hawkes ([9], Theorem 2.2), the mapping (14) is continuous for every $t > 0$ and Borel bounded function f if and only if (AC) holds. In conclusion, the apparently weaker condition involving fine and co-fine continuity reduces to (AC).

5. Analytic continuation. Loosely speaking, we show in this section that Theorem 1 can be extended to some negative q's. To start with, we observe that for every $x \geq 0$ the function $q \to W^{(q)}(x)$ can be extended analytically to $q \in (-\infty, \infty)$. More precisely, from (9), (10) and the continuity of the scale function, the following lemma is immediate.

LEMMA 4. (i) *The mapping* $(x, q) \to W^{(q)}(x)$ *is continuous on* $[0, \infty) \times (-\infty, \infty)$.

(ii) *For every* $x \geq 0$, $q \to W^{(q)}(x)$ *is an entire function.*

(iii) *For every* $x \geq 0$ *and* $r > 0$, *we have*

$$|W^{(q)}(y)| \leq W^{(r)}(x) \quad \text{for } |q| \leq r,\ 0 \leq y \leq x.$$

Next, we introduce the first positive root ρ of $q \to W^{(-q)}(a)$:

$$\rho = \inf\{q \geq 0\colon W^{(-q)}(a) = 0\} \tag{15}$$

with the convention that $\inf\varnothing = \infty$. We observe the following simple lower bound for ρ in terms of the indefinite integral of the scale function $\overline{W}(x) = \int_0^x W(t)\,dt$:

LEMMA 5. *For every* $x > 0$ *and* $q < 1/\overline{W}(x)$, $W^{(-q)}(x) > 0$. *As a consequence, one has* $\rho \geq 1/\overline{W}(a)$.

PROOF. The obvious inequality $W^{*k+1}(x) \leq \overline{W}(x)W^{*k}(x)$ entails that, for every $0 \leq q < 1/\overline{W}(x)$, the series $(-q)^k W^{*k+1}(x)$ is alternating and therefore

$$W^{(-q)}(x) \geq W(x) - qW^{*2}(x) \geq W(x) - q\overline{W}(x)W(x) > 0.$$

This shows our claim. □

We then arrive at the main result of this section.

PROPOSITION 3. *For every* $x \in (0, a)$, $q < \rho$ *and Borel set* $A \subseteq (0, a)$, *we have*

$$\int_0^\infty e^{qt} P^t(x, A)\,dt = \int_A \left\{ \frac{W^{(-q)}(x)W^{(-q)}(a-y)}{W^{(-q)}(a)} - \mathbf{1}_{\{x \geq y\}} W^{(-q)}(x-y) \right\} dy.$$

PROOF. For $q \leq 0$, the formula merely rephrases Theorem 1. Using Lemma 4(iii), we can extend analytically the right-hand side for $q < \rho$. For every integer n, the coefficient c_n of q^n in the corresponding expansion as a power series is given in terms of the nth (left-)derivative at 0 of the left-hand side; specifically

$$c_n = \int_0^\infty \frac{t^n}{n!} P^t(x, A)\,dt.$$

This quantity is nonnegative, and we know that the series $\sum c_n q^n$ converges for $q < \rho$. This establishes the statement. □

COROLLARY 4. *Suppose that* (AC) *holds. For every* $q < \rho$ *and* $x \in (0, a)$, *we have* $W^{(-q)}(x) > 0$.

PROOF. We know from Lemma 5 [or (9) and Lemma 4(i)] that $W^{(-q)}(x) > 0$ whenever $x > 0$ is sufficiently small. Consider $x_0 = \inf\{x \geq 0: W^{(-q)}(x) = 0\}$. If we had $x_0 < a$, then we would have $\int_0^\infty e^{qt} P^t(x_0, (0, x_0))\,dt = 0$ by Proposition 3, and this is absurd since P^t is Lebesgue irreducible. □

6. Exponential decay and ergodic properties. We now state the main result of this paper on the decay and ergodic properties of the transition probabilities P^t under condition (AC). By Corollary 1, Proposition 2 and Theorem 1 in [20], we are entitled to apply the R-theory of irreducible Markov processes developed by Tuominen and Tweedie, to the transition probabilities P^t. We refer to [20] for the terminology used in the next statement, and we recall that ρ has been defined in (15).

THEOREM 2. *Suppose that* (AC) *holds. Then the following hold*:

(i) $\rho \in (0,\infty)$ *and* ρ *is a simple root of the entire function* $q \to W^{(-q)}(a)$;
(ii) P^t *is* ρ-*recurrent and, more precisely*, ρ-*positive*;
(iii) *the function* $W^{(-\rho)}$ *is positive on* $(0,a)$ *and is* ρ-*invariant for* P^t; *that is*,

$$P^t W^{(-\rho)}(x) = e^{-\rho t} W^{(-\rho)}(x) \quad \text{for every } x \in (0,a);$$

(iv) *the measure* $\Pi(dx) = W^{(-\rho)}(a-x)\,dx$ *on* $(0,a)$ *is* ρ-*invariant for* P^t; *that is*,

$$\Pi P^t = e^{-\rho t}\Pi;$$

(v) *there is a constant* $c > 0$ *such that*, *for every* $x \in (0,a)$,

$$\lim_{t\to\infty} e^{\rho t} P^t(x,\cdot) = c W^{(-\rho)}(x)\Pi(\cdot)$$

in the sense of weak convergence.

Before proving this theorem, we first specify the quantities it involves in the stable case. Take $\alpha \in (1,2]$ and $\psi(\lambda) = \lambda^\alpha$, so that X is a standard stable process of index α. Then

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_\alpha(qx^\alpha), \qquad x \geq 0,$$

where E'_α is the derivative of the Mittag–Leffler function of parameter α,

$$E_\alpha(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1+\alpha n)}, \qquad y \in \mathbb{R}$$

(see [2] for the detailed calculation). The root that appears in (15) is thus given by $\rho = a^{-\alpha}\rho(\alpha)$, where $-\rho(\alpha)$ is the first negative root of E'_α. The mapping $\alpha \to \rho(\alpha)$ is depicted in [2]. Specifying (v) gives, in particular,

$$\mathbb{P}_x(T > t) \sim c' W^{(-\rho)}(x) e^{-\rho t} \quad \text{as } t \to \infty$$

(where $c' > 0$ is some constant), which improves Corollary 1 in [2].

In the special case $\alpha = 2$, $X/\sqrt{2}$ is a standard Brownian motion and

$$E'_2(-x) = \frac{\sin\sqrt{x}}{2\sqrt{x}}, \qquad x > 0.$$

In particular, $\rho(2) = \pi^2$, and, in the notation of Theorem 2, we have

$$\rho = a^{-2}\pi^2, \qquad W^{(-\rho)}(x) = \frac{a}{\pi}\sin\left(\frac{\pi}{a}x\right).$$

We now proceed to the proof of Theorem 2.

PROOF OF THEOREM 2. [(i) and (ii)] If ρ were infinite, then according to Proposition 3, $\int_0^\infty e^{qt} P^t(x,A)\,dt$ would be finite for every $x \in (0,a)$ and every $q > 0$; this would not agree with Theorem 2 in [20]. Thus $\rho < \infty$, and we already knew from Lemma 5 that $\rho > 0$.

We then identify ρ as the decay parameter and show that P^t is ρ-recurrent. Take any $x_\rho \in (0, a/2)$ such that $\rho\overline{W}(x_\rho) < 1$. Making use of Lemma 5, we see that $W^{(-\rho)}(x) > 0$ provided that $x \in (0, x_\rho)$. It follows from the continuity of the mapping $(x, q) \to W^{(q)}(x)$ stated in Lemma 4(i) that, for every $x < x_\rho$ and $y \in (a - x_\rho, a)$,

$$\lim_{q\to\rho-} \frac{W^{(-q)}(x)W^{(-q)}(a-y)}{W^{(-q)}(a)} = \infty. \tag{16}$$

Then take any Borel set $A \subseteq (a - x_\rho, a)$ with positive Lebesgue measure. We deduce by the Fatou lemma, Proposition 3 and monotone convergence that

$$\int_0^\infty e^{\rho t} P^t(x, A)\, dt = \infty \text{ for every } x \in (0, x_\rho).$$

Again by Theorem 2 in [20], we deduce that ρ coincides with the decay parameter and that P^t must be ρ-recurrent.

Finally, we prove that the entire function $q \to W^{(-q)}(a)$ has a simple root at ρ and that P^t is ρ-positive. Let $n > 0$ be the multiplicity of the root ρ. We know that there is a positive number c such that $W^{(-\rho+\varepsilon)}(a) \sim \varepsilon^n/c$ as $\varepsilon \to 0+$. We can then refine (16) as

$$\frac{W^{(-\rho+\varepsilon)}(x)W^{(-\rho+\varepsilon)}(a-y)}{W^{(-\rho+\varepsilon)}(a)} \sim W^{(-\rho)}(x)W^{(-\rho)}(a-y)c\varepsilon^{-n} \tag{17}$$

for every $x < x_\rho$ and $y \in (a-x_\rho, a)$. Making use of Lemma 4(iii) and dominated convergence in Proposition 3, we deduce that, for any Borel set $A \subseteq (a-x_\rho, a)$ with positive Lebesgue measure,

$$\int_0^\infty e^{-\varepsilon t} e^{\rho t} P^t(x, A)\, dt \sim c(A)\varepsilon^{-n}$$

for some $c(A) > 0$. It then follows from a Tauberian theorem that

$$\int_0^s e^{\rho t} P^t(x, A)\, dt \sim c'(A)s^n \quad \text{as } s \to \infty.$$

Comparing first with Theorem 6(ii) in [20], we see that P^t cannot be ρ-null; and then with Theorem 5(i) in [20], that we must have $n = 1$. In conclusion, ρ is a simple root and P^t is ρ-positive.

[(iii) and (iv)] Now that we know that ρ is a simple root, the same argument as above based on a Tauberian theorem shows that, for every Borel set $A \subseteq (0, a)$ and $x \in (0, a)$,

$$\int_0^s e^{\rho t} P^t(x, A)\, dt \sim \delta W^{(-\rho)}(x)\left(\int_A W^{(-\rho)}(a-y)\, dy\right)s \quad \text{as } s \to \infty, \tag{18}$$

where $\delta > 0$ is the derivative at $q = -\rho$ of the function $q \to W^{(q)}(a)$. On the other hand, we know from Corollary 4 and Lemma 4 that $W^{(-\rho)}$ is nonnegative on $(0, a)$. Applying Theorems 5 and 3 of [20], (18) thus implies that the measure $\Pi(dy) = W^{(-\rho)}(a-y)\, dy$ on $(0, a)$ is ρ-invariant for P^t, and that

$$P^t W^{(-\rho)}(x) = e^{-\rho t} W^{(-\rho)}(x) \quad \text{for almost-every } x \in (0, a).$$

According to Proposition 2(ii), the left-hand side is a continuous function in the variable x. Applying Lemma 4(i), we deduce that $W^{(-\rho)}$ is ρ-invariant for P^t.

Finally, suppose that $W^{(-\rho)}(x_0) = 0$ for some $x_0 \in (0, a)$. Then we would have

$$\lim_{q\to\rho-} W^{(-q)}(x_0)/W^{(-q)}(a) < \infty,$$

because $q \to W^{(-q)}(x_0)$ is an entire function and $q \to W^{(-q)}(a)$ has a simple root at ρ. Then (applying, e.g., Proposition 3) we would have also $\mathbb{E}_{x_0}(\exp\{\rho T\}) < \infty$, and this would not agree with the fact that P^t is ρ-recurrent. Hence $W^{(-\rho)}$ is positive on $(0, a)$.

(v) Because the ρ-invariant measure Π has a finite mass (it has a bounded density) and $P^t 1$ converges pointwise to 0 as $t \to \infty$, we deduce from Theorem 7 in [20] that, for every Borel set $A \subseteq (0, a)$ and almost every $x \in (0, a)$,

$$\lim_{t\to\infty} P^t(x, A)/P^t(x, (0, a)) = \Pi(A)/\Pi((0, a)).$$

Comparing with Theorem 5(i) in [20], this proves our claim for almost every $x \in (0, a)$. However, (AC) entails that the transition probabilities P^t are absolutely continuous (by the Radon–Nikodym theorem) and a standard argument based on the Markov property enables us to remove the "almost" in the last statement. □

REMARK. The noticeable link between the ρ-invariant function $W^{(-\rho)}$ and the density of the ρ-invariant measure Π is plain from the duality (with respect to the Lebesgue measure) that relates a Lévy process and its negative.

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