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THE RSW THEOREM FOR CONTINUUM PERCOLATION AND THE CLT FOR EUCLIDEAN MINIMAL SPANNING TREES[1]

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We prove a central limit theorem for the length of the minimal spanning tree of the set of sites of a Poisson process of intensity λ in $[0,1]^2$ as $\lambda \to \infty$. As observed previously by Ramey, the main difficulty is the dependency between the contributions to this length from different regions of $[0,1]^2$; a percolation-theoretic result on circuits surrounding a fixed site can be used to control this dependency. We prove such a result via a continuum percolation version of the Russo–Seymour–Welsh theorem for occupied crossings of a rectangle. This RSW theorem also yields a variety of results for two-dimensional fixed-radius continuum percolation already well known for lattice models, including a finite-box criterion for percolation and absence of percolation at the critical point.

1. Introduction. For a finite set $V \subset \mathbb{R}^d$, a *Euclidean minimal spanning tree* (MST) of V is a tree with site set V and minimal total length of all bonds. (Here we use percolation terminology, that is, site = vertex and bond = edge.) Let $\mathscr{L}(V)$ denote this total length; the behavior of $\mathscr{L}(V)$ for random site sets V has been an object of considerable study, particularly its limiting behavior as the cardinality $|V| \to \infty$. For $X_1, X_2, \ldots$ iid uniform in $[0,1]^d$ and $N(\lambda)$ an independent Poisson(λ) r.v., let

$$\mathscr{L}_n := \mathscr{L}(\{X_1, \ldots, X_n\}),$$

so that $\mathscr{L}_{N(\lambda)}$ is $\mathscr{L}(V)$ for V the set of sites of a Poisson process of intensity λ in $[0,1]^d$. Beardwood, Halton and Hammersley [6] proved a strong law for the "Steiner tree" analog of the functional $\mathscr{L}(V)$, in which the site set need only contain V: for some $0 < \beta_d < \infty$,

$$\lim_n \mathscr{L}_n / n^{(d-1)/d} = \beta_d \quad \text{a.s.}$$

[They actually only prove this for a different "traveling salesman" functional $\mathscr{L}(\cdot)$, but, as they point out, the proof for the Steiner tree is similar.] Steele [28] established similar results for the large class of all subadditive Euclidean functionals, and in [30] he proved the strong law for the MST. Steele [30] and Aldous and Steele [1] replaced the length of an edge with a power of that length and considered analogous results. For the MST and/or related func-

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tionals, Rhee and Talagrand [23] established exponential bounds on the tails of $\mathscr{L}_n$, and Alexander [2], Jaillet [14], Redmond and Yukich [21] and Rhee [22] considered the rate of convergence of $E\mathscr{L}_n/n^{(d-1)/d}$ to β_d.

After proving their strong law, Beardwood, Halton and Hammersley suggested that there should be a central limit theorem (CLT) for $\mathscr{L}_{N(\lambda)}$. Similar speculation appears in the paper of Avram and Bertsimas [4], which contains CLT's for some other natural functionals $\mathscr{L}(\cdot)$ of geometric probability, and in Steele [30]. Ramey [20] proved that, for $d = 2$, the CLT was true subject to the validity of a certain percolation-theoretic conjecture. Our main purpose in this paper is essentially to prove Ramey's conjecture, and thereby to prove the CLT for $\mathscr{L}_{N(\lambda)}$. Thus we restrict ourselves henceforth to $d = 2$. We do not prove the CLT for the non-Poissonized quantity $\mathscr{L}_n$, and it does not seem to be any easy consequence of the CLT for $\mathscr{L}_{N(\lambda)}$.

Since the completion of the present work, Kesten and Lee [16] have proved the CLT for both $\mathscr{L}_n$ and $\mathscr{L}_{N(\lambda)}$ in all dimensions, using techniques completely different from those employed here.

Our proof of Ramey's conjecture is based on another result of independent interest: the Russo–Seymour–Welsh (RSW) theorem for occupied crossings in a certain continuum percolation model. The original RSW theorem (see [25]–[27]) was proved for Bernoulli bond percolation on the square lattice, where each bond is independently occupied with some probability p and vacant with probability $1 - p$. This theorem relates the probability of a horizontal crossing by occupied bonds for an $L \times L$ square to that for a $(3L/2) \times L$ rectangle. Crossings are called *short-way* or *long-way* depending on the relative lengths of the sides of the rectangle. Roughly, the RSW theorem says that, for some function f: $(0,1] \to (0,1]$ with $f(p) \to 1$ as $p \to 1$, if the probability is at least p for crossing the square, then it is at least $f(p)$ for a long-way crossing of the rectangle, and the result is uniform in L. This theorem is the foundation for a wide variety of results about two-dimensional percolation, including the existence of a finite-box criterion for percolation, continuity of the percolation probability, nonpercolation at the critical point, noncoexistence of vacant and occupied infinite clusters for fixed p and uniqueness of the critical point, among others; see [7] or [11]. We will establish here some analogous consequences of the RSW theorem in the continuum case. For general background on continuum percolation, see [17].

To formulate the theorem for continuum percolation, let X be a Poisson process in $\mathbb{R}^2$ with intensity λ. For $A \subset \mathbb{R}^2$ and $r \geq 0$ we let

$$A^r := \{x \in \mathbb{R}^2 : d(x, A) \leq r\},$$

where $d(\cdot, \cdot)$ denotes Euclidean distance and $d(x, A) = \inf\{d(x, y): y \in A\}$; for a single point z we abbreviate $\{z\}^r$ as z^r. The set X^r is called the *occupied space* at level r; its complement $(X^r)^c$ is called the *vacant space* at level r. This describes the fixed-radius case of the standard *Poisson blob*, or *Boolean*, model of continuum percolation. We say there is a *horizontal occupied crossing* of a rectangle R at level r if there is a path in $X^r \cap R$ from the left side of R to the right side of R. *Occupied* or *vacant percolation* means the existence of an unbounded component in the occupied or vacant space, respectively.

Kesten [15] generalized the lattice RSW theorem in a way that allows the square to be replaced with a $cL \times L$ rectangle, with $c < 1$. This generalization means roughly that one need only establish a nonnegligible (or high) probability of some short-way crossings in order to prove a nonnegligible (or high) probability of a long-way crossing of a larger rectangle. The corresponding result for continuum models is more difficult, because there is more dependence. Roy [24] (see also [17]) proved an RSW theorem, incorporating Kesten's modifications, for vacant crossings in the bounded-radius Poisson blob model, but without the analog of the property that $f(p) \to 1$ as $p \to 1$. As a consequence, Roy obtained the equality of various critical points for the Poisson blob model. In broad outline Roy's proof of his RSW theorem is similar to Kesten's lattice proof, but adapting the lattice proof to continuum vacant crossings requires numerous new techniques. Roy's techniques for adapting the lattice proof to vacant crossings do not extend naturally to occupied crossings. Roy considered bounded random radii; we consider only the fixed-radius case.

2. The RSW theorem for occupied crossings. Let X be a Poisson process in $\mathbb{R}^2$ with intensity λ; we identify X with the corresopnding set of sites. We assume X is defined on a probability space $(\Omega, \mathscr{A}, P)$; we call each $\omega \in \Omega$ a *configuration.* For $A \subset \mathbb{R}^2$ and $r \geq 0$ we let

$$A^{<r} := \{x \in \mathbb{R}^2 \colon d(x, A) < r\}.$$

Let $B_r(x)$ denote the closed ball of radius r centered at x. We say there is a *horizontal occupied crossing* of a rectangle $R = [a, b] \times [c, d]$ at level r if there is a path in $X^r \cap R$ from $\{a\} \times [c, d]$ to $\{b\} \times [c, d]$; a *vertical occupied crossing* is defined similarly. A horizontal vacant crossing and vertical vacant crossing are define analogously. Define events

$$H_{\text{occ}, r}(R) := [\text{there exists a horizontal occupied crossing of } R \text{ at level } r],$$

$$H_{\text{vac}, r}(R) := [\text{there exists a horizontal vacant crossing of } R \text{ at level } r],$$

$$V_{\text{occ}, r}(R) := [\text{there exists a vertical occupied crossing of } R \text{ at level } r],$$

$$V_{\text{vac}, r}(R) := [\text{there exists a vertical vacant crossing of } R \text{ at level } r].$$

Then

$$H_{\text{occ}, r}(R) = V_{\text{vac}, r}(R)^c \quad \text{and} \quad V_{\text{occ}, r}(R) = H_{\text{vac}, r}(R)^c. \tag{2.1}$$

The following theorem, the main result of this section, is the analog for occupied crossings of the RSW theorem of Roy [24] for vacant crossings in continuum percolation; the techniques we use to adapt the lattice proof are quite different from those of Roy.

THEOREM 2.1. *Let X be a Poisson process in $\mathbb{R}^2$ with intensity λ. Suppose $r > 0$, $l > 0$, $l \geq b > l/2 + 2r$ and $h > 4r$. Then, for some constant $K(\lambda r^2) >$*

0, *with* $K(\cdot)$ *nondecreasing*, *and* $a := b - l/2 + r$,

$$\begin{aligned} &P(H_{\mathrm{occ},r}([0, b+a] \times [0, h-2r])) \\ (2.2) \qquad &\geq K(\lambda r^2) P(H_{\mathrm{occ},r}([0, b+r] \times [0, h-4r]))^4 \\ &\quad \times P(V_{\mathrm{occ},r}([0, l] \times [0, h+3r]))^2. \end{aligned}$$

The most important thing about (2.2) is that the factor $K(\lambda r^2)$ does not depend on the scale of the rectangles, that is, on b, h or l.

The closest analog of the original RSW theorem comparing the crossing probability for an $L \times L$ square to that of a $(3L/2) \times L$ rectangle is obtained by taking $b = L - 3r$, $l = L + 5r$ and $h = L + 2r$, yielding

$$\begin{aligned} &P(H_{\mathrm{occ},r}([0, 3L/2 - 15r/2] \times [0, L])) \\ (2.3) \qquad &\geq K(\lambda r^2) P\Big(H_{\mathrm{occ},r}\big([0, L-2r]^2\big)\Big)^4 P\Big(V_{\mathrm{occ},r}\big([0, L+5r]^2\big)\Big)^2, \end{aligned}$$

which is somewhat similar except for the added/subtracted terms $15r/2$, $2r$ and $5r$ and the factor $K(\lambda r^2)$. Of course, in the last two probabilities in (2.3), $H_{\mathrm{occ},r}$ and $V_{\mathrm{occ},r}$ are interchangeable since the rectangles are squares.

By forming equivalence classes if necessary, we may assume that the random process $X = X(\omega)$ is a one-to-one function on Ω. The ordering of the sets $X(\omega)$, $\omega \in \Omega$, by inclusion then induces an ordering of Ω; an event H is called *increasing* if $\omega \in H$, $\omega \leq \omega'$ imply $\omega' \in H$. The existence of an occupied crossing of a given rectangle is clearly an increasing event. As noted by Roy [24], the FKG inequality of Harris [12] extends straightforwardly to the continuum case, yielding the following result.

LEMMA 2.2. *Any two increasing events have nonnegative correlation.*

Though the following proof is, in principle, self-contained, familiarity with Russo's proof for the lattice case will be invaluable; see the original work ([25] and [26]) or see Lemma 9.73 of [11].

PROOF OF THEOREM 2.1.

Step 1. We begin with a description of some of the geometry associated with occupied crossings and some related definitions.

Given a rectangle $R = [a, b] \times [c, d]$, we define the following modifications:

$$\begin{aligned} \tilde{R} &:= [a-r, b+r] \times [c, d+5r], \\ R_{\mathrm{sup}} &:= (a-r, b+r) \times (c+r, d-r), \\ R_{\mathrm{aug}} &:= [a-r, b+r] \times [c-r, d+r], \\ R_{\mathrm{wide}} &:= [a-r, b+r] \times [c, d], \\ R_{\mathrm{trunc}} &:= [a, b] \times (c+r, d-r). \end{aligned}$$

Observe that any occupied crossing of a rectangle R is contained in $(X \cap R_{\mathrm{aug}})^r$. Let

$$R_{\mathrm{top}} := [a, b] \times \{d\},$$
$$R_{\mathrm{bot}} := [a, b] \times \{c\},$$
$$X_{\mathrm{sup}}(R) := X \cap R_{\mathrm{sup}}.$$

We call a horizontal occupied crossing of R at level r *supported* if it is contained in $X_{\mathrm{sup}}(R)^r$. Loosely, this means that none of the balls $B_r(x)$ necessary for the crossing intersect the bottom or the top of the rectangle. The relevant "vacant space" in a rectangle when supported crossings are considered is

$$S_{\mathrm{sup}}(R) := R \setminus X_{\mathrm{sup}}(R)^r.$$

We define the event

$H^s_{\mathrm{occ}, r}(R) :=$ [there is a supported horizontal occupied crossing of R at level r].

Clearly,

(2.4) $\quad H_{\mathrm{occ}, r}([a, b] \times [c, d]) \subset H^s_{\mathrm{occ}, r}([a, b] \times [c - 2r, d + 2r])$

and

(2.5) $\quad H^s_{\mathrm{occ}, r}(R) \subset H_{\mathrm{occ}, r}(R_{\mathrm{trunc}}).$

We say $B \to C$ *in* A if there is a path γ from some point $x \in B$ to some point $y \in C$ with $\gamma \setminus \{x\} \subset A$; for a point x, $x \to B$ means $\{x\} \to B$. Define

$$L(R) := \{x \in R: x \to R_{\mathrm{bot}} \text{ in } S_{\mathrm{sup}}(R)\};$$

see Figure 1. It is clear that

(2.6) $\quad L(R)$ is a connected set containing R_{bot}.

Now let

$$Y(R) := \{x \in X_{\mathrm{sup}}(R): \partial B_r(x) \cap \partial L(R) \neq \varnothing\}.$$

Suppose there is a supported horizontal occupied crossing of $R = [a, b] \times [c, d]$. Then the boundary of $L(R)$ consists of R_{bot}, portions of the left and right sides of R and some arcs of the circles $\partial B_r(x)$, $x \in Y(R)$. Define

$$U(R) := \{x \in R: x \to R_{\mathrm{top}} \text{ in } R \setminus Y(R)^r\},$$
$$Z(R) := \{x \in Y(R): \partial B_r(x) \cap \partial U(R) \neq \varnothing\}$$
$$= \{x \in Y(R): \partial B_r(x) \to R_{\mathrm{top}} \text{ in } R \setminus Y(R)^r\}.$$

Loosely, $U(R)$ is the region above $Y(R)^r$ in R; see Figure 1. Note that $U(R)$ and $L(R)$ are separated since $Y(R)^r$ contains an occupied crossing. The boundary of $U(R)$ consists of R_{top}, portions of the left and right sides of R and some arcs of the circles $\partial B_r(x)$, $x \in Z(R)$. Let u_{lt} and u_{rt} denote the lowest points of $U(R)$ on the left and right sides of R, respectively; then

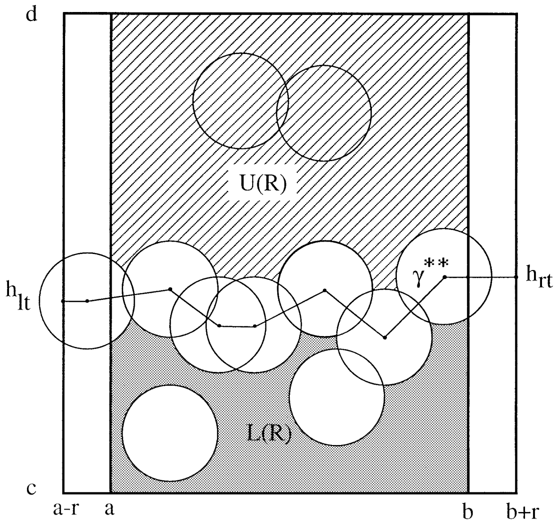

FIG. 1. *Typical configuration with a supported horizontal occupied crossing. In this example,* $z_0 = z_1$ *but* $z_{n+1} \neq z_n$.

$\partial U(R)$ includes a horizontal occupied crossing $\gamma_{U(R)}$ of R from u_{lt} to u_{rt} consisting entirely of arcs of the circles $\partial B_r(x)$, $x \in Z(R)$. To each such arc in $\gamma_{U(R)}$ there corresponds a center in $Z(R)$ of the corresponding r-ball; the sequence of arcs comprising $\gamma_{U(R)}$ thus gives rise to a sequence $z_1, \dots, z_n$ in $Z(R)$ of ball centers. From the definition of $Z(R)$, each point of $Z(R)$ appears at least once in this sequence; that is, $Z(R) = \{z_1, \dots, z_n\}$ as sets. (In fact, it is easy to show that each point appears exactly once, but we will not make use of this fact.) Let h_{lt} and h_{rt} denote the vertical coordinates of z_1 and z_n, respectively, and let

$$z_0 := \begin{cases} z_1, & \text{if } z_1 \notin R, \\ (a, h_{\mathrm{lt}}), & \text{if } z_1 \in R, \end{cases}$$

$$z_{-1} := (a - r, h_{\mathrm{lt}}),$$

and similarly

$$z_{n+1} := \begin{cases} z_n, & \text{if } z_n \notin R, \\ (b, h_{\mathrm{rt}}), & \text{if } z_n \in R, \end{cases}$$

$$z_{n+2} := (b + r, h_{\mathrm{rt}}).$$

We can then define the *canonical low occupied crossing*, denoted γ^*, to be the piecewise linear path $z_0 \to z_1 \to \cdots \to z_{n+1}$. It is clear that γ^* contains a horizontal crossing of R; let us verify that this crossing is indeed occupied. For $1 \le i < n$, it follows from the definition of the sequence $z_1, \dots, z_n$ that

$$|z_{i+1} - z_i| \le 2r; \tag{2.7}$$

therefore, the line segment from z_i to z_{i+1} is contained in the occupied space. The same definition also yields that $B_r(z_1)$ meets the left side of R, so the line segment from z_0 to z_1 is also contained in the occupied space, and similarly for the segment from z_n to z_{n+1}. Define the strips

$$S_{\mathrm{lt}}([a,b] \times [c,d]) := [a-r,a] \times [c,d],$$
$$S_{\mathrm{rt}}([a,b] \times [c,d]) := [b,b+r] \times [c,d].$$

The *extended canonical low crossing*, denoted γ^{**}, is the piecewise linear path $z_{-1} \to z_0 \to \cdots \to z_{n+2}$, which is a crossing of $R_{\mathrm{wide}} = S_{\mathrm{lt}}(R) \cup R \cup S_{\mathrm{rt}}(R)$, not necessarily entirely occupied.

We have seen that if there is a supported horizontal occupied crossing of R, then there is a canonical one γ^* contained in $Z(R)^r$. Here γ^* will play roughly the role for us that the "lowest occupied crossing" plays in the proof of the lattice RSW theorem.

We write $L(R,\omega)$ for $L(R)$ when we wish to designate the underlying configuration ω giving rise to the set $L(R)$, and similarly for other random sets. Let Ω_0 be the event that there do not exist two sites $x, y \in X$ with $|x-y| = 2r$, nor three sites $x, y, z \in X$ with $\partial B_r(x) \cap \partial B_r(y) \cap \partial B_r(z) \neq \emptyset$. Then $P(\Omega_0) = 1$; we tacitly henceforth assume that all configurations come from Ω_0.

Step 2. We give a description of the conditioning that occurs when some of the sets L, Y, U and Z are specified.

Let $\mathscr{X}_{\mathrm{sup}}$, $\mathscr{Y}$ and $\mathscr{Z}$ denote the ranges of the functions $X_{\mathrm{sup}}(R,\cdot)$, $Y(R,\cdot)$ and $Z(R,\cdot)$ on Ω_0, respectively. We obtain topologies on $\mathscr{X}_{\mathrm{sup}}$, $\mathscr{Y}$ and $\mathscr{Z}$ by viewing $X_{\mathrm{sup}}(R)$, $Y(R)$ and $Z(R)$ as elements of the disjoint union $\bigcup_{k \geq 0} (R_{\mathrm{sup}})^k$; this makes $Y(R)$ and $Z(R)$ continuous functions of $X_{\mathrm{sup}}(R)$ at every $\omega \in \Omega_0$. For configurations in Ω_0 one can determine $L(R)^{<r}$ from $Y(R)$; more precisely, there exists a map $\Lambda\colon \mathscr{Y} \to \mathscr{L}$ such that $\Lambda(Y(R,\omega)) = L(R,\omega)^{<r}$ for all $\omega \in \Omega_0$.

Suppose that, for some set D and configurations ω, ω', we have $L(R,\omega)^{<r} = D$ and $X_{\mathrm{sup}}(R,\omega) \cap \overline{D} = X_{\mathrm{sup}}(R,\omega') \cap \overline{D}$. Here $\overline{D}$ denotes the closure of D. It is easy to see that then $D = L(R,\omega)^{<r} = L(R,\omega')^{<r}$ as well, with also the equality for Y, U and Z in place of $L(\cdot)^{<r}$. We therefore call $L(R)^r \cap R_{\mathrm{sup}}$ the *conditioned region*, because changes to X outside $L(R)^r \cap R_{\mathrm{sup}}$ do not affect the values of L, Y, U and Z. It follows that

(2.8) given $L(R)^{<r} = D$, or given $\Lambda(Y(R)) = D$, $X \cap (\overline{D} \cap R_{\mathrm{sup}})^c$ remains a Poisson process in $(\overline{D} \cap R_{\mathrm{sup}})^c$ with intensity λ.

Observe also that the only sites of $X_{\mathrm{sup}}(R)$ inside $L(R)^r$ are the sites in $Y(R)$, that is,

$$X_{\mathrm{sup}}(R) \cap L(R)^r = Y(R).$$

Step 3. Note that the conditioned region may intersect $U(R)$ or $S_{\mathrm{lt}}(R)$ or $S_{\mathrm{rt}}(R)$, in contrast to the lattice analog; see Figures 1 and 2. We need to show that this intersection is not too big. More precisely, suppose, for the remain-

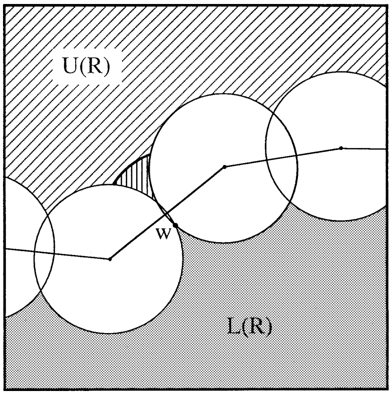

FIG. 2. *Detail of an example in which the conditioned region intersects $U(R)$; the intersection is vertically striped and is part of an r-ball centered at w.*

der of Step 3, that there is a horizontal supported occupied crossing of R, and define

$$\tilde{U}(R) := \left\{x \in \tilde{R}: x \to \tilde{R}_{\text{top}} \text{ in } \tilde{R} \setminus Z(R)^{2\sqrt{2}r}\right\},$$
$$\tilde{L}(R) := \left\{x \in \tilde{R}: x \nrightarrow \tilde{R}_{\text{top}} \text{ in } \tilde{R} \setminus \gamma^{**}\right\}.$$

Loosely, $\tilde{U}(R)$ is the region above $Z(R)^{2\sqrt{2}r}$ in a slight enlargement $\tilde{R}$ of R, and $\tilde{L}(R)$ is the region below γ^{**} in $\tilde{R}$. Since γ^* is occupied, from the definition of $L(R)$ we have $L(R) \subset \tilde{L}(R)$. Let $[x, y]$ denote the (closed) line segment with endpoints x and y, and analogously for $(x, y]$ and so on. Now suppose the following:

(2.9) $x = (x^1, x^2) \in \tilde{U}(R)$, $z = (z^1, z^2)$ is the closest point to x in $Z(R)$ (with ties broken arbitrarily), $y = (y^1, y^2) \in [x, z]$ with $|y - z| \geq 31r/16$

(see Figure 3). We claim that

(2.10) $d(y, L(R)) > 17r/16$ and therefore $B_{r/16}(y) \cap L(R)^r = \varnothing$.

For fixed x, z and a finite number of sites y as in (2.9), sites in the balls $B_{r/16}(y)$ can be used to create an occupied path from x to z; see Figure 4. The latter part of (2.10) ensures that the probability of the existence of such sites does not depend on $Y(R)$. To connect x to z, at least one such site must be in $B_{2r}(z)$, but, under (2.9) and (2.10), that is not a problem since $31r/16 + r/16 = 2r$. The proof of (2.10) is fairly straightforward when x and y are in R, but since x and/or y can be in $S_{\text{lt}}(R)$ or $S_{\text{rt}}(R)$, a number of cases and subcases must be covered separately. We will make use of the following preliminary results.

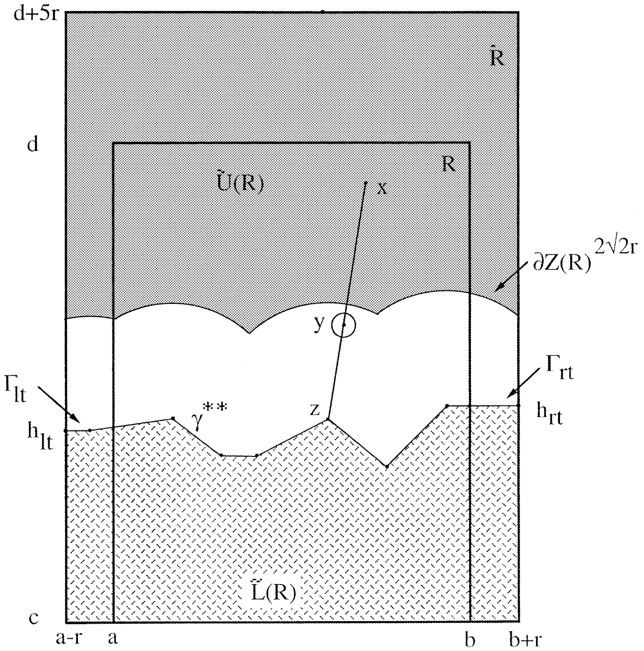

FIG. 3. *Illustration for* (2.9) *and* (2.10). *The figure is not drawn to scale.*

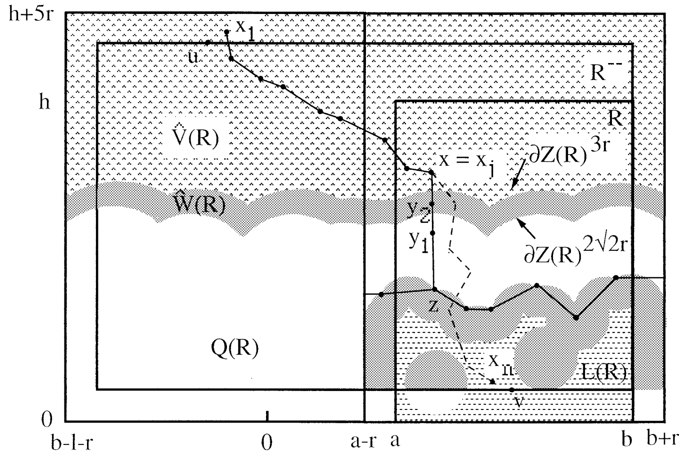

FIG. 4. *Illustration for Steps* 4 *and* 5. *The entire rectangle is* R_{aug}^{--}. *The point* x *is a close final visible site. The sites in the dashed path from* x_j *to* x_n *are hidden sites in* $X(\omega_2)$ *which do not appear in* $X(\omega)$. *The shaded region at the lower right, partly covered by* $L(R)$, *is* $L(R)^r \cap R_{\mathrm{sup}}$. *The figure is not drawn to scale.*

Claim 1. γ^{**} separates $\tilde{U}(R)$ from $\tilde{L}(R)$, and hence also from $L(R)$.

This is an easy consequence of the fact that $\gamma^{**} \subset Z(R)^{2r}$.

Claim 2. If $p, q, s, t \in \mathbb{R}^2$, $|p - q| \le 17r/16$, $|s - t| \le 2r$ and $[p, q] \cap [s, t] \ne \varnothing$, then $\min(|p - s|, |p - t|) \le 3r/2$.

This is proved by noting that if $s, t \notin B_{3r/2}(p)$ and $[p, q] \cap [s, t] \ne \varnothing$, then $[s, t]$ includes a chord of the circle $\partial B_{3r/2}(p)$ of length less than $2r$, for which the distance from the chord to the circle center is less than $17r/16$. This is impossible since $\sqrt{(17r/16)^2 + r^2} < 3r/2$.

For $q = (q^1, q^2)$ with $b \le q^1 \le b + r$, let $q_{\mathrm{rt}} := (b + r, q_2)$.

Claim 3. If $y^1 \ge b$, then $[y, y_{\mathrm{rt}}] \cap \gamma^* = \varnothing$.

By Claim 2, if $y^1 \ge b$ and $[y, y_{\mathrm{rt}}] \cap \gamma^* \ne \varnothing$, then $d(y, Z(R)) \le 3r/2$, contradicting (2.9).

Let w be the closest point to y in $L(R)$ (breaking ties arbitrarily.)

Claim 4. If $|y - w| \le 17r/16$, then $[y, w] \cap \gamma^* = \varnothing$. The proof is the same as that of Claim 3.

Let u denote the closest point to y in γ^{**}, breaking ties arbitrarily. Let Γ_{lt} and Γ_{rt} denote the line segments $[a - r, a] \times \{h_{\mathrm{lt}}\}$ and $[b, b + r] \times \{h_{\mathrm{rt}}\}$, respectively. Observe that

$$d([x, y], Z(R)) = d(y, Z(R)) \ge 31r/16, \tag{2.11}$$

so, since $|z_n - z_{n+2}| \le 2r$ and $|z_1 - z_{-1}| \le 2r$,

(2.12) $[x, y] \cap \gamma^{**}$, if nonempty, is a single point in the outer 1/16 part of Γ_{lt} or Γ_{rt}. If this point is in Γ_{rt}, then $z_n \in R$.

Since $w \in L(R)$, there is a path σ from w to R_{bot} with $\sigma \setminus \{w\} \subset S_{\mathrm{sup}}(R)$. From Claims 3 and 4,

$$\text{if } y^1 \ge b \text{ and } |y - w| \le 17r/16, \quad \text{then } (\sigma \cup [w, y] \cup [y, y_{\mathrm{rt}}]) \cap \gamma^* = \varnothing. \tag{2.13}$$

Since $\sigma \cup [w, y] \cup [y, y_{\mathrm{rt}}]$ separates $[y_1, b + r] \times [c, y_2]$ from z_0 when $y^1 \ge b$, it follows from (2.13) that

$$\text{if } y^1 \ge b \text{ and } |y - w| \le 17r/16, \quad \text{then } [y^1, b + r] \times [c, y^2] \cap Z(R) = \varnothing. \tag{2.14}$$

We now proceed with the various cases and subcases needed to prove (2.10).

Case 1. $y \in \tilde{L}(R) \cup \gamma^{**}$. Then since, by Claim 1, $x \notin \tilde{L}(R)$, by (2.12), $[x, y] \cap \gamma^{**}$ must be a single point q in Γ_{lt} or Γ_{rt} and $[x, q) \cap \tilde{L}(R) = \varnothing$ while $(q, y] \subset \tilde{L}(R)$, so y must be below x, that is, $y^2 < x^2$. Suppose $|y - w|$

$\leq 17r/16$ and $q \in \Gamma_{\mathrm{rt}}$; then $x^2 > h_{\mathrm{rt}} \geq y^2$ and, by (2.12) and (2.14), we must have $x^1 > q^1 \geq y^1 > z^1$ (i.e., x above and right of y and z) and $z_n \in R$. Therefore, since $|x - z_n| \geq 2\sqrt{2}\,r$ we have $x^2 - h_{\mathrm{rt}} \geq 2r$, so, using (2.11),

$$\begin{aligned} |x - z|^2 &\geq (|x - q| + |y - z|)^2 \\ &\geq \left(x^2 - h_{\mathrm{rt}}\right)^2 + 2|y - z|\left(x^2 - h_{\mathrm{rt}}\right) \\ &> \left(x^2 - h_{\mathrm{rt}}\right)^2 + (2r)^2 \\ &\geq \left(x^2 - h_{\mathrm{rt}}\right)^2 + \left(x^1 - z_n^1\right)^2 \\ &= |x - z_n|^2, \end{aligned}$$

which contradicts the definition of z. Similarly, we cannot have $|y - w| \leq 17r/16$ and $q \in \Gamma_{\mathrm{lt}}$. Thus $|y - w| > 17r/16$, proving (2.10).

Case 2. $u \in Z(R)$ and $y \notin \tilde{L}(R) \cup \gamma^{**}$. Then $[y, w] \cap \gamma^{**} \neq \varnothing$ since $L(R) \subset \tilde{L}(R)$, so, by (2.9), $|y - w| \geq |y - u| \geq 31r/16$, proving (2.10).

Case 3. $u \notin Z(R)$ and $y \notin \tilde{L}(R) \cup \gamma^{**}$. There then exists i, $-1 \leq i \leq n + 1$, for which $u \in [z_i, z_{i+1}]$, and $y - u$ is perpendicular to $z_{i+1} - z_i$. Further, we have $|y - w| \geq |y - u|$, so we may assume

$$|y - u| \leq 17r/16. \tag{2.15}$$

Case 3a. $0 \leq i \leq n$. Let v denote the closest point to u in $\{z_i, z_{i+1}\} \cap Z(R)$, breaking ties arbitrarily. Then $|u - v| \leq r$, so $|y - w|^2 \geq |y - u|^2 = |y - v|^2 - |u - v|^2 \geq (31r/16)^2 - r^2 > (17r/16)^2$ and (2.10) follows.

Case 3b. $i = n + 1$. Then $y^1 = u^1 \geq b$; since $y \notin \tilde{L}(R) \cup \gamma^{**}$, by Claim 3 and (2.15), it follows that

$$h_{\mathrm{rt}} + 17r/16 > y_2 > h_{\mathrm{rt}}. \tag{2.16}$$

By Claim 1 and the fact that $\gamma^{**} \subset Z(R)^{2r}$, we have $x \notin \tilde{L}(R) \cup \gamma^{**}$. Hence, by (2.12),

$$[x, y] \cap \gamma^{**} = \varnothing. \tag{2.17}$$

Case 3b(i). $z^1 \geq y^1$ and $z^2 \leq y^2$. Then, by (2.14), we have $|y - w| > 17r/16$ and (2.10) follows.

Case 3b(ii). $z^1 < y^1$ and $z^2 > y^2$. Then, from (2.17) and (2.15), we have

$$x \in \left[y^1, b + r\right] \times \left[h_{\mathrm{rt}}, y^2\right] \subset [b, b + r] \times [h_{\mathrm{rt}}, h_{\mathrm{rt}} + 17r/16] \subset B_{5r/2}(z_n),$$

contradicting (2.9).

Case 3b(iii). $z^1 \geq y^1$ and $z^2 > y^2$. Since $x \in \tilde{U}(R)$ there is a path τ from x to $\tilde{R}_{\mathrm{top}}$ with $\tau \setminus \{x\} \subset \tilde{R} \setminus Z(R)^{2\sqrt{2}r}$. Then $\tau \cap [z, z_{\mathrm{rt}}] = \varnothing$ so $\tau \cup [x, y] \cup [y, y_{\mathrm{rt}}]$ separates z from z_0 in $\tilde{R}$, so γ^* must cross $\tau \cup [x, y] \cup [y, y_{\mathrm{rt}}]$. But, by Claim 3 and (2.17), $(\tau \cup [x, y] \cup [y, y_{\mathrm{rt}}]) \cap \gamma^{**} = \varnothing$, so we have a contradiction.

Case 3b(iv). $z^1 < y^1$ and $z^2 \leq y^2$. Then

$$x^1 \geq b,\ x^2 \geq y^2 > h_{\mathrm{rt}} \text{ and } |x - z_n| \geq 2\sqrt{2}\,r, \quad \text{so } x^2 \geq h_{\mathrm{rt}} + 2r. \tag{2.18}$$

[This last conclusion is the reason for choosing the constant $2\sqrt{2}$ in the definition of $\tilde{U}(R)$.]

Since $z \in B_{|x-z_n|}(x)$, it is easy to see that the minimum possible value for y^2 under (2.18) is $h_{\rm rt} + 2r - 2\sqrt{2}\,r + 31r/16$, achieved when $z_n = (b - r, h_{\rm rt})$, $x = (b + r, h_{\rm rt} + 2r)$, $z = (b + r, h_{\rm rt} + 2r - 2\sqrt{2}\,r)$ and $y = (b + r, h_{\rm rt} + 2r - 2\sqrt{2}\,r + 31r/16)$. Since $h_{\rm rt} + 2r - 2\sqrt{2}\,r + 31r/16 > h_{\rm rt} + 17r/16$, this contradicts (2.16).

Case 3c. $i = -1$. This is equivalent to Case 3b by reflection.

This completes the proof of (2.10) in all cases. Let us next show that

$$d\big(\tilde{U}(R), \gamma^{**}\big) > r; \tag{2.19}$$

by Claim 1 a consequence of (2.19) is that

$$\tilde{U}(R) \cap L(R)^r = \varnothing. \tag{2.20}$$

To prove (2.19), suppose $x \in \tilde{R}$ and $d(x, \gamma^{**}) \le r$. Let u be the closest point to x inγ^{**}, so $|x - u| \le r$, and fix $-1 \le i \le n + 1$ such that $u \in [z_i, z_{i+1}]$. If $u \in Z(R)$, then $d(x, Z(R)) \le r$ so $x \notin \tilde{U}(R)$. If $u \notin Z(R)$, then $x - u$ is perpendicular to $z_i - z_{i+1}$. Defining z to be z_i if $1 \le i \le n$, z_n if $i = n + 1$ and z_1 if $i = -1$ or 0, we have $z \in Z(R)$ and $|z - u| \le 2r$, so $|x - z|^2 = |x - u|^2 + |u - z|^2 \le r^2 + (2r)^2 < (2\sqrt{2}\,r)^2$, so again $x \notin \tilde{U}(R)$. Thus (2.19) is proved.

Step 4. Given a region $A \subset \tilde{R}$, let $A_{\rm ref}$ denote the reflection of A across the line $\{a - r\} \times \mathbb{R}$ and let $\hat{A} := A \cup A_{\rm ref}$.

It follows from (2.8) and (2.20) that we may use the following special construction of the process X. Let $V(R) := \tilde{U}(R) \setminus Z(R)^{3r}$ and let $W(R) := \tilde{U}(R) \cap Z(R)^{3r}$. Then let

$$Q(R) := \Big[\big(L(R)^r \cap R_{\rm sup}\big) \cup \hat{W}(R) \cup \hat{V}(R)\Big]^c;$$

see Figure 4. It follows easily from (2.20) that

$$\hat{V}(R), L(R)^r \cap R_{\rm sup}, \hat{W}(R) \text{ and } Q(R) \text{ form a partition of } \mathbb{R}^2.$$

Let ω_1, ω_2 and ω_3 be independent configurations and let $\omega = (\omega_1, \omega_2, \omega_3)$. Let C be an event, to be specified later, in the σ-algebra generated by $Y(R, \omega_1)$ and $X(\omega_2) \cap \hat{V}(R, \omega_1)$, with $C \subset [\omega_1 \in H^s_{{\rm occ}, r}(R)]$. Then define

$$X(\omega) := \begin{cases} \big(X(\omega_1) \cap L(R, \omega_1)^r \cap R_{\rm sup}\big) \cup \big(X(\omega_2) \cap \hat{V}(R, \omega_1)^r\big) \\ \qquad \cup \big(X(\omega_3) \cap \hat{W}(R, \omega_1)\big) \cup \big(X(\omega_3) \cap Q(R, \omega_1)\big), \quad \text{if } \omega \in C, \\ \big(X(\omega_1) \cap L(R, \omega_1)^r \cap R_{\rm sup}\big) \cup \big(X(\omega_2) \cap \hat{V}(R, \omega_1)^r\big) \\ \qquad \cup \big(X(\omega_2) \cap \hat{W}(R, \omega_1)\big) \cup \big(X(\omega_3) \cap Q(R, \omega_1)\big), \\ \qquad\qquad \text{if } \omega \in \big[\omega_1 \in H^s_{{\rm occ}, r}(R)\big] \setminus C, \\ X(\omega_1), \quad \text{if } \omega_1 \notin H^s_{{\rm occ}, r}(R). \end{cases}$$

Roughly, this means that, to form $X(\omega)$, we first look at $X(\omega_1)$ to determine the region $L(R, \omega_1)$. Provided there is a supported horizontal occupied crossing of R, this immediately determines the four disjoint regions $\hat{V}(R, \omega_1)$, $L(R, \omega_1)^r \cap R_{\text{sup}}$, $\hat{W}(R, \omega_1)$ and $Q(R, \omega_1)$ into which $\mathbb{R}^2$ is divided, but it does not in any way condition sites outside $L(R, \omega_1)^r \cap R_{\text{sup}}$. We can therefore use $X(\omega_2)$ in the region $\hat{V}(R, \omega_1)$ and $X(\omega_3)$ in the region $Q(R, \omega_1)$. The sites of $X(\omega_2)$ in the region $\hat{V}(R, \omega_1)$ then determine whether $X(\omega_2)$ or $X(\omega_3)$ is used in the region $\hat{W}(R, \omega_1)$.

We call the sites of $X(\omega_2)$ in $\hat{V}(R, \omega_1)$ *visible*, and call other sites of $X(\omega_2)$ *hidden*; thus visible sites of $X(\omega_2)$ always appear in $X(\omega)$. Hidden sites of $X(\omega_2)$ in $\hat{W}(R, \omega_1)$ are *uncovered* [i.e., are sites of $X(\omega)$] when the event $[\omega_1 \in H^s_{\text{occ}, r}(R)] \setminus C$ occurs.

Step 5. Let $a := b - l/2 + r$. We now consider $R = [a, b] \times [0, h]$, $R^- := [0, b + r] \times [r, h - r]$, $R^{--} := [b - l, b] \times [r, h + 4r]$, $R^+ := [a - r, a + b] \times [r, h - r]$, $R^{++} := [a, a + l] \times [r, h + 4r]$ and $R_0 := [b - l, a + l] \times [0, h + 4r]$. If $r = 0$, one can visualize these rectangles as follows: dividing the rectangle R_0 into five blocks using vertical lines at 0, a, b, and $b + a$, R is the center block, R^- consists of the center and left center blocks, R^{--} consists of the three leftmost blocks, R^+ consists of the center and right center blocks and R^{++} consists of the three rightmost blocks; see Figure 5. The picture for $r > 0$ is a slight modification of this. Roughly, we wish to build a horizontal crossing of the three central blocks out of the canonical low crossing of R, horizontal crossings of R^- and R^+ and vertical crossings of R^{--} and R^{++}. Note that $\hat{V}(R) \subset R^{--}_{\text{aug}}$; in fact (see Figure 4), the left edge of $\hat{V}(R)$ is part of the left edge of R^{--}_{aug}. Also, $\tilde{R}$ is the right half of R^{--}_{aug}.

Let E denote the event that there exist (i) a horizontal occupied crossing γ of R_{trunc} with initial point in $\{a\} \times [0, h/2]$ and final point in $\{b\} \times [0, h/2]$, (ii) an occupied path β^{--} from γ to R^{--}_{top} in R^{--}_{wide} and (iii) an occupied path

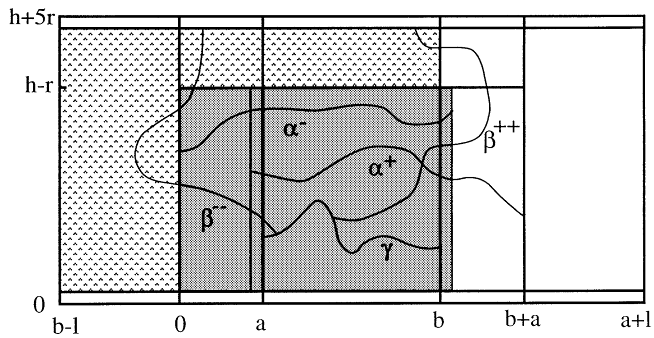

FIG. 5. *Illustration of occupied paths in the event $E \cap G^- \cap G^+$. The shaded rectangle is R^-; the dotted rectangle which it partly covers is R^{--}. Note that α^- does not meet β^{--}, but this forces β^{--} to cross the vertical line at* 0. *In contrast, β^{++} does not cross the vertical line at $b + a$, but does meet α^+. The five paths together include a horizontal occupied crossing of* $[0, b + a] \times [r, h - r]$.

β^{++} from γ to R^{++}_{top} in R^{++}_{wide}; see Figure 5. Suppose α^- is a horizontal occupied crossing of R^-. Following α^- from the left to right side of R^-, let $(a, y^-(\alpha^-))$ be the last point of α^- in the vertical line $\{a\} \times [r, h - r]$. Let G^- denote the event that there exists such crossing α^- with $y^-(\alpha^-) \geq h/2$. Similarly, for α^+ a right-to-left crossing of R^+, we let $(b, y^+(\alpha^+))$ be the last point of α^+ in the vertical line $\{b\} \times [r, h - r]$, and let G^+ denote the event that there exists such a crossing with $y^+(\alpha^+) \geq h/2$. When $E \cap G^- \cap G^+$ occurs, the path α^- may or may not meet $\gamma \cup \beta^{--}$, but if it does not, then β^{--} must cross the line $\{0\} \times [r, h - r]$; either way, γ is connected by an occupied path to the left side $\{0\} \times [r, h - r]$ of R^-. Similarly, γ is connected to the right side $\{b + a\} \times [r, h - r]$ of R^+; see Figure 5. Thus, using Lemma 2.2, we have

$$P(H_{\text{occ}, r}(R^- \cup R^+)) \geq P(E)P(G^-)P(G^+).$$

From symmetry we have $P(G^-) \geq P(H_{\text{occ}, r}(R^-))/2$, and similarly for G^+, so it follows that

$$P(H_{\text{occ}, r}(R^- \cup R^+)) \geq P(E)P(H_{\text{occ},}(R^-))^2/4. \tag{2.21}$$

We thus need to obtain a lower bound for $P(E)$. Using the methods of the lattice case, it is not difficult to obtain paths "almost like" β^{--} and β^{++}, with sufficient probability, which come close to the path γ; the problem comes in actually connecting the three paths together without difficulties related to conditioning.

If $x_1, \ldots, x_j$ are sites in $X(\omega_2)$ with $|x_{i+1} - x_i| < 2r$ for all $i < j$, we call the corresponding piecewise linear path $x_1 \to \cdots \to x_j$ an *occupied PL-path*. If there is an occupied PL-path $x_1 \to \cdots \to x_j$ in $\hat{V}(R, \omega_1)$ such that $B_r(x_1)$ intersects R^{--}_{top} and $x_j \in \hat{Z}(R, \omega_1)^{(2\sqrt{2}+2)r}$, we call x_j a *frontier site*; see Figure 4.

Suppose $\omega_1 \in H^s_{\text{occ}, r}(R)$ and there is an occupied path in $X(\omega_2)^r \cap R^{--}$ from a point u of R^{--}_{top} to a point v of $R^{--} \setminus \hat{\hat{U}}(R, \omega_1)$. [Note that $\hat{\hat{U}}(R, \omega_1) = \hat{W}(R, \omega_1) \cup \hat{V}(R, \omega_1)$; see Figure 4.] Corresponding to such a path, there is an occupied PL-path $x_1 \to \cdots \to x_n$, with all sites in $X(\omega_2) \cap R^{--}_{\text{aug}}$, such that $u \in B_r(x_1)$ and $v \in B_r(x_n)$. Since $v \notin \hat{\hat{U}}(R, \omega_1)$ and since [using $Z(R, \omega_1) \subset R_{\text{sup}}$] we have $d(R^{--}_{\text{aug}} \setminus \hat{V}(R, \omega_1), R^{--}_{\text{top}}) > r$, we conclude that $x_1 \in \hat{V}(R, \omega_1)$. Therefore, there exists a largest index j such that $\{x_1, \ldots, x_j\} \subset \hat{V}(R, \omega_1)$; we call x_j a *final visible site* of $X(\omega_2)$. A final visible site is necessarily in $\hat{Z}(R, \omega_1)^{5r}$. If $j < n$ we call x_{j+1} a *first hidden site* of $X(\omega_2)$; necessarily $x_{j+1} \in \hat{Z}(R, \omega_1)^{3r}$, and, for such j,

$$x_j \notin \hat{Z}(R, \omega_1)^{(2\sqrt{2}+2)r} \quad \text{implies} \quad x_{j+1} \in \hat{W}(R, \omega_1). \tag{2.22}$$

We call a final visible site *close* if it lies in $\hat{Z}(R, \omega_1)^{(2\sqrt{2}+2)r}$, and *distant* otherwise. A close final visible site is thus a special case of a frontier site.

We can now specify C of Step 3 to be the event that $\omega_1 \in H^s_{\text{occ}, r}(R)$ and there exists a frontier site of $X(\omega_2)$ in $\hat{V}(R, \omega_1)$. Let D be the event that $\omega_1 \in H^s_{\text{occ}, r}(R)$ and C does not occur, but there exists a distant final visible site of $X(\omega_2)$ in $\hat{V}(R, \omega_1)$ and a first hidden site of $X(\omega_2)$, necessarily in $\hat{W}(R, \omega_1)$ by (2.22). Thus, when D occurs, the first hidden site is uncovered. If there is a vertical occupied crossing of R^{--} in $X(\omega_2)^r$, then there is an occupied PL-path with $d(x_n, R^{--}_{\text{bot}}) \le r$. Since, by (2.19), $d(\hat{\tilde{U}}(R, \omega_1), R^{--}_{\text{bot}}) \ge d(\gamma^{**}, R^{--}_{\text{bot}}) + r > r$, we then have $x_n \notin \hat{\tilde{U}}(R, \omega_1)$, so there must then be a final visible site with index $j < n$; see Figure 4. Thus we have

$$(2.23) \qquad [\omega_1 \in H^s_{\text{occ},r}(R)] \cap [\omega_2 \in V_{\text{occ},r}(R^{--})] \subset C \cup D.$$

If C occurs and x is a frontier site of $X(\omega_2)$, or if D occurs and x is a (necessarily uncovered) first hidden site of $X(\omega_2)$, then we call x a *terminal site* of $X(\omega_2)$. Let C_{rt} (resp. D_{rt}) denote the event that C (resp. D) occurs with a terminal site of $X(\omega_2)$ in $\tilde{R}$, the right half of R^{--}_{aug}. From symmetry, on the event $[\omega_1 \in H^s_{\text{occ},r}(R)]$,

$$(2.24) \qquad P(C_{\text{rt}} \cup D_{\text{rt}} \mid X(\omega_1)) \ge P(C \cup D \mid X(\omega_1))/2 \quad \text{a.s.}$$

If x is a terminal site of $X(\omega_2)$ in $\tilde{R}$, then x is a site of $X(\omega)$, $x \in \tilde{U}(R, \omega_1)$ and $2\sqrt{2}\,r < d(x, \hat{Z}(R)) \le (2\sqrt{2} + 2)r$.

Suppose $\omega \in C_{\text{rt}} \cup D_{\text{rt}}$, x is a terminal site of $X(\omega_2)$ in $\tilde{R}$ and z is a closest site to x in $Z(R)$; see Figure 4. (If there is more than one such x, we take the first one in lexicographic order, to be concrete.) Suppose first that $\omega \in C_{\text{rt}}$ and this x is a frontier site. Let y_1 and y_2 be points on the line segment from x to z with $|y_2 - z| = 47r/16$ and $|y_1 - z| = 31r/16$. We have $\omega \in C$ and, by (2.10), $B_{r/16}(y_i) \subset Q(R, \omega_1) \cup \hat{W}(R, \omega_1)$, so

$$(2.25) \qquad X(\omega) \cap B_{r/16}(y_i) = X(\omega_3) \cap B_{r/16}(y_i), \qquad i = 1, 2;$$

that is, the sites in $B_{r/16}(y_i)$ come from $X(\omega_3)$. [Note that $B_{r/16}(y_2) \subset \hat{W}(R, \omega_1)$, so (2.25) would be false if we uncovered the sites of $X(\omega_2)$ in $\hat{W}(R, \omega_1)$ when $\omega \in C$; this is the reason for making the distinction between $\omega \in C$ and $\omega \notin C$ in Step 4.] Further, since $x, z \in \tilde{R}$ we have $y_i \in \tilde{R}$ and the area of $B_{r/16}(y_i) \cap \tilde{R}$ is at least $\pi r^2/1024$. Letting C'_{rt} denote the event $C_{\text{rt}} \cap [X(\omega_3) \cap B_{r/16}(y_1) \cap \tilde{R} \ne \varnothing] \cap [X(\omega_3) \cap B_{r/16}(y_2) \cap \tilde{R} \ne \varnothing]$, it follows that

$$(2.26) \qquad \begin{aligned} &P(C'_{\text{rt}} \mid X(\omega_1), X(\omega_2)) \\ &\qquad \ge \left(1 - \exp(-\lambda \pi r^2/1024)\right)^2 \quad \text{a.s. on the event } C_{\text{rt}}. \end{aligned}$$

We call sites of $X(\omega_3) \cap B_{r/16}(y_i)$ *auxiliary sites*. Letting y'_i denote such an auxiliary site for $i = 1, 2$, we have $|x - y_2| = |x - z| - |y_2 - z| \le (2\sqrt{2} + 2)r - 47r/16 < 31r/16$ and therefore

$$(2.27) \qquad |x - y'_2| < 2r, \qquad |y'_2 - y'_1| < 2r, \qquad |y'_1 - z| < 2r.$$

Note that this computation might fail if x were a distant final visible site; $|x - z|$ could then be as large as $5r$, so obtaining $|x - y'_2| < 2r$ would require that $|y'_2 - z|$ be at least $3r$, which places y'_2 in the region $\hat{V}(R, \omega_1)$ where sites come from $X(\omega_2)$, causing (2.25) and therefore perhaps (2.26) to fail. This again is the reason for "uncovering" the sites of $X(\omega_2)$ in

$\hat{W}(R, \omega_1)$, but only when there is no frontier site, in the special construction of X in Step 4—we need to find a terminal site closer than a distant final visible site.

Similarly, suppose alternatively that $\omega \in D_{\mathrm{rt}}$ and x is a first uncovered site. Again let y_1 be a point on the line segment from x to z with $|y_1 - z| = 31r/16$. We have $\omega \in [\omega_1 \in H^s_{\mathrm{occ},r}(R)] \setminus C$ and, by (2.10), $B_{r/16}(y_1) \subset Q(R, \omega_1)$, so $X(\omega) \cap B_{r/16}(y_1) = X(\omega_3) \cap B_{r/16}(y_1)$; that is, the sites in $B_{r/16}(y_1)$ again come from $X(\omega_3)$. Letting D'_{rt} denote the event $D_{\mathrm{rt}} \cap [X(\omega_3) \cap B_{r/16}(y_1) \cap \tilde{R} \neq \varnothing]$, it follows that

$$P(D'_{\mathrm{rt}} \mid X(\omega_1), X(\omega_2)) \geq 1 - \exp(-\lambda\pi r^2/1024) \quad \text{a.s. on the event } D_{\mathrm{rt}}. \tag{2.28}$$

If y'_1 is an auxiliary site, we have $|x - y_1| = |x - z| - |z - y_1| \leq 3r - 31r/16 < 31r/16$ and therefore

$$|x - y'_1| < 2r \quad \text{and} \quad |y'_1 - z| < 2r. \tag{2.29}$$

Because of (2.27) and (2.29), occurrence of the event $C'_{\mathrm{rt}} \cup D'_{\mathrm{rt}}$ implies that there exists an occupied path in $(X(\omega) \cap (L(R, \omega_1)^r)^c)^r$ from $Z(R)^r$ to R^{--}_{top} inside R^{--}_{wide}. The fact that the paths in these events are in $(X(\omega) \cap (L(R, \omega_1)^r)^c)^r$ means that the relevant sites of $X(\omega)$ are all outside $L(R, \omega_1)^r$ so are from $X(\omega_2) \cup X(\omega_3)$. Define the events

$$J^{--} := H^s_{\mathrm{occ},r}(R) \cap \Big[\text{there exists an occupied path in } \big(X \cap (L(R)^r)^c\big)^r \text{ from } Z(R)^r \text{ to } R^{--}_{\mathrm{top}} \text{ inside } R^{--}_{\mathrm{wide}}\Big],$$

$$J^{++} := H^s_{\mathrm{occ},r}(R) \cap \Big[\text{there exists an occupied path in } \big(X \cap (L(R)^r)^c\big)^r \text{ from } Z(R)^r \text{ to } R^{++}_{\mathrm{top}} \text{ inside } R^{++}_{\mathrm{wide}}\Big].$$

For $\omega_1 \in H^s_{\mathrm{occ},r}(R)$ we then have, using Lemma 2.2,

$$P(J^{--} \cap J^{++} \mid X(\omega_1)) \geq P(J^{--} \mid X(\omega_1)) P(J^{++} \mid X(\omega_1)) = P(J^{--} \mid X(\omega_1))^2 \quad \text{a.s.,} \tag{2.30}$$

while, by (2.26) and (2.28), on the event $C_{\mathrm{rt}} \cup D_{\mathrm{rt}}$,

$$P(J^{--} \mid X(\omega_1), X(\omega_2)) \geq P(C'_{\mathrm{rt}} \cup D'_{\mathrm{rt}} \mid X(\omega_1), X(\omega_2)) \geq \big(1 - \exp(-\lambda\pi r^2/1024)\big)^2 \quad \text{a.s.}$$

Therefore, by (2.23) and (2.24), on the event $[\omega_1 \in H^s_{\mathrm{occ},r}(R)]$,

$$\begin{aligned} P(J^{--} \mid X(\omega_1)) &\geq \big(1 - \exp(-\lambda\pi r^2/1024)\big)^2 P(C_{\mathrm{rt}} \cup D_{\mathrm{rt}} \mid X(\omega_1)) \\ &\geq \big(1 - \exp(-\lambda\pi r^2/1024)\big)^2 P(C \cup D \mid X(\omega_1))/2 \\ &\geq \big(1 - \exp(-\lambda\pi r^2/1024)\big)^2 P(\omega_2 \in V_{\mathrm{occ},r}(R^{--}))/2 \quad \text{a.s.} \end{aligned} \tag{2.31}$$

Let F denote the event that there exists a supported horizontal occupied crossing of R starting in the lower half of the left side of R and ending in the lower half of the right side of R. Then, by (2.30) and (2.31), integrating over the event $[\omega_1 \in F]$,

$$\begin{aligned} P(E) &\geq P(J^{--} \cap J^{++} \cap F) \\ &\geq \left(1 - \exp(-\lambda\pi r^2/1024)\right)^4 P(V_{\text{occ},r}(R^{--}))^2 P(F)/4. \end{aligned} \tag{2.32}$$

To bound $P(F)$ from below, let F_{lt} (F_{rt}) denote the event that there exists a supported horizontal occupied crossing of R starting (ending) in the lower half of the left (right) side of R. Then F_{lt} and F_{rt} are increasing events, and, from (2.4) and symmetry,

$$P(F_{\text{lt}}) = P(F_{\text{rt}}) \geq P\left(H^s_{\text{occ},r}(R)\right)/2 \geq P\left(H_{\text{occ},r}(R^-_{\text{trunc}})\right)/2,$$

so, by Lemma 2.2,

$$P(F) \geq P(F_{\text{lt}} \cap F_{\text{rt}}) \geq P\left(H_{\text{occ},r}(R^-_{\text{trunc}})\right)^2/4.$$

Therefore, by (2.32) and (2.4), for $K(t) = (1 - \exp(-\pi t/1024))^4/16$,

$$P(E) \geq K(\lambda r^2) P\left(V_{\text{occ},r}(R^{--})\right)^2 P\left(H_{\text{occ},r}(R^-_{\text{trunc}})\right)^2.$$

With (2.21) this shows

$$P\left(H_{\text{occ},r}(R^- \cup R^+)\right) \geq K(\lambda r^2) P\left(V_{\text{occ},r}(R^{--})\right)^2 P\left(H_{\text{occ},r}(R^-_{\text{trunc}})\right)^4,$$

which is equivalent to the conclusion of the theorem. □

3. Consequences of the RSW theorem. The RSW theorem on a lattice has a variety of by-now-standard consequences; see Chapter 9 of [11] or see [7]. For most of these consequences, the extension to the present continuum case is reasonably straightforward, but some technicalities do arise due to the presence of the constant K in Theorem 2.1. In addition to these standard consequences, we will derive a new result from the RSW theorem—essentially Ramey's conjecture [18]—which will be applied in the proof of the CLT for minimal spanning trees.

Let $\lceil x \rceil$ denote the least integer greater than or equal to x.

LEMMA 3.1. *Let $r, l > 0$ and suppose $\alpha > \delta > 0$ and $k \geq \lceil \alpha/\delta \rceil$. Then the following conclusions hold*:

(i) $P(H_{\text{occ},r}([0,(1+\alpha)l] \times [0,l])) \geq P(H_{\text{occ},r}([0,(1+\delta)l] \times [0,l]))^{2k-1}$.

(ii) *If* $l > 2r$, $0 < \varepsilon \leq 1/(2k-1)$, $\beta \geq 3$ *and* $P(H_{\text{occ},r}([0,(1+\delta)l] \times [0,l]) > 1 - \varepsilon$, *then*

$$P\left(H_{\text{occ},r}([0,(1+\alpha)l] \times [0,\beta l])\right) > 1 - \left((2k-1)\varepsilon\right)^2 \geq 1 - \varepsilon.$$

The same conclusions hold with "occ" *replaced throughout by* "vac."

PROOF. If there are horizontal occupied crossings of each of the rectangles $[j\delta l, (1+(j+1)\delta)l] \times [0, l]$, $0 \le j \le k-1$, and vertical occupied crossings of each of the rectangles $[j\delta l, (1+j\delta)/l] \times [0, (1+\delta)l]$, $1 \le j \le k-1$, then there is necessarily a horizontal occupied crossing of the rectangle $[0, (\alpha+1)l] \times [0, l]$. Each of these $2k-1$ crossings has the probability on the right-hand side of (i). This and Lemma 2.2 prove (i).

Turning to (ii), since $l > 2r$ and $\beta \ge 3$, the events $H_{\text{occ}, r}([0, (1+\alpha)l] \times [0, l])$ and $H_{\text{occ}, r}([0, (1+\alpha)l] \times [(\beta-1)l, \beta l])$ are independent. Statement (ii) now follows from the fact that the union of these two events is contained in $H_{\text{occ}, r}([0, (1+\alpha)l] \times [0, \beta l])$, while, by (i), each of these two events has probability at least $1-(2k-1)\varepsilon$. □

Given L and ε, we will later wish to choose a value of r so that $P(H_{\text{occ}, r}([0, L] \times [0, 3L])) = \varepsilon$. This motivates the following two results.

LEMMA 3.2. *Suppose* $0 < \varepsilon < 1/363$, $0 < 2r < l \le L/3$ *and* $\delta \ge 1/7$. *Then the following conclusions hold*:

(i) *If*

$$P(H_{\text{vac}, r}([0, (1+\delta)l] \times [0, l])) > 1-\varepsilon, \tag{3.1}$$

then $P(H_{\text{vac}, r}([0, 3L] \times [0, L])) > 1-\varepsilon$.

(ii) *If*

$$P(H_{\text{occ}, r}([0, L] \times [0, 3L])) = \varepsilon, \tag{3.2}$$

then $P(H_{\text{occ}, r}([0, l] \times [0, (1+\delta)l])) \ge \varepsilon$.

The same results are valid with "occ" *and* "vac" *interchanged throughout*.

PROOF. Suppose first that (3.1) holds with $\delta = 2$ and $l = 3^{-j}L$ for some $j \ge 0$. If $j = 0$ there is nothing to prove. If $j \ge 1$, then applying Lemma 3.1(ii) with the α, δ and β there assigned to be 8, 2 and 3, respectively, we see that (3.1) is also true with these same α, δ and β for $l = 3^{-j+1}L$. Iterating on j then proves (i). For general δ and l, define $j \ge 0$ by $3^{-(j+2)}L < l \le 3^{-(j+1)}L$. Set $\beta := 3^{-j}L/l$, so $3 \le \beta < 9$, and define α by $1+\alpha = 3^{-j+1}L/l$, so $8 \le \alpha < 26$ and $\alpha/\delta < 182$. By Lemma 3.1(ii) with $k = 182$, we have $P(H_{\text{vac}, r}([0, 3^{-j+1}L] \times [0, 3^{-j}L])) > 1-\varepsilon$. But this is just (3.1) with $\delta = 2$ and $l = 3^{-j}L$, which are hypotheses under which we have already proved (i), so the proof of (i) is complete.

Because of the duality relation (2.1) and rotational invariance, (ii) is essentially just the contrapositive of (i). □

LEMMA 3.3. *Suppose* r, L *and* ε *satisfy* (3.2), *suppose* $0 < \varepsilon < 1/363$ *and suppose* $45r \le s < L/3 - 5r$. *Then*

$$P(H_{\text{occ}, r}([0, 3s] \times [0, s])) \ge (K(\lambda r^2)\varepsilon^6)^{27}.$$

PROOF. Define b, h, t and a by

$$s = 8b/7 + 22r/7,$$
$$h = 8b/7 + 36r/7 \qquad [\text{equivalently, } h - 4r = 8(b + r)/7],$$
$$t = 64b/49 + 456r/49 \quad [\text{equivalently, } t = 8(h + 3r)/7],$$
$$a = b - t/2 + r.$$

Then $b > 36r$ and $t \geq b > t/2 + 2r$. Applying Lemma 3.2(ii) with $l = b + r$ and $(1 + \delta)l = h - 4r$, we see that $\delta = 1/7$ and

$$P(H_{\mathrm{occ}, r}([0, b + r] \times [0, h - 4r])) \geq \varepsilon.$$

Similarly, applying Lemma 3.2(ii) with $l = h + 3r$ and $(1 + \delta)l = t$ and using rotational invariance, we obtain

$$P(V_{\mathrm{occ}, r}([0, t] \times [0, h + 3r])) \geq \varepsilon.$$

Since $b > 36r$, we have $b + a \geq 8(h - 2r)/7 = 8s/7$. Therefore, by Theorem 2.1,

$$P(H_{\mathrm{occ}, r}([0, 8s/7] \times [0, s])) \geq P(H_{\mathrm{occ}, r}([0, b + a] \times [0, h - 2r])) \geq K(\lambda r^2)\varepsilon^6.$$

Lemma 3.1(i), with $\alpha = 2$, $\delta = 1/7$and $k = 14$, now completes the proof. □

THEOREM 3.4. *There exists $\theta > 0$ such that, for each $r > 0$, the following are equivalent*:

(i) *There is occupied percolation at level r.*
(ii) $\lim_{L \to \infty} P(H_{\mathrm{occ}, r}([0, L] \times [0, L])) = 1$.
(iii) $\lim_{L \to \infty} P(H_{\mathrm{occ}, r}([0, 3L] \times [0, L])) = 1$.
(iv) *There exists $L > 2r$ such that $P(H_{\mathrm{occ}, r}([0, 3L] \times [0, L])) > 1 - \theta$.*

The same result is valid with "occ" *replaced throughout by* "vac."

PROOF. We show first that (i) ⇒ (iii) and (ii) ⇒ (iii). Suppose (iii) is false; that is, there exists $\varepsilon > 0$ such that $P(H_{\mathrm{occ}, r}([0, 3L] \times [0, L])) \leq 1 - \varepsilon$ for arbitrarily large values of L. We may assume $\varepsilon < 1/363$; then, by Lemma 3.2(i), we have $P(H_{\mathrm{occ}, r}([0, 8l/7] \times [0, l])) \leq 1 - \varepsilon$ or, equivalently, $P(H_{\mathrm{vac}, r}([0, l] \times [0, 8l/7])) \geq \varepsilon$ for all $l > 2r$. We now apply Roy's RSW theorem for vacant crossings [24], with Roy's l_1, l_2 and l_3 set to be $7l/8$, l and $8l/7$, respectively, and Roy's $k = 24/7$. This yields

$$P(H_{\mathrm{vac}, r}([0, 3l] \times [0, l])) \geq K_k(\lambda r^2) f(\varepsilon) \quad \text{for all } l > 5r, \tag{3.3}$$

where $f(\varepsilon)$ is a constant not depending on λ, r or l and $K_k(\lambda r^2)$ is a constant which does not depend on ε or l. This means that the probability there is a vacant circuit surrounding $(0, l)^2$ in the annulus $[-l, 2l]^2 \setminus (0, l)^2$ is at least $(K_k(\lambda r^2) f(\varepsilon))^4$, so, with probability 1, such circuits exist for arbitrarily large l. This means that there is no occupied percolation; that is, (i) is false. From

(3.3) we also obtain that $P(H_{\text{vac},r}([0,l]\times[0,l])) \geq K_k(\lambda r^2)f(\varepsilon)$ for all $l > 5r$, so that (ii) is false. Thus we have shown (i) ⇒ (iii) and (ii) ⇒ (iii).

Clearly, (iii) implies (ii) and (iv). That (iv) implies (i) for sufficiently small θ follows from a Peierls-type argument using renormalized bonds of size $3L \times L$; details are similar to the proof of Proposition 3.1 of [13].

The proof for vacant percolation and vacant crossings is similar, using Theorem 2.1 in place of Roy's RSW theorem [24] and Lemma 3.3 in place of (3.3). □

By monotonicity in r, there exists a constant $r_c(\lambda)$ such that there is occupied percolation a.s. when $r > r_c(\lambda)$, and a.s. no occupied percolation when $r < r_c(\lambda)$. Zuev and Sidorenko [31] proved that $0 < r_c(\lambda) < \infty$.

There is a natural extension of the notion of a Euclidean minimal spanning tree to the notion of a "minimal spanning forest" of an infinite site set; see [1] or [3]. The following corollary is applied in [3] to show that, for the set of sites of a Poisson process in $\mathbb{R}^2$, this minimal spanning forest actually consists of a single infinite tree with one topological end.

COROLLARY 3.5. *At level $r = r_c(\lambda)$, there is a.s. neither occupied nor vacant percolation.*

PROOF. Clearly, $P(H_{\text{occ},r}([0,3L]\times[0,L]))$ is a continuous function of r for fixed L. It follows that the set of r for which (iv) of Theorem 3.4 is satisfied is open. The same is true with "occ" replaced by "vac." □

We say a set B *surrounds* a set A if $B \subset A^c$ but every path from A to ∞ in $\mathbb{R}^2$ intersects B. For $A \subset \Lambda \subset \mathbb{R}^2$, and V a subset of $\mathbb{R}^2$, we say a finite set $D \subset V$ is a *barrier set in V around A in Λ at level r* if D^r is connected, D^r surrounds A and the connected component of D^r in V^r is contained in Λ. We will see in Section 4 that the existence of such a barrier set essentially means that the portion of a minimal spanning tree inside D^r is not affected by the sites outside Λ. It is easy to see that if there is an occupied path γ in Λ which surrounds A^r and is not connected to $\partial\Lambda$ in the occupied space V^r, then there must be a barrier set in V around A in Λ. In [20] a structure similar to a barrier set was called a magic circle, and the following result was conjectured.

PROPOSITION 3.6. *For each $\varepsilon > 0$ there exists $L > 0$ (not depending on λ) such that, for each $x \in \mathbb{R}^2$ and for $r = r_c(\lambda)$,*

$$P\Big(\textit{there exists a barrier set in } X \textit{ around } x^r \textit{ in } x + \big(-L/\sqrt{\lambda}, L/\sqrt{\lambda}\big)^2 \textit{ at level } r\Big) \geq 1 - \varepsilon.$$

PROOF. By rescaling we may assume $\lambda = 1$. By Corollary 3.5 there is no vacant percolation at level r, so there exists a.s. a finite $D \subset X$ such that D^r surrounds x^r; we may choose D^r with D^r connected. Since there is also no occupied percolation a.s., the connected component of D^r in X^r is bounded. Therefore, with probability 1, for sufficiently large L there exists a barrier set in X around x^1 in $x + (-L, L)^2$ at level r, and the proposition follows. □

4. The CLT for the minimal spanning tree. If all interpoint distances in a finite set V are distinct, then the Euclidean minimal spanning tree of V is unique, and we denote it MST(V). Of course, this is the case a.s. if V is the set of sites of a Poisson process. Note that MST(V) is a subgraph of the complete graph CG(V) on V; CG(V) has a bond, denoted $\langle x, y\rangle$, between each pair of distinct sites $x, y \in V$. There are a number of standard characterizations describing which bonds of CG(V) are in MST(V); see [18] and [19]. To describe the ones we need, we begin with some definitions. By a *path* in a graph G we implicitly mean a self-avoiding one, that is, a sequence of distinct sites $v_1, \dots, v_n$ such that $\langle v_i, v_{i+1}\rangle \in G$ for all $i < n$. We also identify a path with the corresponding sequence of bonds $\langle v_i, v_{i+1}\rangle$. For sites u and v of a path γ in CG(V), we let γ_{uv} denote the segment of γ from u to v. We call γ *locally minimax* if, for every pair u, v of sites in γ and every path α in CG(V) from u to v,

$$\max\{|x - y|: \langle x, y\rangle \in \gamma_{uv}\} \le \max\{|x - y|: \langle x, y\rangle \in \alpha\}.$$

By viewing the radius r as representing time, one may think of V^r as a growing set. At certain times r, one component of V^r "bumps into" another one and they merge into a single component. With this picture in mind, for x and y distinct sites in V, we call $\langle x, y\rangle$ a *contact bond* for V if, for $r = |x - y|/2$, x and y are in distinct components of V^s for all $s < r$, but are in the same component of V^r.

PROPOSITION 4.1. *Let V be a finite subset of $\mathbb{R}^2$ with all interpoint distances distinct, and let $x, y \in V$. The following are equivalent*:

(4.1) $\langle x, y\rangle \in \mathrm{MST}(V)$.

(4.2) *For some $A \subset V$, $\langle x, y\rangle$ is the shortest bond from A to $V \setminus A$.*

(4.3) *There exists no path from x to y in* CG(V) *with all bonds strictly shorter than $\langle x, y\rangle$.*

(4.4) *$\langle x, y\rangle$ is a bond in some locally minimax path in* CG(V).

(4.5) *$\langle x, y\rangle$ is a contact bond for V.*

Further,

(4.6) *every path in* MST(V) *is locally minimax*.

PROOF. Statement (4.6) and the equivalence of (4.1)–(4.4) are well known, with portions appearing in [18] and [19]; a full proof is given in [3]. The equivalence of (4.5) and (4.3) is straightforward. □

The construction of the MST as the set of all contact bonds is equivalent to the standard "greedy algorithm"; see [18].

Though many of the details have been changed, the general outline of the remaining results in this section appeared in the dissertation of Ramey [20].

In view of the characterization (4.5), the following two results underlie the relation between barrier sets and the MST.

LEMMA 4.2. *Let V be a finite subset of $\mathbb{R}^2$ with all interpoint distances distinct. Suppose $r > 0$, $A \subset V$, $x \in V$, $y \in V$ and A^r, x^r and y^r are disjoint. If A^r surrounds x^r but not y^r, then $\langle x, y \rangle$ is not a contact bond for V.*

PROOF. Let $D \subset A$ be such that D^r is connected and surrounds x^r. Then $r < d(x, D^r) < d(x, y^r) = |x - y| - r$ and $r < d(y, D^r) < d(y, x^r) = |x - y| - r$, so, for $s := \max(d(x, D), d(y, D))/2$, we have $r < s < |x - y|/2$ and x, y and D are all part of the same connected component of V^s. Thus the lemma follows. $\square$

Given V a finite subset of $\mathbb{R}^2$ and $D \subset V$, let

$$S = S(V, D) := \{x \in V: x^r \text{ is surrounded by } D^r\},$$
$$C = C(V, D) := \{x \in V \setminus S: x \text{ is connected to } D \text{ in } V^r\},$$
$$E = E(V, D) := V \setminus (S \cup C).$$

Consider also the set

$$\tilde{C} = \tilde{C}(V, D) := \{x \in V \setminus S: x \text{ is connected to } D \text{ in } (V \setminus S)^r\}.$$

Clearly, $\tilde{C} \subset C$. If $x \in C$ with $x^r \cap y^r \neq \emptyset$ for some $y \in D$, then (using $x, y \notin S$) the straight line from x to y is in $(V \setminus S)^r$, so we have $x \in \tilde{C}$. On the other hand, if $x \in C$ with $x^r \cap D^r = \emptyset$, then D^r separates x from S^r, so again $x \in \tilde{C}$. Thus $C \subset \tilde{C}$, so we have

$$C = \tilde{C}.$$

LEMMA 4.3. *Let V be a finite subset of $\mathbb{R}^2$ with all interpoint distances distinct, and let $D \subset V$ be a set of sites such that D^r is connected. Then, for C, S and E as above,*

(i) *for $x, y \in C \cup S$, $\langle x, y \rangle \in \mathrm{MST}(V)$ if and only if $\langle x, y \rangle \in \mathrm{MST}(C \cup S)$;*
(ii) *for $x \in E$ and $y \in C \cup E$, $\langle x, y \rangle \in \mathrm{MST}(V)$ if and only if $\langle x, y \rangle \in \mathrm{MST}(C \cup E)$.*

PROOF. We begin with (i). If $\langle x, y \rangle \in \mathrm{MST}(V)$, then it follows from the characterization (4.3) that $\langle x, y \rangle \in \mathrm{MST}(C \cup S)$. Thus suppose $\langle x, y \rangle \notin \mathrm{MST}(V)$. Then, by (4.3), there exists $t < |x - y|/2$ such that there is a path γ from x to y in $\mathrm{CG}(V)$ in which every bond has length at most $2t$. By (4.4) and (4.6), we may assume γ is a path in $\mathrm{MST}(V)$. We claim that there is a path from x to y in $\mathrm{CG}(C \cup S)$ in which every bond has length at most $2t$.

Suppose γ contains a site $z \in E$; if not, then the claim is proved. Let v be the first site in γ_{zy} such that all sites in γ_{vy} are in $C \cup S$, and let u be the site immediately preceding v in γ. Then $u \in E$ and $\langle u, v \rangle \in \mathrm{MST}(V)$ so, by Lemma 4.2, $v \notin S$ and hence $v \in C$. Thus γ_{vy} is a path in $\mathrm{CG}(C \cup S)$ from C to y in which every bond has length at most $2t$. Similarly, there is a path γ_{xw} in $\mathrm{CG}(C \cup S)$ from x to some $w \in C$ in which every bond has length at most $2t$. Now $\tilde{C}^r$ $(= C^r)$ is connected since D^r is, so there is a path α from w to v in $\mathrm{CG}(C)$ in which every bond has length at most $2r$. Further, $|u - v| > 2r$ since $u \in E$ and $v \in C$, while $|u - v| \le 2t$ since $\langle u, v \rangle \in \gamma$. Therefore, $r < t$, so $\gamma_{xw} \cup \alpha \cup \gamma_{vy}$ is a path in $\mathrm{CG}(C \cup S)$ in which every bond has length at most $2t$, proving the claim. Thus, by (4.3), $\langle x, y \rangle \notin \mathrm{MST}(C \cup S)$, and (i) is proved.

The proof of (ii) is similar with S and E interchanged, except that v and w are now the first and last sites of γ which are in $C \cup S$. □

COROLLARY 4.4. *Suppose V is a finite subset of $\mathbb{R}^2$ with all interpoint distances distinct, $r > 0$, $x \in \Lambda \subset \mathbb{R}^2$ and there is a barrier set in V around x^r in Λ at level r. Then, for $W := V \cap \Lambda$,*

$$\mathscr{L}(V \cup \{x\}) - \mathscr{L}(V) = \mathscr{L}(W \cup \{x\}) - \mathscr{L}(W).$$

PROOF. Let D be the barrier set. Then

$$\begin{aligned} S(V \cup \{x\}, D) &= S(V, D) \cup \{x\}, \\ C(V \cup \{x\}, D) &= C(V, D), \\ E(V \cup \{x\}, D) &= E(V, D) \end{aligned}$$

and

$$C(W, D) = C(V, D), \qquad S(W, D) = S(V, D). \tag{4.7}$$

Hence, by Lemma 4.3(i) and (ii),

$$\begin{aligned} &\mathrm{MST}(V) \cap \{\langle z, y \rangle : z \in E(V, D),\ y \in C(V, D) \cup E(V, D)\} \\ &\quad = \mathrm{MST}(V \cup \{x\}) \cap \{\langle z, y \rangle : z \in E(V, D),\ y \in C(V, D) \cup E(V, D)\} \end{aligned}$$

and

$$\begin{aligned} &\mathrm{MST}(V) \cap \{\langle z, y \rangle : z, y \in C(V, D) \cup S(V, D)\} \\ &\quad = \mathrm{MST}(C(V, D) \cup S(V, D)), \\ &\mathrm{MST}(V \cup \{x\}) \cap \{\langle z, y \rangle : z, y \in C(V \cup \{x\}, D) \cup S(V \cup \{x\}, D)\} \\ &\quad = \mathrm{MST}(C(V, D) \cup S(V, D) \cup \{x\}), \end{aligned}$$

while, by Lemma 4.2,

$$\begin{aligned} &\mathrm{MST}(V) \cap \{\langle z, y \rangle : z \in S(V, D),\ y \in E(V, D)\} \\ &\quad = \mathrm{MST}(V \cup \{x\}) \cap \{\langle z, y \rangle : z \in S(V \cup \{x\}, D),\ y \in E(V \cup \{x\}, D)\} \\ &\quad = \varnothing, \end{aligned}$$

so

$$\mathscr{L}(V \cup \{x\}) - \mathscr{L}(V) = \mathscr{L}(C(V,D) \cup S(V,D) \cup \{x\}) - \mathscr{L}(C(V,D) \cup S(V,D)). \tag{4.8}$$

By the same argument, (4.8) is true with V replaced by W. But, by (4.7), the right-hand side of (4.8) remains the same when V is replaced by W, and the corollary follows. □

LEMMA 4.5. *The sequence $n^{\alpha} E(\mathscr{L}_n - \mathscr{L}_{n-1})^{2\alpha}$, $n \geq 2$, is bounded for each $\alpha > 0$.*

PROOF. Let U_n denote the total length of all bonds in MST($\{X_1, \dots, X_n\}$) for which X_n is an endpoint, and let i be such that X_i is the closest site to X_n in $\{X_1, \dots, X_{n-1}\}$. By (4.2), $\langle X_n, X_i \rangle \in$ MST($\{X_1, \dots, X_n\}$), so $|X_n - X_i| \leq U_n$. Since MST($\{X_1, \dots, X_{n-1}\}$) $\cup \{\langle X_n, X_i \rangle\}$ spans $\{X_1, \dots, X_n\}$, we have

$$\mathscr{L}_n \leq \mathscr{L}_{n-1} + |X_n - X_i| \leq \mathscr{L}_{n-1} + U_n. \tag{4.9}$$

In the opposite direction, let $J := \{j \leq n-1: \langle X_n, X_j \rangle \in \text{MST}(\{X_1, \dots, X_n\})\}$. Deleting X_n and $\langle X_n, X_i \rangle$ from MST($\{X_1, \dots, X_n\}$) and replacing $\langle X_n, X_j \rangle$ with $\langle X_i, X_j \rangle$ for each $j \in J$ yields a graph which spans $\{X_1, \dots, X_{n-1}\}$ for which the total length of all bonds is at most $\mathscr{L}_n + |J|\,|X_n - X_i| \leq \mathscr{L}_n + U_n$. Hence $\mathscr{L}_{n-1} \leq \mathscr{L}_n + U_n$, which with (4.9) yields

$$|\mathscr{L}_{n-1} - \mathscr{L}_n| \leq U_n. \tag{4.10}$$

Let $R_n := \max\{|X_n - X_j|: j \in J\}$. If $\langle X_n, X_j \rangle \in$ MST($\{X_1, \dots, X_n\}$), it is well known that $\langle X_n, X_j \rangle$ is a bond in the Delaunay triangulation of $\{X_1, \dots, X_n\}$ (see [18] for the definition and this result), and for each point u of $\langle X_n, X_j \rangle$, the closest site to u among $\{X_1, \dots, X_n\}$ is either X_n or X_j, or both. In particular, X_n and X_j are the closest sites to the midpoint $(X_n + X_j)/2$. Therefore, for the open ball, denoted B_{nj}, which has X_n and X_j as diameter endpoints,

$$\{X_1, \dots, X_{n-1}\} \cap B_{nj} = \varnothing. \tag{4.11}$$

Given $k \geq 1$, we can divide $[0,1]^2$ into a collection $\mathscr{C}_k$ of 4^k squares of side 2^{-k}, with disjoint interiors. Given $0 < t < \sqrt{2}$, let k be the least integer such that $t/9 \geq 2^{-k}$, and let C_{nk} denote the square in $\mathscr{C}_k$ which contains X_n. Consider a 5×5 array $\mathscr{A}_{nk}$ of squares of side 2^{-k} with C_{nk} as the center square. Suppose $|X_n - X_j| > t$ for some $j \in J$. Then the line from X_n to X_j passes through at least one of the squares of $\mathscr{C}_k$ on the outer perimeter of the array; let Q_{njk} be such a square (see Figure 6). Note $(X_n + X_j)/2$ is outside the 5×5 array on this same line, since $t/9 \geq 2^{-k}$. It follows easily that

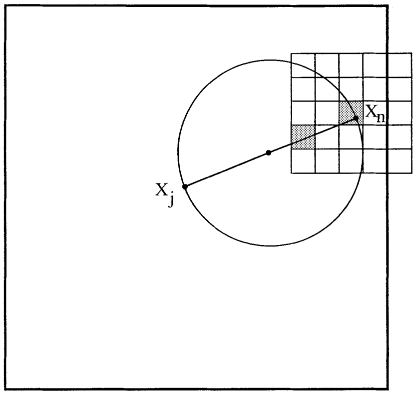

FIG. 6. *Illustration for the proof of Lemma* 4.5. *The squares C_{nk} and Q_{njk} are shaded.*

$Q_{njk} \subset B_{nj}$, so, by (4.11), we have $\{X_1, \dots, X_n\} \cap Q_{njk} = \varnothing$. Thus

$$\begin{aligned} P(R_n > t \mid X_n) &\le P(\{X_1, \dots, X_{n-1}\} \cap Q = \varnothing \text{ for some } Q \in \mathscr{A}_{nk} \mid X_n) \\ &\le 25(1 - 4^{-k})^{n-1} \\ &\le 25 \exp(-(n-1)t^2/576), \end{aligned}$$

so $\{n^\alpha E R_n^{2\alpha},\ n \ge 2\}$ is bounded. But, as observed in [20] and [30], the angle between two MST bonds emanating from the same site cannot be less than 60°, so, from (4.10),

$$|\mathscr{L}_{n-1} - \mathscr{L}_n| \le |J| R_n \le 6 R_n,$$

and the lemma follows. □

We next define an approximation to $\mathscr{L}(\cdot)$ for which the CLT is relatively easy to prove. For $k \ge 1$, divide $[0,1)^2$ into a $k \times k$ array $\{G_i,\ i \le k^2\}$ of equal-sized squares, each of the form $[a, b) \times [c, d)$. Let $I(n, k)$ be such that $X_n \in G_{I(n,k)}$. Now, for V a finite subset of $[0,1]^2$, define

$$\tilde{\mathscr{L}}_{k,n} := \sum_{i \le k^2} \mathscr{L}(\{X_1, \dots, X_n\} \cap G_i).$$

Let $k(\lambda) \to \infty$ with $k(\lambda) = o(\lambda^{1/2})$. Note that

(4.12) $\tilde{\mathscr{L}}_{k(\lambda), N(\lambda)}$ is the sum of $k(\lambda)^2$ independent copies of $k(\lambda)^{-1} \mathscr{L}_{N(\lambda / k(\lambda)^2)}$.

By a result of Few [10], there exists a constant c_0 such that

$$\mathscr{L}(V) \le c_0 |V|^{1/2} \quad \text{for every finite } V \subset [0,1]^2. \tag{4.13}$$

It follows readily from (4.12) that

$$\tilde{\mathscr{L}}_{k,n} \le c_0 n^{1/2} \quad \text{for all } n, k \ge 1. \tag{4.14}$$

LEMMA 4.6. *There exist constants $r_n > 0$ such that $\inf_{n \ge \lambda/2} P($there exists a barrier set in $\{X_1, \dots, X_{n-1}\}$ around $X_n^{r_n}$ in $G_{I(n, k(\lambda))}$ at level $r_n) \to 1$ as $\lambda \to \infty$.*

PROOF. Let $\varepsilon, \lambda > 0$ and $n \geq \lambda/2$. Let M be Binomial$(n-1, k(\lambda)^{-2})$ and let M^* be Poisson$((n-1)/k(\lambda)^2)$. If λ is large, then M and M^* can be coupled so that

$$P(M \neq M^*) < \varepsilon; \tag{4.15}$$

see [5]. Let $Y_1, Y_2, \ldots$ be iid uniform in $G_{I(n,k(\lambda))}$, independent of X_n, M and M^*. Then

$$\{Y_1, \ldots, Y_M\} = \{X_1, \ldots, X_{n-1}\} \cap G_{I(n,k(\lambda))} \quad \text{in distribution.} \tag{4.16}$$

By Proposition 3.6, there exist $L > 0$ and $r_n > 0$ such that, for $n \geq \lambda/2$ and λ large,

$$\begin{aligned} &P\big(\text{there does not exist a barrier set in } \{Y_1, \ldots, Y_{M^*}\} \text{ around } X_n^{r_n} \\ &\qquad\qquad\qquad\qquad \text{in } G_{I(n,k(\lambda))} \text{ at level } r_n\big) \\ &\quad \leq P\Big(X_n + \big(-L/\sqrt{n} - 1, L/\sqrt{n} - 1\big)^2 \not\subset G_{I(n,k(\lambda))}\Big) + \varepsilon \\ &\quad \leq 2\varepsilon, \end{aligned}$$

since $L/\sqrt{n} - 1 = o(1/k(\lambda))$. Since ε is arbitrary, with (4.15) and (4.16) this proves the lemma. $\square$

We next use the results on barrier sets (Corollary 4.4 and Lemma 4.6) to show that $\tilde{\mathscr{L}}_{k(\lambda), N(\lambda)}$ is indeed a good approximation to $\mathscr{L}_{N(\lambda)}$.

LEMMA 4.7. (i) $\sup_{n \geq \lambda/2} \mathrm{Var}(\tilde{\mathscr{L}}_{k(\lambda), n} - \mathscr{L}_n) \to 0$ *as* $\lambda \to \infty$.
(ii) $\mathrm{Var}(\tilde{\mathscr{L}}_{k(\lambda), N(\lambda)} - \mathscr{L}_{N(\lambda)}) \to 0$ *as* $\lambda \to \infty$.

PROOF. (i) Fix λ and let

$$H_{n,\lambda} := \tilde{\mathscr{L}}_{k(\lambda), n} - \mathscr{L}_n.$$

By the Efron–Stein [9] inequality, we have

$$\mathrm{Var}(H_{n-1,\lambda}) \leq nE(H_{n,\lambda} - H_{n-1,\lambda})^2. \tag{4.17}$$

By Lemma 4.5, $\{n(\mathscr{L}_n - \mathscr{L}_{n-1})^2,\ n \geq 2\}$ is uniformly integrable. By the same proof, $\{n(\tilde{\mathscr{L}}_{k,n} - \tilde{\mathscr{L}}_{k,n-1})^2,\ n \geq 2,\ k \geq 1\}$ is uniformly integrable. Hence $\{n(H_{n,\lambda} - H_{n-1,\lambda})^2,\ n \geq 2,\ \lambda > 0\}$ is uniformly integrable. But, by Corollary 4.4, applied with $x = X_n$ and $\Lambda = G_{I(n,k(\lambda))}$, when the event in Lemma 4.6 occurs, we have

$$H_{n,\lambda} - H_{n-1,\lambda} = \big(\tilde{\mathscr{L}}_{k,n} - \tilde{\mathscr{L}}_{k,n-1}\big) - (\mathscr{L}_n - \mathscr{L}_{n-1}) = 0,$$

so, by Lemma 4.6 and uniform integrability,

$$\sup_{n \geq \lambda/2} nE(H_{n,\lambda} - H_{n-1,\lambda})^2 \to 0 \quad \text{as } \lambda \to \infty. \tag{4.18}$$

The result now follows from (4.17).

(ii) Observe that

$$\psi_\lambda(n) := E\Big(\tilde{L}_{k(\lambda), N(\lambda)} - L_{N(\lambda)} \,|\, N(\lambda) = n\Big) = EH_{n,\lambda},$$

$$\varphi_\lambda(n) := \mathrm{Var}\Big(\tilde{L}_{k(\lambda), N(\lambda)} - L_{N(\lambda)} \,|\, N(\lambda) = n\Big) = \mathrm{Var}(H_{n,\lambda}).$$

Therefore,

$$\mathrm{Var}\Big(\tilde{\mathscr{L}}_{k(\lambda), N(\lambda)} - \mathscr{L}_{N(\lambda)}\Big) = E\varphi_\lambda(N(\lambda)) + \mathrm{var}(\psi_\lambda(N(\lambda))). \tag{4.19}$$

By (4.13) and (4.14), we have $\varphi_\lambda(n) \le 4c_0^2 n$ for all n, while, by part (i), $\sup_{n \ge \lambda/2} \varphi_\lambda(n) \to 0$ as $\lambda \to \infty$. It follows easily that

$$E\varphi_\lambda(N(\lambda)) \to 0 \quad \text{as } \lambda \to \infty.$$

By (4.13) and (4.14) again, we have $\psi_\lambda(n) \le 2c_0 n^{1/2}$, while, by (4.18), we have

$$\varepsilon_\lambda := \sup_{n \ge \lambda/2} n^{1/2} |\psi_\lambda(n) - \psi_\lambda(n-1)| \to 0 \quad \text{as } \lambda \to \infty.$$

It follows easily that

$$\begin{aligned}\mathrm{var}(\psi_\lambda(N(\lambda))) &\le E(\psi_\lambda(N(\lambda)) - \psi_\lambda(\lfloor \lambda \rfloor))^2 \\ &\le \lambda^{-1}\varepsilon_\lambda^2 E(N(\lambda) - \lfloor \lambda \rfloor)^2 + o(1) \\ &\to 0 \quad \text{as } \lambda \to \infty,\end{aligned}$$

so the lemma follows from (4.19). □

In the next lemma, part (ii) is motivated by (4.12).

LEMMA 4.8. (i) *There exists* $v > 0$ *such that* $\mathrm{Var}(\mathscr{L}_n) \ge v$ *for all* $n \ge 1$, *and* $\mathrm{Var}(\mathscr{L}_{N(\lambda)}) \ge v$ *and* $\mathrm{Var}(\tilde{\mathscr{L}}_{k(\lambda), N(\lambda)}) \ge v$ *for all* $\lambda \ge 1$.

(ii) $k(\lambda)^2 E\left| k(\lambda)^{-1} (\mathscr{L}_{N(\lambda/k(\lambda)^2)} - E\mathscr{L}_{N(\lambda/k(\lambda)^2)}) / \sqrt{\mathrm{Var}\Big(\tilde{\mathscr{L}}_{k(\lambda), N(\lambda)}\Big)} \right|^3 \to 0$ as $\lambda \to \infty$.

PROOF. (i) The lower bound for $\mathrm{Var}(\mathscr{L}_n)$ is essentially similar to the proof of Proposition 5 of [4]; a full proof appears in [20]. The lower bound for $\mathrm{Var}(\mathscr{L}_{N(\lambda)})$ is an easy consequence, and the lower bound for $\mathrm{Var}(\tilde{\mathscr{L}}_{k(\lambda), N(\lambda)})$ then follows from (4.12).

(ii) By part (i), it is sufficient to show that $E|\mathscr{L}_{N(\beta)} - E\mathscr{L}_{N(\beta)}|^3$ is bounded in β. By Lemma 4.5 and the proof of Theorem 3 in [29], there exists c such that

$$\kappa(n) := E|\mathscr{L}_n - E\mathscr{L}_n|^3 \le c \quad \text{for all } n. \tag{4.20}$$

Let $\tau(n) := E\mathscr{L}_n$. Then, using $E\tau(N(\beta)) = E\mathscr{L}_{N(\beta)}$ and $|a+b|^3 \le 8|a|^3 + 8|b|^3$, we obtain

$$\begin{aligned}&E|\mathscr{L}_{N(\beta)} - E\mathscr{L}_{N(\beta)}|^3 \\ &\quad \le 8E|\mathscr{L}_{N(\beta)} - \tau(N(\beta))|^3 + 8E|\tau(N(\beta)) - E\tau(N(\beta))|^3 \\ &\quad = 8E\kappa(N(\beta)) + 8E|\tau(N(\beta)) - E\tau(N(\beta))|^3.\end{aligned} \tag{4.21}$$

Let $N'(\beta)$ be an independent copy of $N(\beta)$; then

$$E|\tau(N(\beta)) - E\tau(N(\beta))|^3 \le E|\tau(N(\beta)) - \tau(N'(\beta))|^3. \tag{4.22}$$

By Lemma 4.5, there exists c_1 such that $|\tau(n) - \tau(n-1)| \le c_1/\sqrt{n}$ for all n, while, by (4.13), $\tau(n) \le c_0\sqrt{n}$ for all n. Therefore, for some constant c_2,

$$E|\tau(N(\beta)) - \tau(N'(\beta))|^3$$
$$\le \left(2c_1/\sqrt{\beta}\right)^3 E|N(\beta) - N'(\beta)|^3 + c_2\beta^{3/2}P(N(\beta) < \beta/4),$$

which is bounded in β. Combining this with (4.20)–(4.22) proves the result. □

The following is our main result.

THEOREM 4.9. *The quantity* $(\mathscr{L}_{N(\lambda)} - E\mathscr{L}_{N(\lambda)})/\sqrt{\mathrm{Var}(\mathscr{L}_{N(\lambda)})}$ *converges in distribution to a standard normal* $N(0,1)$ *as* $\lambda \to \infty$.

PROOF. From (4.12), Lemma 4.8(ii) and Liapounov's CLT (see [8], page 200), it follows that $\left(\tilde{\mathscr{L}}_{k(\lambda),N(\lambda)} - E\tilde{\mathscr{L}}_{k(\lambda),N(\lambda)}\right)/\sqrt{\mathrm{Var}\left(\tilde{\mathscr{L}}_{k(\lambda),N(\lambda)}\right)}$ converges in distribution to a standard normal. By Lemmas 4.8(i) and 4.7(ii),

$$\mathrm{Var}\left(\tilde{\mathscr{L}}_{k(\lambda),N(\lambda)}\right)/\mathrm{Var}\left(\mathscr{L}_{N(\lambda)}\right) \to 1,$$

while

$$\left[\left(\tilde{\mathscr{L}}_{k(\lambda),N(\lambda)} - E\tilde{\mathscr{L}}_{k(\lambda),N(\lambda)}\right) - \left(\mathscr{L}_{N(\lambda)} - E\mathscr{L}_{N(\lambda)}\right)\right]/\sqrt{\mathrm{Var}\left(\tilde{\mathscr{L}}_{k(\lambda),N(\lambda)}\right)} \to 0$$

in probability, and the theorem follows. □

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