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THE NET OUTPUT PROCESS OF A SYSTEM WITH INFINITELY MANY QUEUES[1]

BY P. A. FERRARI AND L. R. G. FONTES

Universidade de São Paulo

We study a system of infinitely many queues with Poisson arrivals and exponential service times. Let the net output process be the difference between the departure process and the arrival process. We impose certain ergodicity conditions on the underlying Markov chain governing the customer path. These conditions imply the existence of an invariant measure under which the average net output process is positive and proportional to the time. Starting the system with that measure, we prove that the net output process is a Poisson process plus a perturbation of order 1. This generalizes the classical theorem of Burke which asserts that the departure process is a Poisson process. An analogous result is proven for the net input process.

1. Introduction. We consider a system of infinitely many queues with arrivals and departures. To describe it, let $S = \{-1, 0, 1, \dots\}$ and consider that at each $x \geq 0$ there is a queue formed by a nonnegative number of indistinguishable customers. If this number is positive, we can think that one of the customers is being served and the others are waiting for service. The service time of queue $x \geq 0$ is exponential with rate $\mu(x)$; that is, it depends on the queue label but does not depend on the configuration of customers in the system. Once the customer is served it jumps to queue y with probability $p(x, y)$. The matrix p is called the routing matrix. "Queue" -1 is out of the system and it is considered as a queue only for notational convenience. We assume that there are infinitely many customers at -1 at all times. This implies that customers are entering from the outside of the system to queue $y \geq 0$ according to a Poisson process of rate $\mu(-1)p(-1, y)$. This system is a particular case of the so-called zero range process with an external source/sink of customers. The process takes values in $\Omega = \mathbb{N}^{\mathbb{N}}$ and its generator is given by

$$(1) \qquad \mathbf{L}f(\eta) = \sum_{x,\, y \geq -1} 1\{\eta(x) > 0\}\mu(x)p(x, y)[f(\eta^{xy}) - f(\eta)],$$

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where $\eta \in \Omega$ is a state of the system, $\eta(-1)$ is defined as $+\infty$, η^{xy} is defined by

$$\eta^{xy}(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y, \\ \eta(x) - 1, & \text{if } z = x, \\ \eta(y) + 1, & \text{if } z = y, \end{cases}$$

and f is a local function in Ω. There are two differences between the system studied here and the zero range process studied by Andjel (1982). The first one is that we have a sink/source of customers, which is absent in Andjel's paper. The other is that our service rate $\mu(x)$ does not depend on the configuration, but it may depend on the queue label. The service rate considered by Andjel is a function $g(k)$, where k is the number of customers in the queue, independent of the queue. Since our rates do not depend on the configuration and may vary from queue to queue, it is more convenient to use the notation

$$q(x, y) = \mu(x)p(x, y), \quad x \neq y, \qquad q(x, x) = -\mu(x).$$

If the total arrival and departure rates are uniformly bounded, that is,

$$\sup_y \sum_x q(x, y) < \infty, \qquad \sup_y \sum_x q(y, x) < \infty, \tag{2}$$

then it can be shown as in Andjel (1982) that the process is well defined, at least for a class of initial configurations. We call η_t the resulting process in $\mathbb{N}^{\mathbb{N}}$, where for $x \geq 0$, $\eta_t(x)$ is the number of customers in queue x by time t.

We are interested in the case when on average there is a positive net flux of customers out of the system. In order to guarantee this, the underlying continuous time Markov process on S with jump rates matrix q must satisfy some conditions. First we assume that the transition rate matrix $q(x, y)$ determines a unique finite invariant measure m on S satisfying

$$\mu(y)m(y) = \sum_{x \geq -1} m(x)q(x, y), \qquad y \geq -1, \tag{3}$$

$$m(-1) = 1 > m(x), \qquad x \geq 0. \tag{4}$$

In particular, the process with transition rate matrix q must be positive recurrent. We discuss later the null recurrent and the transient cases. We also assume that for each $\lambda > 0$, there exists a sigma-finite measure $\rho = \rho_\lambda$ on S satisfying

$$\mu(y)\rho(y) = \sum_{x \geq -1} \rho(x)q(x, y), \qquad y \geq 0, \tag{5}$$

$$\rho(-1) = 1 > \rho(x) > m(x), \qquad x \geq 0, \tag{6}$$

$$\lambda = \sum_{x \geq 0} (\rho(x)q(x, -1) - q(-1, x)). \tag{7}$$

One can interpret the parameter λ as the "rate of entrance of customers at infinity." In equilibrium this rate must equal the difference between the rate of exiting to -1 and the rate of entrance from -1. This is the meaning of (7).

Let ν_ρ be the product measure on $\mathbb{N}^{\mathbb{N}}$ whose marginals are geometric with parameter $\rho(x)$. In other words, for any finite set $A \subset \mathbb{N}$, and any nonnegative numbers $k(x) \geq 0$, $x \in A$,

$$\nu_\rho(\eta(x) = k(x), x \in A) = \prod_{x \in A} \rho(x)^{k(x)}(1 - \rho(x)). \tag{8}$$

If ρ satisfies the conditions above, then ν_ρ is invariant for the system, as a consequence of Proposition 1 below. This was proven by Jackson (1963) for finite networks, by Andjel (1982) for infinite conservative systems and by Ferrari (1986) for a special case of $q(x, y)$ satisfying our conditions. The condition $\rho(x) < 1$ is necessary for the invariant measure ν_ρ to concentrate on configurations with a finite number of customers at each queue.

Let the departure process D_t be the number of customers leaving the system in $[0, t]$ and let the arrival process A_t be the number of customers entering the system in $[0, t]$. The departure (respectively, arrival) process at time t counts the number of customers jumping to (respectively, from) -1 in the interval $[0, t]$. Let the net output process be $X_t = D_t - A_t$. Burke (1956) showed that for one queue in equilibrium with Poisson arrivals and exponential service times, the departure process D_t is a Poisson process. This result was extended to systems with a finite number of queues; see Kelly (1979) and references therein. The extension to a system with infinitely many queues is demonstrated by the following theorem.

THEOREM 1. *Let $q(x, y)$ be the transition rates of a continuous time Markov process on S satisfying* (2) *to* (7) *and*

$$\sup_y \rho(y) \sum_{x \geq -1} \frac{q(y, x)}{\rho(x)} < \infty. \tag{9}$$

Let η_t be the system of queues with rates $q(x, y)$ whose generator is given by (1) *with initial distribution ν_ρ given by* (8). *Then the departure process D_t is a Poisson process with parameter $\sum_{x \geq 0} \rho(x) q(x, -1)$.*

SKETCH OF THE PROOF. The easiest way to prove Theorem 1 is to follow Reich (1957) [see also Kelly (1979)]. Construct η_t^*, the reverse process of η_t with respect to the invariant measure ν_ρ and verify that $\{D_t\} = \{A_t^*\}$ as processes in distribution, where A_t^* is the number of customers entering the reverse process. By construction, A_t^* is a Poisson process with rate $\sum_{x \geq 0} \rho(x) q(x, -1)$. Condition (9) guarantees the existence of the reverse process η_t^*. □

If no customers enter the system from -1, that is, $q(-1, y) \equiv 0$, then the net output process equals the departure process: $X_t = D_t$ and by Theorem 1, X_t is a Poisson process. When $q(-1, y)$ is not identically zero, customers enter the system producing memory, which prevents X_t from being a Poisson process. In this case, we have the following theorem, which is our main result.

THEOREM 2. *Let $q(x, y)$ be the transition rates of a continuous time Markov process on S satisfying conditions (2) to (7) and*

$$\sum_{x \geq 0} \frac{m(x)}{1 - \rho(x) + m(x)} < \infty, \tag{10}$$

$$\sup_y m(y) \sum_{x \geq -1} \frac{q(y, x)}{m(x)} < \infty, \tag{11}$$

$$\sup_y (\rho(y) - m(y)) \sum_{x \geq 0} \frac{q(y, x)}{\rho(x) - m(x)} < \infty. \tag{12}$$

Let η_t be the system of infinitely many queues on $\mathbb{N}^{\mathbb{N}}$ with rates $q(x, y)$, whose generator is given by (1) *and whose initial distribution is the invariant measure ν_ρ defined in* (8). *Then the net output process X_t can be expressed as the sum*

$$X_t = R_t - B_t + B_0, \tag{13}$$

where R_t is a Poisson process with rate λ given in (7) *and B_t is a stationary process on $\mathbb{N}$ such that the distribution of B_t decays exponentially. In other words, the distribution of B_t does not depend on t and there are positive constants C and β such that*

$$P(B_t > k) \leq Ce^{-\beta k}, \qquad k, t \geq 0. \tag{14}$$

In general, B_t is not a Markov process and it is not independent of R_t. The exact distribution of B_t is given in the proof of the theorem.

We sketch now the proof of Theorem 2, in which we also use the reverse process but in a somewhat more subtle way. The main point is to distinguish between two types of customers: those that enter the system from -1 that we call black customers and the others that we call red customers. We can think that the red customers come from "infinity." The motion of this two species system is based on the assumption that black and red customers behave in the same way in the queues $x \geq 0$. When the server in queue x finishes a service—this happens at rate $\mu(x)$—it chooses uniformly one customer from among those in its queue. Hence the probability of choosing a black customer is the quotient of the number of black customers and the total number of customers in that queue. The chosen customer jumps to queue $y \neq x$ with probability $p(x, y) = q(x, y)/\mu(x)$. The only difference between black and red customers occurs at the boundary: only black customers enter into the system from -1 and they enter queue y at rate $q(-1, y)$. We call (σ_t, ξ_t) the resulting system, where $\sigma_t(x)$ [respectively, $\xi_t(x)$] is the number of black (respectively, red) customers in queue x by time t. By construction $\eta_t = \sigma_t + \xi_t$ coordinatewise. This means that if we disregard colors, we recover the original system η_t. Denote by R_t the departure process of red customers and by B_t the number of black customers in the system at time t. Since no red customers enter the system, we have $X_t = R_t - B_t + B_0$.

The next step is to find the invariant measure ν_2 for (σ_t, ξ_t) and the reverse process $(\sigma_t, \xi_t)^* = (\sigma_t^*, \xi_t^*)$ with respect to ν_2. We simply exhibit them and show the assertion. Call R_t^* the arrival process of red customers in the reverse system. Call B_t^* the number of black customers by time t in the reverse system. The reverse process is still a queueing system with service rates $\mu^*(x) = \mu(x)$ depending only on the queue label. The main difference is that the reversed routing matrices are different for red and black customers. Nevertheless the reverse routing matrices still do not depend on the configuration. The most important property of the reverse process is that R_t^*, the arrival process of red customers is a Poisson process. On the other hand, under the invariant distribution ν_2, B_t^*, the number of black customers at time t, is stationary and its distribution has an exponential tail. To conclude the proof of Theorem 2, we use the same idea as in the proof of Theorem 1. By reversing the process under initial distribution ν_2, $\{B_t\} = \{B_t^*\}$ and $\{R_t\} = \{R_t^*\}$ as processes in distribution.

In this light, we can explain where the conditions of Theorem 2 come from. Condition (3) is necessary in order to have an invariant measure concentrating mass on configurations with a finite number of customers. In turn, this together with (10) implies that B_t, the number of black customers by time t, has an exponential tail uniformly in t. Conditions (5) and (6) guarantee the existence of an invariant measure with infinitely many customers. This implies that the departure process of red customers is not trivial. Conditions (11) and (12) are sufficient to show the existence of the reverse two species process. They are automatically satisfied if $q(x, y)$ is of finite range.

We now discuss the cases when q is not positive recurrent. In the null recurrent case it is not possible to find a *finite* m satisfying (3). Hence the number of black customers will be infinite and we cannot expect an analog of Theorem 2 to hold.

More interesting is the transient case. We assume that there exists a sigma-finite solution ρ of (5) satisfying also

$$\rho(-1) = 1 > \rho(x) > 0, \qquad x \geq 0. \tag{15}$$

This solution is not summable because a finite solution would imply positive recurrence. Under the invariant distribution ν_ρ the net input process $Y_t = A_t - D_t$ has a positive mean. Using the sketched proof of Theorem 2, we prove that the net input process can be expressed as the sum of a Poisson process and an error of order 1. To state the result, let $\alpha(y)$ be the probability that a process on S with transition rate matrix q starting from y eventually hits -1. Since q is transient, $\alpha(y) < 1$, and by ergodicity, $\alpha(y) > 0$. Of course $\alpha(-1) = 1$.

THEOREM 3. *Let q be the transition rate matrix corresponding to a transient Markov process satisfying* (2). *Let ρ be the unique* (*sigma-finite*) *solution of* (5) *and* (15). *Assume* (10),

$$\sum_{y \geq 0} \frac{\rho(y)\alpha(y)}{1 - \rho(y) + \rho(y)\alpha(y)} < \infty, \tag{16}$$

$$\sup_y \alpha(y) \sum_{x \geq 0} \frac{q(y,x)}{\alpha(x)} < \infty, \tag{17}$$

$$\sup_y (1 - \alpha(y)) \sum_{x \geq 0} \frac{q(y,x)}{1 - \alpha(x)} < \infty. \tag{18}$$

Then the net input process Y_t can be expressed as

$$Y_t = R_t + B_t - B_0,$$

where the process R_t is a Poisson process of rate

$$\sum_{y \geq 0} p(-1, y)(1 - \alpha(y))$$

and the process B_t is stationary and has an exponential decay as in (14).

In Section 2 we introduce the two species process, the corresponding invariant measure and the reverse two species process. In Section 3 we show Theorems 2 and 3. In Section 4 we compare the infinite system with finite systems, discuss some extensions of our results and show simple examples of systems satisfying the conditions of Theorems 2 and 3.

2. The two species system. We introduce the process $(\sigma_t, \xi_t) \in \Omega_2 = \Omega \times \Omega$ as follows. Each queue may have two types of customers: black and red. The number of black (respectively, red) customers at queue x at time t is $\sigma_t(x)$ [respectively, $\xi_t(x)$]. Let the system evolve with the provision that only black customers enter it and they do so at queue y with rate $q(-1, y)$. The evolution in the system is the following. Given that a service time finished at queue x, the probability that a black customer has been served is the number of black customers in x over the total number of customers in x. The service corresponds to a red customer with complementary probability. Then the chosen customer jumps to queue y with probability $p(x, y) = q(x, y)/\mu(x)$. Notice that this specification implies that our system is not a classical system of queues with two types of customers. In a classical system the customer is chosen according to some rule (that may be random, as is our rule) at the beginning of the service time interval. In our system, the customer is chosen at the end of this interval. In this way a customer that has arrived after the starting of the service time interval can be the one served by the end of the interval. The generator of this process is given by

$$\mathbf{L}_2 f(\sigma, \xi) = \sum_{x, y \geq -1} q(x, y) \left\{ \frac{\sigma(x)}{\sigma(x) + \xi(x)} [f(\sigma^{xy}, \xi) - f(\sigma, \xi)] + \frac{\xi(x)}{\sigma(x) + \xi(x)} [f(\sigma, \xi^{xy}) - f(\sigma, \xi)] \right\},$$

where (σ, ξ) is the initial state of the system [with $\sigma(-1) = +\infty$ and $\xi(-1) = 0$], σ^{xy} and ξ^{xy} are defined analogously to η^{xy} with the exception

$\xi^{x,-1}(-1) \equiv 0$. f is a local function in Ω_2 and we adopt the convention that $\infty/\infty = 1$ and $0/0 = 0$. Then we have the following lemma, whose proof is straightforward.

LEMMA 1. *Let the process* (σ_t, ξ_t) *have initial configuration* (σ, ξ). *Then the process* $\{\sigma_t + \xi_t\}$ *has the same distribution as the process* $\{\eta_t\}$ *with initial configuration* $\sigma + \xi$. *In other words, if f is a local function defined on* Ω, *then* $\mathbf{L}_2 f(\sigma + \xi) = \mathbf{L} f(\sigma + \xi)$.

Define a measure ν_2 on Ω_2 as follows. Let η be chosen from ν_ρ. Let $\alpha(x) = m(x)/\rho(x)$. For all $x \geq 0$, each η customer in queue x will be called black, with probability $\alpha(x)$, or red, with probability $1 - \alpha(x)$, independently of each other and of other queues. For each x let $\sigma(x)$ and $\xi(x)$ be the number of black and red customers, respectively, in queue x. The distribution of (σ, ξ) so obtained is called ν_2. Under ν_2, $\{\sigma(x), \xi(x)\}_x$ is a collection of independent random vectors with marginal distribution

$$\nu_2(\sigma(x) = k, \xi(x) = j - k) \tag{19}$$
$$= \rho(x)^j (1 - \rho(x)) \binom{j}{k} \alpha(x)^k (1 - \alpha(x))^{j-k}, \qquad j \geq k \geq 0$$

A short computation shows that under ν_2, $\sigma(x)$ is geometric with parameter $(\alpha(x)\rho(x))/(1 - \rho(x) + \alpha(x)\rho(x))$:

$$\nu_2(\sigma(x) \geq k) = \left(\frac{\alpha(x)\rho(x)}{1 - \rho(x) + \alpha(x)\rho(x)} \right)^k, \qquad k \geq 0. \tag{20}$$

From Lemma 1 and the construction of ν_2 we have

$$\int d\nu_2(\sigma, \xi) \mathbf{L}_2 f(\sigma + \xi) = \int d\nu_\rho(\eta) \mathbf{L} f(\eta). \tag{21}$$

Define new Markov transition rates $q^*(x, y)$ on S as

$$q^*(x, y) = \frac{q(y, x)\rho(y)}{\rho(x)} \tag{22}$$

and a new system of queues η_t^* in Ω with transition rates $q^*(x, y)$ and service rates $\mu^*(x) = \sum_y q^*(x, y)$. [Notice that $\mu^*(x) = \mu(x)$, $x \in \mathbb{N}$.] It is easy to check that $\alpha(x)$ is the probability that the process on S with transition rates q^* eventually hits -1, given that the initial state is x.

As in the system η_t, we distinguish between two kinds of customers in this new system, which we call black* and red*. Let

$$q_b^*(x, y) = q^*(x, y)\alpha(y)/\alpha(x), \qquad x \geq -1,$$
$$q_r^*(x, y) = q^*(x, y)(1 - \alpha(y))/(1 - \alpha(x)), \qquad x \geq 0, \tag{23}$$
$$q_r^*(-1, y) = q^*(-1, y)(1 - \alpha(y)).$$

If we consider a Markov process on S with transition rates q^* and condition it to be eventually absorbed at -1 we get a new Markov process with transition rates given by q_b^*. If we condition on nonabsorption, we get rates q_r^*. Notice that $\Sigma_y q_b^*(x, y) = \Sigma_y q_r^*(x, y) = \mu^*(x) = \mu(x)$. We define a process (σ_t^*, ξ_t^*), where $\sigma_t^*(x)$ and $\xi_t^*(x)$, respectively, count the number of black* and red* customers in queue x at time t. This process has the following evolution. At rate $\mu^*(x)$, a customer is selected in queue x uniformly among the $\sigma_t^*(x) + \xi_t^*(x)$ customers. If it is a black* one, then the customer jumps to y with probability $q_b^*(x, y)/\mu^*(x)$ and if it is a red* one, then the same jump occurs with probability $q_r^*(x, y)/\mu^*(x)$. The black* customers enter the system to queue y at rate $q_b^*(-1, y)$ and red* ones do it at rate $q_r^*(-1, y)$. The process (σ_t^*, ξ_t^*) takes values in Ω_2 and has the generator

$$\mathbf{L}_2^* f(\sigma, \xi) = \sum_{x, y \geq -1} \left\{ q_b^*(x, y) \frac{\sigma(x)}{\xi(x) + \sigma(x)} [f(\sigma^{xy}, \xi) - f(\sigma, \xi)] \right.$$
$$\left. + q_r^*(x, y) \frac{\xi(x)}{\sigma(x) + \xi(x)} [f(\sigma, \xi^{xy}) - f(\sigma, \xi)] \right\},$$

where we use the convention

$$\frac{\sigma(-1)}{\sigma(-1) + \xi(-1)} = \frac{\xi(-1)}{\sigma(-1) + \xi(-1)} = 1.$$

Condition (12) insures the existence of this process. Here is the main ingredient in the proof of Theorems 2 and 3.

PROPOSITION 1. *The processes (σ_t, ξ_t) and (σ_t^*, ξ_t^*) are the reverse of one another with respect to ν_2.*

PROOF. We want to show that

$$\int f \mathbf{L}_2 g \, d\nu_2 = \int g \mathbf{L}_2^* f \, d\nu_2 \tag{24}$$

for local functions f and g on Ω_2. The above equality will be established if we prove the equalities

$$\begin{aligned} &q(x, y) \int \frac{\sigma(x)}{\sigma(x) + \xi(x)} f(\sigma^{xy}, \xi) g(\sigma, \xi) \, d\nu_2(\sigma, \xi) \\ &\quad = q_b^*(y, x) \int \frac{\sigma(y)}{\sigma(y) + \xi(y)} f(\sigma, \xi) g(\sigma^{yx}, \xi) \, d\nu_2(\sigma, \xi) \end{aligned} \tag{25}$$

and

$$\begin{aligned} &q(x, y) \int \frac{\xi(x)}{\sigma(x) + \xi(x)} f(\sigma, \xi^{xy}) g(\sigma, \xi) \, d\nu_2(\sigma, \xi) \\ &\quad = q_r^*(y, x) \int \frac{\xi(y)}{\sigma(y) + \xi(y)} f(\sigma, \xi) g(\sigma, \xi^{yx}) \, d\nu_2(\sigma, \xi), \end{aligned} \tag{26}$$

for $x, y \in S$ and

$$\begin{aligned}
&\sum_{x,y} q(x,y) \int 1\{\sigma(x) + \xi(x) > 0\} f(\sigma,\xi) g(\sigma,\xi)\, d\nu_2(\sigma,\xi) \\
(27)\qquad &= \sum_{x,y} q_b^*(x,y) \int \frac{\sigma(x)}{\sigma(x) + \xi(x)} f(\sigma,\xi) g(\sigma,\xi)\, d\nu_2(\sigma,\xi) \\
&\quad + \sum_{x,y} q_r^*(x,y) \int \frac{\xi(x)}{\sigma(x) + \xi(x)} f(\sigma,\xi) g(\sigma,\xi)\, d\nu_2(\sigma,\xi)
\end{aligned}$$

where the sums have index set $\{x, y\colon x \text{ or } y \in [-1, \dots, N]\}$, where N is large enough so that $[0, N]$ contains the supports of f and g. To prove (25) and (26), we can take f, g of the type

$$\prod_{v \in A} 1\{\sigma(v) = k_v\} \prod_{w \in B} 1\{\xi(w) = l_w\},$$

where A and B are arbitrary finite subsets of $\mathbb{N}$ and k_v and l_w are arbitrary nonnegative integers. Since ν_2 is a product measure, it is sufficient to consider the cases

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x,\ \sigma(y) = k_y + 1,\ \xi(x) = l_x,\ \xi(y) = l_y\},$$
$$g(\sigma,\xi) = 1\{\sigma(x) = k_x + 1,\ \sigma(y) = k_y,\ \xi(x) = l_x,\ \xi(y) = l_y\}$$

and

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x,\ \sigma(y) = k_y,\ \xi(x) = l_x,\ \xi(y) = l_y + 1\},$$
$$g(\sigma,\xi) = 1\{\sigma(x) = k_x,\ \sigma(y) = k_y,\ \xi(x) = l_x + 1,\ \xi(y) = l_y\},$$

respectively, for (25) and (26), when $x, y \in \mathbb{N}$. When $y = -1$, take

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x,\ \xi(x) = l_x\},$$
$$g(\sigma,\xi) = 1\{\sigma(x) = k_x + 1,\ \xi(x) = l_x\}$$

and

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x,\ \xi(x) = l_x\},$$
$$g(\sigma,\xi) = 1\{\sigma(x) = k_x,\ \xi(x) = l_x + 1\},$$

respectively, for (25) and (26). When $x = -1$, (26) is trivially satisfied (both sides vanish); for (25), one should take

$$f(\sigma,\xi) = 1\{\sigma(y) = k_y + 1,\ \xi(y) = l_y\},$$
$$g(\sigma,\xi) = 1\{\sigma(y) = k_y,\ \xi(y) = l_y\}.$$

For all these cases, it is a simple calculation to verify (25) and (26). The equality (27) is easily verified for the sums taken in the set $\{0 \le x \le N,$

$y \geq -1\}$ from the equalities

$$\sum_y q(x,y) = \sum_y q_b^*(x,y) = \sum_y q_r^*(x,y), \qquad x \geq 0.$$

So it suffices to show that

$$\int f(\sigma,\xi) g(\sigma,\xi) \sum_y q(-1,y)\, d\nu_2(\sigma,\xi) + \sum_{x>N} \sum_{y \leq N} q(x,y) \times \int 1\{\sigma(x)+\xi(x)>0\} f(\sigma,\xi) g(\sigma,\xi)\, d\nu_2(\sigma,\xi) \tag{28}$$

$$= \int fg \sum_y q_b^*(-1,y)\, d\nu_2 + \int fg \sum_y q_r^*(-1,y)\, d\nu_2$$

$$+ \sum_{x>N} \sum_{y \leq N} q_b^*(x,y) \int \frac{\sigma(x)}{\sigma(x)+\xi(x)} f(\sigma,\xi) g(\sigma,\xi)\, d\nu_2(\sigma,\xi) \tag{29}$$

$$+ \sum_{x>N} \sum_{y \leq N} q_r^*(x,y) \int \frac{\xi(x)}{\sigma(x)+\xi(x)} f(\sigma,\xi) g(\sigma,\xi)\, d\nu_2(\sigma,\xi).$$

The second term in the l.h.s. and the third and fourth terms in the r.h.s. equal, respectively,

$$\int fg \sum_{x>N} \sum_{y \leq N} \rho(x) q(x,y)\, d\nu_2,$$

$$\int fg \sum_{x>N} \sum_{y \leq N} \alpha(x) \rho(x) q_b^*(x,y)\, d\nu_2,$$

$$\int fg \sum_{x>N} \sum_{y \leq N} (1-\alpha(x)) \rho(x) q_r^*(x,y)\, d\nu_2,$$

so (29) can be rewritten (when $\int fg\, d\nu_2 \neq 0$; otherwise we are done) as

$$\sum_y q(-1,y) + \sum_{x>N} \sum_{y \leq N} \rho(x) q(x,y)$$
$$= \sum_y q_b^*(-1,y) + \sum_y q_r^*(-1,y) + \sum_{x>N} \sum_{y \leq N} \rho(y) q(y,x).$$

So, it suffices to prove the equality

$$\sum_{x>N} \sum_{y \leq N} \rho(x) q(x,y) - \sum_{x>N} \sum_{y \leq N} \rho(y) q(y,x) = \sum_y \rho(y) q(y,-1) - \sum_y q(-1,y), \tag{30}$$

which we do by rewriting the l.h.s. as

$$\sum_{x \geq -1} \sum_{y \leq N} \rho(x) q(x,y) - \sum_{x \geq -1} \sum_{y \leq N} \rho(y) q(y,x).$$

This equals the r.h.s. of (30) plus

$$\sum_{x\geq -1}\sum_{0\leq y\leq N}\rho(x)q(x,y) - \sum_{x\geq -1}\sum_{0\leq y\leq N}\rho(y)q(y,x),$$

and the equality of these two terms follows from (5). □

3. Proofs of Theorems.

PROOF OF THEOREM 2. From Proposition 1 and (21) it follows that ν_ρ is invariant for the process η_t. From Lemma 1, we have $\eta_t = \sigma_t + \xi_t$, and

$$X_t = R_t - B_t + B_0, \tag{31}$$

where B_t is the number of black customers in the system at time t and R_t is the number of red customers leaving the system in the interval $[0, t]$ for the process (σ_t, ξ_t) under initial distribution ν_2. Proposition 1 implies

$$\{(R_t, B_t)\} = \{(R_t^*, B_t^*)\} \quad \text{as processes in distribution,} \tag{32}$$

where B_t^* is the number of black* customers at time t and R_t^* is the number of red* customers entering the system in $[0, t]$ for the reverse process. By definition, R_t^* is a Poisson process of rate

$$\sum_{y\geq 0} p_r^*(-1,y) = \sum_{x\geq 0}[\rho(x) - m(x)]q(x,-1). \tag{33}$$

However, this is just λ given in (7) because

$$\sum_{x\geq 0} m(x)q(x,-1) = \sum_{x\geq 0} q(-1,x)$$

is the balance equation that m must satisfy for state -1. On the other hand, under initial distribution ν_2, B_t^*, B_t, $|\sigma_0^*|$ and $|\sigma_t|$ all have the same distribution, which is also independent of t. This is also the same as the distribution of

$$B_0 = \sum_{x\geq 0}\sigma(x), \tag{34}$$

where $\{\sigma(x)\}_x$ is a collection of independent random variables with geometric distribution given by (20). Hence

$$\sum_{x\geq 0}\nu_2(\sigma(x)\geq 1) = \sum_{x\geq 0}\frac{\alpha(x)\rho(x)}{1-\rho(x)+\alpha(x)\rho(x)} < \infty \tag{35}$$

by (10). This suffices to get the exponential decay required in (14). □

PROOF OF THEOREM 3. Let

$$q_b(x,y) = q(x,y)\alpha(y)/\alpha(x), \qquad x\geq -1,$$
$$q_r(x,y) = q(x,y)(1-\alpha(y))/(1-\alpha(x)), \qquad x\geq 0,$$
$$q_r(-1,y) = q(-1,y)(1-\alpha(y))$$

and consider the process (σ_t, ξ_t) in Ω_2 generated by

$$\mathbf{L}_2 f(\sigma, \xi) = \sum_{x, y \geq -1} \left\{ q_b(x, y) \frac{\sigma(x)}{\sigma(x) + \xi(x)} [f(\sigma^{xy}, \xi) - f(\sigma, \xi)] \right.$$
$$\left. + q_r(x, y) \frac{\xi(x)}{\sigma(x) + \xi(x)} [f(\sigma, \xi^{xy}) - f(\sigma, \xi)] \right\},$$

with the usual conventions. This exists by (17) and (18). The difficulty here is that the process $(\sigma_t + \xi_t)$ is not Markovian, so we have to use the reverse process. Define new rates $q^*(x, y)$ as in (22) and consider the processes η_t^* in Ω and (σ_t^*, ξ_t^*) in Ω_2, respectively, generated by

$$\mathbf{L}^* f(\eta) = \sum_{x, y \geq -1} 1\{\eta(x) > 0\} q^*(x, y)[f(\eta^{xy}) - f(\eta)]$$

and

$$\mathbf{L}_2^* f(\sigma, \xi) = \sum_{x, y \geq -1} q^*(x, y) \left\{ \frac{\sigma(x)}{\sigma(x) + \xi(x)} [f(\sigma^{xy}, \xi) - f(\sigma, \xi)] \right.$$
$$\left. + \frac{\xi(x)}{\sigma(x) + \xi(x)} [f(\sigma, \xi^{xy}) - f(\sigma, \xi)] \right\},$$

which exist by condition (9) and Lemma 1.

Changing labels in the proof of Proposition 1, we get that (σ_t, ξ_t) and (σ_t^*, ξ_t^*) are the reverse of one another with respect to ν_2. An argument in the same vein shows that η_t and η_t^* are the reverse of one another with respect to ν_ρ. Then, calling Y_t^* the net output process of η_t^*, starting with ν_ρ,

(36) $$\{Y_t\} = \{Y_t^*\} \quad \text{as processes in distribution,}$$

and calling R_t^* the output process of ξ_t^* and $B_t^* = |\sigma_t^*|$, starting with ν_2,

(37) $$\{(R_t^*, B_t^*)\} = \{(R_t, B_t)\} \quad \text{as processes in distribution,}$$

where R_t is the input process of ξ_t and $B_t = |\sigma_t|$.

Since the generator of the reverse process is the same as the generator of the direct process of Lemma 1, this lemma implies $\eta_t^* = \sigma_t^* + \xi_t^*$. Then Y_t^* can be written as

(38) $$Y_t^* = R_t^* - B_t^* + B_0^*.$$

From (36), (37) and (38) it is clear that starting η_t with ν_ρ and (σ_t, ξ_t) with ν_2,

$$\{Y_t\} = \{R_t - B_t + B_0\} \quad \text{as processes in distribution.}$$

However, R_t is indeed just the input process, since ξ particles do not leave the system, so that it is a Poisson process (with rate given in the statement of Theorem 3). Finally, the distribution of B_t does not depend on t and has the required exponential tail as shown at the end of the proof of Theorem 2. Here enters (16). □

4. Final remarks. The method of proof of Theorem 1 allows one to show that the departure process from queue y is a Poisson process of parameter $\rho(y)q(y, -1)$. Furthermore, the departure processes from different queues are independent [Kelly (1979)]. One is tempted to extend the result to the net flux out of queue y, but we think that this does not necessarily hold. It is true that the departure processes of red customers from different queues are independent Poisson processes with rate $[\rho(y) - m(y)]q(y, -1)$, but it is hard to control the net output process of black customers because they can enter into other queues different from y and the compensation that allows one to prove Theorem 2 does not occur.

Another possible extension would be to consider a zero range process with service rates depending on the number of customers in the queue. For this process there are product invariant measures as shown by Andjel (1982). However, the reverse process with respect to any of these measures is not a zero range process, because the rate of jump of the reverse process depends also on the number of customers of the destination queue. This implies in particular that not even Theorem 1 is true because the arrival process for the reverse system is not Poisson.

When there is a unique source/sink of customers at -1, the main difference between a system with a finite number of queues and a system with infinitely many queues is the following. For ergodic $q(x, y)$, the finite system admits at most one invariant measure. In contrast, the infinite system may admit infinitely many invariant measures. These measures are indexed by the parameter λ that represents the "rate of entrance of customers at infinity." The relation between λ and ρ is given in (7). Notice also that $m = \rho_0$, the measure obtained when the entrance rate at infinity is null. Hence $m(x)$ can be interpreted as the stationary probability that queue x is occupied when no customers enter from infinity ($\lambda = 0$) and $\rho(x)$ is the stationary probability that queue x is occupied when the rate of entrance at infinity is λ. Finite systems with N queues and a unique source/sink of customers at -1 correspond roughly to the infinite case $\lambda = 0$. For these systems the analog of Theorem 2 is trivial. The only hypothesis needed is the existence of a finite invariant measure m_N satisfying (3). To show the theorem in this case, observe that the net output at time t is the same as the number of customers in the system at time 0 minus the number of customers at time t:

$$D_t - A_t = |\eta_0| - |\eta_t|,$$

where $|\eta_t|$ is the total number of customers in the system by time t. This is of order 1 because in equilibrium $|\eta_t|$ has the same distributions as $|\eta_0|$, which in turn is a finite sum of independent geometric random variables with finite means:

$$\mathbf{E}|\eta_0| = \sum_{x=1}^{N} \frac{m_N(x)}{1 - m_N(x)}.$$

In this case, R_t is the trivial Poisson process of rate 0 and $B_t = |\eta_t|$.

Another possibility is to consider a finite system in $\{0,\dots,N-1\}$ with two boundaries: one at -1 and the other at N. To make a parallel with the infinite system, we do not allow customers to exit to N and we put a source at N. The entrance rates are determined by an infinite transition matrix q satisfying the conditions of Theorem 2 and by the invariant measure for the infinite system ρ_λ as follows. For $x, y \in \{-1, 0, \dots, N\}$ define

$$q_N(x,y) = \begin{cases} q(x,y), & \text{if } x, y \le N-1, \\ \sum_{z \ge N} \rho(z) q(z,y), & \text{if } x = N, 0 \le y \le N-1, \\ 0, & \text{if } y = N. \end{cases}$$

Let ρ_N be the unique invariant measure of the matrix q_N and let ν_N be the corresponding invariant measure for the process $\eta_{N,t}$ constructed with the rates q_N. Using the technique of proof of Theorem 2, one can prove that the net output process $X_{N,t}$ of this system can be written as

$$X_{N,t} = R_{N,t} - B_{N,t} + B_{N,0}, \tag{39}$$

where $R_{N,t}$ is a Poisson process with rate

$$\lambda_N = \sum_{0 \le y \le N} q_N(N+1, y) = \sum_{0 \le x \le N} \rho_N(x) q_N(x, -1)$$

and $B_{N,t}$ is a stationary process on $\mathbb{N}$ whose distribution decays exponentially uniformly in N. Furthermore, one can show that, as $N \to \infty$, λ_N converges to λ. Appropriately modifying the entrance rates, one could get finite approximations $\eta_{N,t}$ of the process η_t satisfying $\lambda_N \equiv \lambda$ and still get the uniform (in N) exponential bound on the error $B_{N,t}$. We leave this to the reader.

One particular case of a (positive recurrent) transition function satisfying the conditions of Theorem 2 is the nearest neighbor asymmetric random walk

$$q(x,y) = \begin{cases} a, & \text{if } x \ge -1, y = x+1, \\ b, & \text{if } x \ge 0, y = x-1, \\ 0, & \text{otherwise,} \end{cases}$$

where $b > a \ge 0$. In this case

$$m(x) = (a/b)^{x+1},$$

and since $q(x,y)$ satisfies

$$\sum_{y:\, y \ne x} q(x,y) = \sum_{x:\, x \ne y} q(x,y) = a + b,$$

we can take

$$\rho(x) = m(x) + (1 - m(x))\rho,$$

where ρ is any number in the interval $(0,1)$ [Ferrari (1986)]. Here $\lambda = (b-a)\rho$. When $a = 0$, Theorem 1 applies and X_t is a Poisson process. For nearest neighbor jumps, the system of queues is isomorphic to the simple exclusion process as seen from the leftmost particle. This is mentioned for this model by Spitzer (1970), quoting an oral communication of Kesten, who noticed that Theorem 1 holds for this last process. Liggett [(1985), Theorem 4.7] proves Theorem 1 by a direct computation for the exclusion process. Kipnis (1986) makes the relationship between Burke's result and Kesten's observation.

It is possible to prove Theorem 2 for the net flux of customers between two consecutive queues in a doubly infinite system. To be more precise, let ζ_t be the process on $\mathbb{Z}^{\mathbb{N}}$ with transition rates a and b, respectively, for customer jumps to the right (respectively, left) nearest neighbor. Let ν_ρ be the product measure of geometrics of parameter ρ. Let Y_t be the net flux of customers between queues 0 and -1. Then Y_t is a Poisson process of parameter $(b-a)\rho$ plus a perturbation of order 1 as in Theorem 2. The case $a = 0$ is immediate. The proof for the case $a > b > 0$ is based on a coupling between the semiinfinite process η_t and the doubly infinite process ζ_t. This proves a conjecture of Arratia (1983) and will appear in Ferrari and Fontes (1994).

A simple case of null recurrent $q(x, y)$ is

$$q(x,y) = \begin{cases} a, & \text{if } x = -1,\ y = 0, \\ b, & \text{if } x \geq 0,\ y = x \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $a \geq b$, there are no solutions ρ satisfying (5) and (6). If $a < b$, there is a unique sigma-finite solution of (5) and (6), but there is no finite solution of (3) and (4). Hence our method to show Theorem 2 does not work.

A (transient) case satisfying Theorem 3 is

$$q(x,y) = \begin{cases} a, & \text{if } x = -1,\ y = 0, \\ b, & \text{if } x \geq 0,\ y = x + 1, \\ c, & \text{if } x \geq 0,\ y = x - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Taking $c < a < b$, the unique solution of (5) and (6) is $\rho(x) \equiv a/b$. In this case,

$$\alpha(x) = \frac{c}{c+b}\left(\frac{c}{b}\right)^x \quad \text{and} \quad \lambda = \frac{ca}{b} - a.$$

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INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE DE SÃO PAULO
CX. POSTAL 20570
01452-990 SÃO PAULO SP
BRASIL