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PARKING ARCS ON THE CIRCLE WITH APPLICATIONS TO ONE-DIMENSIONAL COMMUNICATION NETWORKS

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Let $(r_1, s_1), \dots, (r_n, s_n)$ be a sequence of requests to place arcs on the unit circle, where $0 \le r_i, s_i \le 1$ are endpoints relative to some origin on the circle. The first request is always satisfied by reserving, or parking, the shorter of the two arcs between r_1 and s_1 (either arc can be parked in case of ties). Thereafter, one of the two arcs between r_i and s_i is parked if and only if it does not overlap any arc already parked by the first $i-1$ requests. Assuming that the r_i, s_i are $2n$ independent uniform random draws from $[0, 1]$, what is the expected number $E(N_n)$ of parked arcs as a function of n? By an asymptotic analysis of a relatively complicated exact formula, we prove the estimate for large n:

$$E[N_n] = cn^\alpha + o(1), \qquad n \to \infty,$$

where $\alpha = (\sqrt{17} - 3)/4 = 0.28078\dots$ and where the evaluation of an exact formula gives $c = 0.98487\dots$. We also derive a limit law for the distribution of gap lengths between parked arcs as $n \to \infty$. The problem arises in a model of one-dimensional loss networks: The circle is a continuous approximation of a ring network and arcs are paths between communicating stations. The application suggests open problems, which are also discussed.

1. Introduction. Recent research in one-dimensional loss networks has rekindled an interest in new versions of classical *parking problems*; Mannion (1976) reviews much of the early literature on these problems. In a broad survey on loss networks, Kelly (1991) discusses stochastic generalizations of the following simple model of circuit-switching ring communication networks. A large number of identical stations are equally spaced along a closed communication path, or ring. A request by station r for communication with station s is denoted by the pair (r, s), and is *satisfied* when one of the two paths (arcs) connecting the two stations is allocated for their exclusive use. For simplicity, we adopt a standard continuous approximation in which the ring has unit length and requests (r, s) are pairs of real-valued coordinates, $0 \le r, x \le 1$, relative to some fixed origin on the ring. We assume that r and s are independent samples from the uniform distribution on $[0, 1]$.

In the problem studied here, the ring is initially idle and a list $(r_1, s_1), \dots, (r_n, s_n), \dots$ of independent requests is to be scanned in the order given. The shorter of the two arcs between r_1 and s_1 is always allocated to the first request. Thereafter, for each $n \ge 2$, the nth request is satisfied if and only if

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one of the two arcs between r_n and s_n does not overlap any arc already allocated to the first $n - 1$ requests. Requests not satisfied are simply lost. This paper studies the distribution of the number N_n of satisfied requests and the empirical size distribution of the gaps between satisfied requests, just after the nth request has been inspected.

In analogy with classical parking problems, the operation of satisfying a request will be referred to as *parking an arc* in some gap. Note that, after parking the first arc, our problem on the circle reduces to parking intervals in some gap $[0, x]$. The discussion below refers to the problem on an interval and continues with the parking terminology.

Suppose we modify the problem on $[0, x]$ so that all intervals have the same length and the request sequence consists of single numbers $q_1, q_2, \dots$ drawn independently and uniformly at random from $[0, x]$. The ith interval is parked with its midpoint at q_i if and only if, in that position, the interval lies entirely within $[0, x]$ and overlaps none of the intervals already parked. We then have a version of Renyi's (1958) *car parking problem*, where cars are the intervals being parked. Note that, in contrast to our parking process, Renyi's process terminates almost surely; eventually all gaps between parked intervals will have lengths less than the given interval length. Mannion (1976, 1979) studies much more general parking problems in which general distributions govern interval sizes and parking positions. However, the generalized process retains the property that eventually all gaps will become indivisible in a terminal state. The analysis characterizes the terminal state as a function of the original gap size x. Note that our analysis is quite different in that, for the nonterminating process, we characterize the state as a function of the number n of parking attempts. Results for classical parking problems are typically asymptotic in x, whereas our main results will be asymptotic in n for given x.

Past research equally closely related to ours is that of Justicz, Scheinerman and Winkler (1990), who consider the problem instance on $[0, 1]$ with the same generator of random intervals. They pose the optimization question: Given all intervals in advance, select a maximum cardinality subset that can be parked in $[0, 1]$. In an analysis quite different from ours, they show, among other interesting results, that the expected number selected is asymptotically $(2/\sqrt{\pi})n^{1/2}$. This is to be compared with the asymptotically smaller $c_0 n^{(\sqrt{17}-3)/4}$ result in Section 3 (see Corollary 1) for the on-line, sequential selection version of the problem studied here.

Interval splitting problems or, to use the more colorful phrase, *stick breaking problems* are also related to our parking problem. Brennan and Durrett (1986, 1987) give a unified treatment of the work on these problems and provide many references. To establish the connection, we construct a continuous-time version of our problem on $[0, x]$ which is interesting in its own right. Let the requests (r_n, s_n), $n \geq 1$, arrive by a rate-1 Poisson process and consider a gap of size y in the parking process at some time t. Since the (r_n, s_n) are i.i.d. uniforms on $[0, 1]$, the probability that the next request parks an interval in the gap is y^2. Then the number of requests that have to be inspected before one parks an interval in the gap is geometric with parameter y^2, so the waiting time for an interval to be parked in the gap is exponential with rate y^2. Thus, we can formulate our problem on a gap of size x as follows. After a rate-x^2 exponential time, the gap splits into two gaps of sizes x_1, x_2. Then, after random, independent rate-x_1^2 and rate-x_2^2 exponential times, respectively, each of these gaps further subdivides into two more gaps. The process continues: Whenever a gap of size y is created, it divides into two gaps after a random rate-y^2 exponential time that is independent of all other splitting times.

So far, the description is merely that of Brennan and Durrett's (1986) stick breaking process in continuous time; in fact, it is the special case of their rate parameter x^α when $\alpha = 2$. The fundamental difference between stick breaking and our problem lies in the splitting mechanism. In stick breaking, a stick (gap) of length y breaks into two pieces of sizes Zy and $(1 - Z)y$, where Z is a random variable with a given distribution on $[0, 1]$. In our problem, a gap of size y breaks into two gaps whose sum is less than y w.p.1. Of course, it is just this fact that converts a stick breaking problem into a parking problem; a gap effectively breaks into three pieces with one reserved thereafter as a parked object. In our specific case, a gap of size y breaks into three parts equal in distribution to the spacings induced by two i.i.d. uniforms on $[0, y]$. If $U_{(1)}, U_{(2)}$ denote the order statistics of two i.i.d. uniforms on $[0, 1]$, then the two new gaps are the first and third spacings $U_{(1)}y$ and $(1 - U_{(2)})y$, while the middle spacing $(U_{(2)} - U_{(1)})y$ is the parked interval.

Section 2 keeps with the discrete-time model and develops the formulas on which the asymptotic analyses of N_n and the empirical gap-size distribution in Sections 3 and 4 are based. The final remarks in Section 5 point to a number of open problems. The remainder of this section comments briefly on cognate problems.

Parking problems are fragmentation problems, as are dynamic packing problems, in which the intervals to be packed are commonly called items. In the latter problem, items are always placed against the item (or the origin) that follows the left boundary of the gap in which the item is packed. Fragmentation is created in packing problems by the random arrival and departure of items. Coffman, Kadota and Shepp (1985) and Aldous (1986) analyzed the problem with unit item sizes; several other variants in dynamic storage allocation are surveyed by Coffman and Mitrani (1988).

The circle covering problems studied by Flatto and Konheim (1962) and Shepp (1972), among others, are duals of the circle fragmentation problems. The text by Kahane [(1985), Chapter 11] gives a modern treatment of the theory. More generally, the parking problems studied here fall in a larger class of sequential, or on-line selection problems. A single list of items is scanned in a given order fixed in advance. As each item is inspected it is selected or rejected once and for all. Commonly, the items are positive reals and the selection rule attempts to optimize some property of the selected subset. For example, the objective could be to select a maximum length monotone sequence or a maximum cardinality subset subject to a sum constraint. An analysis of these problems and references to many others can be found in Samuels and Steele (1981), Mallows, Nair, Shepp and Vardi (1985), Coffman, Flatto and Weber (1987) and Justicz, Scheinerman and Winkler (1990).

2. Exact formulas. To calculate the expected number EN_n of arcs parked, or equivalently the expected number of gaps between parked arcs, we first reduce the circle problem to the problem on an interval. On the circle, the first arc is always parked and its length is distributed uniformly on $[0, 1/2]$, since we take the shorter of the two arcs with the specified endpoints. Hence the gap remaining is uniform on $[1/2, 1]$ and we have, considering EN_n now as the expected number of gaps,

$$EN_n = 2\int_{1/2}^{1} E_{n-1}(x)\,dx, \qquad n \ge 1, \tag{2.1}$$

where $E_n(x) = E(N_n(x))$ and $N_n(x) = N_n[0, x]$ is the number of gaps in $[0, x]$ after successively attempting to park n random intervals with endpoints chosen independently and uniformly at random from $[0, 1]$. More generally, for integers $k \ge 0$, let $E_n^{(k)}(x)$ denote the expected value of the sum of the kth powers of the gap lengths after attempting to park n intervals in $[0, x]$. Then $E_n(x) = E_n^{(0)}(x)$.

Extending the approach of Renyi (1958), a recurrence for $E_n^{(k)}(x)$ develops as follows. The first request (r, s) fails to fit in $[0, x]$ with probability $1 - x^2$, in which case the problem reduces to trying the remaining $n - 1$ requests in the same interval $[0, x]$. If (r, s) fits, that is, if $0 \le r, s \le x$, then two disjoint gaps of lengths r and $x - s$ (if $r < s$) or s and $x - r$ (if $s < r$) are formed, and the problem reduces to trying the remaining $n - 1$ requests in these gaps. Taking expected values and using the symmetry in r and s leads to

$$\begin{aligned} E_n^{(k)}(x) &= (1 - x^2)E_{n-1}^{(k)}(x) \\ &\quad + 2\int_{r=0}^{x}\int_{s=r}^{x} dr\,ds\left[E_{n-1}^{(k)}(r) + E_{n-1}^{(k)}(x - s)\right] \end{aligned} \tag{2.2}$$

for $n \ge 1$. Now integrate, interchanging the order of integration for the second term of the integrand, to obtain for $n \ge 1$,

$$E_n^{(k)}(x) = (1 - x^2)E_{n-1}^{(k)}(x) + 4\int_0^x (x - r)E_{n-1}^{(k)}(r)\,dr. \tag{2.3}$$

Together with $E_0^{(k)}(x) = x^k$, this specifies $E_n^{(k)}(x)$ completely.

By induction on n, it is easily verified from (2.3) that $E_n^{(k)}(x)$ is a polynomial of the form

$$E_n^{(k)}(x) = \sum_{j=0}^{n} u_{n,j}^{(k)} x^{k+2j}, \tag{2.4}$$

where $u_{n,0}^{(k)} = 1$, $n \ge 0$, $u_{0,j}^{(k)} = 0$ for $j \ne 0$, and where the coefficient of x^{k+2j} in (2.3) yields

$$\begin{aligned} u_{n,j}^{(k)} &= u_{n-1,j}^{(k)} - u_{n-1,j-1}^{(k)}\left[1 - \frac{4}{k+2j-1} + \frac{4}{k+2j}\right] \\ &= u_{n-1,j}^{(k)} - h_{k+2j} u_{n-1,j-1}^{(k)}, \qquad j \ge 1, \end{aligned} \tag{2.5}$$

where

$$h_l = 1 - \frac{4}{l(l-1)}.$$

Let

$$\pi_0^{(k)} = 1, \qquad \pi_j^{(k)} = \prod_{l=1}^{j} h_{k+2l}, \qquad j \ge 1,$$

and

$$v_{n,j}^{(k)} = (-1)^j u_{n,j}^{(k)} / \pi_j^{(k)}.$$

Then (2.5) becomes

$$v_{n,j}^{(k)} = v_{n-1,j}^{(k)} + v_{n-1,j-1}^{(k)}, \qquad 1 \le j \le n, \tag{2.6}$$

with $v_{n,0}^{(k)} = 1$, $n \ge 0$, $v_{0,j}^{(k)} = 0$ for $j \ne 0$, so for $0 \le j \le n$,

$$v_{n,j}^{(k)} = \binom{n}{j}, \tag{2.7}$$

$$u_{n,j}^{(k)} = (-1)^j \binom{n}{j} \pi_j^{(k)} \tag{2.8}$$

and, hence,

$$E_n^{(k)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \pi_j^{(k)} x^{k+2j}, \qquad n \ge 0, \tag{2.9}$$

which is the desired result. The next section derives more easily interpreted estimates from this formula.

A similar analysis will give a formula for A_n, the expectation of the square of the number of gaps after n tries. We have, as in (2.1),

$$A_n = E(N_n^2) = 2\int_{1/2}^{1} A_{n-1}(x)\, dx, \tag{2.10}$$

where

$$A_n(x) = E\big((N_n(x))^2\big).$$

As above, we find a recurrence

$$\begin{aligned} A_n(x) &= (1-x^2) A_{n-1}(x) \\ &\quad + 2\int_0^x dr \int_r^x ds \{A_{n-1}(r) + 2B_{n-1}(r, x-s) + A_{n-1}(x-s)\} \\ &= (1-x^2) A_{n-1}(x) \\ &\quad + 4\int_0^x dr (x-r) A_{n-1}(r) + 4\int_0^x dr \int_0^{x-r} dt\, B_{n-1}(r,t), \end{aligned} \tag{2.11}$$

where $B_n(x, y) = E(N_n[0, x]N_n[1-y, 1])$, $x + y < 1$. Further, we have

$$B_n(x,y) = (1-x^2-y^2)B_{n-1}(x,y)$$
$$+ 2\int_0^x dr\int_r^x ds\, E(N_n[0,r] + N_n[s,x])N_n[1-y,1])$$
$$+ 2\int_0^y dr\int_r^y dx\, E(N_n[1-x,1](N_n(0,r) + N_n(y-r,y))$$
$$= (1-x^2-y^2)B_{n-1}(x,y) + 4\int_0^x (x-r)B_{n-1}(r,y)\,dr$$
$$+ 4\int_0^y (y-s)B_{n-1}(x,s)\,ds.$$

By induction, we find

$$B_n(x,y) = \sum_{i+j\le n} b_{ij}^{(n)}x^{2i}y^{2j}, \qquad A_n(x) = \sum_{i=0}^{n} a_i^{(n)}x^{2i}, \tag{2.12}$$

where

$$b_{ij}^{(n)} = b_{ij}^{(n-1)} - h_{2i}b_{i-1,j}^{(n-1)} - h_{2j}b_{i,j-1}^{(n-1)},$$

whence

$$b_{ij}^{(n)} = (-1)^{i+j}\frac{n!}{i!j!(n-i-j)!}\pi_i\pi_j,$$

where $\pi_i = \pi_i^{(0)}$. Now (2.11) gives

$$a_i^{(n)} = a_i^{(n-1)} - h_{2i}a_{i-1}^{(n-1)} + b_i^{(n-1)},$$

where

$$b_i^{(n)} = 4\sum_{j+k=i-1}(-1)^{j+k}b_{jk}^{(n)}\frac{(2j)!(2k)!}{(2i)!}, \qquad 0 \le i \le n.$$

Hence

$$a_i^{(n)} = (-1)^i\pi_i\left(\binom{n}{i} + \sum_{m=1}^{i}(-1)^m\sum_{j=m-1}^{n-i+m-1}\binom{n-j-1}{i-m}\frac{b_m^{(j)}}{\pi_m}\right), \qquad 0 \le i \le n.$$

Substituting, we find

$$a_i^{(n)} = (-1)^i\pi_i\binom{n}{i}(1-\sigma_i), \tag{2.13}$$

where

$$\sigma_i = 4\sum_{j+k\le i-1}\frac{\pi_j\pi_k}{\pi_{j+k+1}}\frac{(2j)!(2k)!}{(2j+2k+2)!}\binom{j+k}{j}. \tag{2.14}$$

Assembling results, (2.10), (2.12) and (2.13) give

$$A_n = 2\sum_{i=0}^{n}\frac{(-1)^i}{2i+1}[1-2^{-(2i+1)}]\pi_i\binom{n}{i}(1-\sigma_i).$$

As yet, we have been unable to find asymptotics for $A_n(x)$ and A_n.

3. Asymptotics. In an application of Cauchy's integral formula, we obtain asymptotics for $E_n^{(k)}(x)$, $n \to \infty$, by interpreting the terms on the right-hand side of (2.9) as the residues of a meromorphic function, an approach attributed to S. O. Rice [see Knuth (1973), page 138]. For further discussion of the method and its applications, see Knuth (1973), Flajolet and Sedgewick (1986), Szpankowski (1988) and Kirschenhofer, Prodinger and Szpankowski (1989).

First note that

$$1 - \frac{4}{(2m-1)2m} = \frac{m^2 - (1/2)m - 1}{m^2 - (1/2)m} = \frac{(m-\alpha-1)(m-\beta-1)}{m(m-(1/2))}, \tag{3.1}$$

where $\alpha = (\sqrt{17} - 3)/4 = 0.28\ldots$, $\beta = -\alpha - 3/2 = -1.78\ldots$. Thus (2.9) is a (terminating) generalized hypergeometric series [see Erdélyi (1953)],

$$\begin{aligned} E_n^{(k)}(x) &= x^k {}_3F_2\left(-n, \frac{k}{2} - \alpha, \frac{k}{2} - \beta; \frac{k}{2} + \frac{1}{2}, \frac{k}{2} + 1; x^2\right) \\ &= \sum_{j=0}^{n} (-1)^n \binom{n}{j} \frac{\phi(k/2 + j)}{\phi(k/2)} x^{k+2j}, \end{aligned} \tag{3.2}$$

where $\phi(z)$ is the meromorphic function

$$\phi(z) = \frac{\Gamma(z-\alpha)\Gamma(z-\beta)}{\Gamma(z+1)\Gamma(z+(1/2))}. \tag{3.3}$$

Furthermore, it is easily seen that $(-1)^j\binom{n}{j}$ is the residue of the function

$$\psi_n(z) = \frac{(-1)^n n!}{z(z-1)\cdots(z-n)} \tag{3.4}$$

at the simple pole $z = j$, so for $x > 0$,

$$E_n^{(k)}(x) = \sum_{j=0}^{n} \operatorname{Res}_{z=j}\left(\psi_n(z) \frac{\phi(z + k/2)}{\phi(k/2)} x^{k+2z}\right). \tag{3.5}$$

[Note that $\phi(z + k/2)$ is holomorphic at $z = 0, 1, \ldots, n$.] Using this we will prove the following theorem.

THEOREM 1. *For fixed $k \geq 0$ and $\delta \in (0,1)$,*

$$E_n^{(k)}(x) = c_k n^{\alpha - k/2} x^{2\alpha} + O(n^{-1/2-k/2}) \tag{3.6}$$

uniformly for $x \in [\delta, 1]$ as $n \to \infty$, where

$$c_k = \frac{\Gamma(2\alpha + 3/2)\Gamma(k/2 + 1/2)\Gamma(k/2 + 1)}{\Gamma(\alpha + 1/2)\Gamma(\alpha + 1)\Gamma(k/2 + \alpha + 3/2)}. \tag{3.7}$$

PROOF. Consider the contour γ shown in Figure 1, consisting of an arc of a circle centered at 0 (slightly more than a semicircle) and a vertical segment

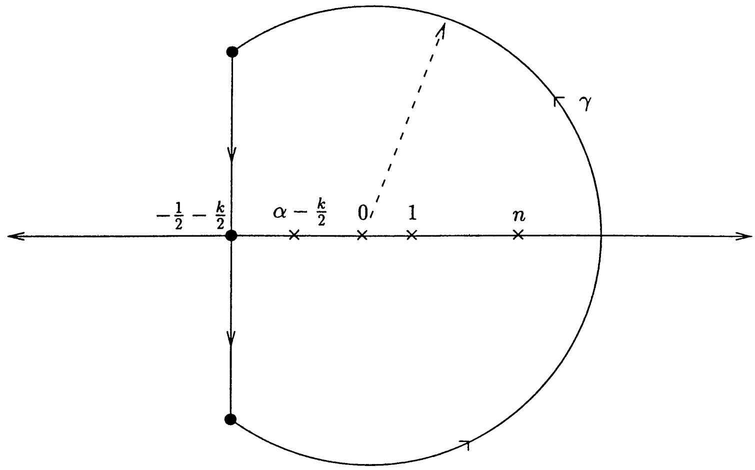

FIG. 1. *The contour* γ.

along the line $\operatorname{Re} z = -1/2 - k/2$. The radius of the circle is $R \gg n, k$. By Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_\gamma \psi_n(z)\frac{\phi(z+k/2)}{\phi(k/2)}x^{k+2z}\,dz = \sum_{j=0}^{n}\operatorname{Res}_{z=j}\left[\psi_n(z)\frac{\phi(z+k/2)}{\phi(k/2)}x^{k+2z}\right] + \operatorname{Res}_{z=\alpha-k/2}\left[\psi_n(z)\frac{\phi(z+k/2)}{\phi(k/2)}x^{k+2z}\right], \tag{3.8}$$

since the poles of $\phi(z+k/2)$ are $\alpha - k/2, \alpha - 1 - k/2, \dots, \beta - k/2, \beta - 1 - k/2, \dots,$ all of which are outside γ, except for $\alpha - k/2$.

Since $(-\alpha) + (-\beta) = 1 + 1/2$, it follows as in Erdélyi [(1953), page 7] that $\phi(z)$ is bounded everywhere in the complex plane after deleting small balls of radius ε around the poles. Hence $\phi(z + k/2)$ is bounded on γ and the bound is independent of R. The same is true of x^{k+2z}, since $x \in (0,1]$. But $\psi_n(z) \sim z^{-(n+1)}$ as $z \to \infty$, so the circular part of the integral tends to 0 as $R \to \infty$. So if we take the limit as $R \to \infty$ of (3.8) and substitute (3.5) we get

$$\frac{1}{2\pi i}\int_{-1/2-k/2+i\infty}^{-1/2-k/2-i\infty}\psi_n(z)\frac{\phi(z+k/2)}{\phi(k/2)}x^{k+2z}\,dz = E_n^{(k)}(x) + \operatorname{Res}_{z=\alpha-k/2}\left[\psi_n(z)\frac{\phi(z+k/2)}{\phi(k/2)}x^{k+2z}\right]. \tag{3.9}$$

We show next that the left-hand side is $O(n^{-1/2-k/2})$. As remarked before, $(\phi(z+k/2)/\phi(k/2))x^{k+2z}$ is bounded on the contour, so it will suffice to bound

$$\int_{-\infty}^{\infty} |\psi_n(-1/2-k/2+iy)|\,dy, \tag{3.10}$$

as $n \to \infty$. From the product defining $\psi_n(z)$ we see that

$$|\psi_n(x+iy)| \le |\psi_n(x)| \qquad \forall\, x, y \in \mathbb{R}, \quad x<0. \tag{3.11}$$

Furthermore,

$$\frac{|\psi_n(x+iy)|}{|\psi_n(x)|} = \prod_{j=0}^{n} \left|\frac{x-j}{x+iy-j}\right| \le \left|\frac{x}{x+iy}\right| \left|\frac{x-1}{x+iy-1}\right| \le \frac{c(x)}{y^2} \tag{3.12}$$

for some constant $c(x) > 0$ independent of n. From (3.11) and (3.12) we deduce that

$$\int_{-\infty}^{\infty} |\psi_n(-1/2-k/2+iy)|\,dy \le c'(k)|\psi_n(-1/2-k/2)| \tag{3.13}$$

for some constant $c'(k) > 0$ independent of n. However, for fixed z,

$$\psi_n(z) = -\frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(-z+n+1)} = -n^z\Gamma(-z)\left(1+O\left(\frac{1}{n}\right)\right) \tag{3.14}$$

as $n \to \infty$, by (4) of Erdélyi [(1953), page 47], and this is $O(n^{-1/2-k/2})$ when $z = -1/2-k/2$, as desired. Now by (3.9) we see that

$$\begin{aligned} E_n^{(k)}(x) &= -\operatorname{Res}_{z=\alpha-k/2}\left[\psi_n(z)\frac{\phi(z+k/2)}{\phi(k/2)}x^{k+2z}\right] \\ &\quad + O(n^{-1/2-k/2}) \\ &= -\psi_n\left(\alpha-\frac{k}{2}\right)\frac{\operatorname{Res}_{z=\alpha}\phi(z)}{\phi(k/2)}x^{k+2(\alpha-k/2)} \\ &\quad + O(n^{-1/2-k/2}) \\ &= n^{\alpha-k/2}\Gamma\left(\frac{k}{2}-\alpha\right)\left(1+O\left(\frac{1}{n}\right)\right)\frac{\operatorname{Res}_{z=\alpha}\phi(z)}{\phi(k/2)}x^{2\alpha} \\ &\quad + O(n^{-1/2-k/2}), \end{aligned} \tag{3.15}$$

by (3.14). The error in the first product gets absorbed into the big-O term at the end, and we are left with $n^{\alpha-k/2}x^{2\alpha}$ multiplied by a constant (depending on k) which is

$$\begin{aligned} &\frac{\Gamma(k/2-\alpha)\operatorname{Res}_{z=\alpha}\phi(z)}{\phi(k/2)} \\ &\quad = \frac{\Gamma(\alpha-\beta)\Gamma(k/2+1/2)\Gamma(k/2+1)}{\Gamma(\alpha+1/2)\Gamma(\alpha+1)\Gamma(k/2-\beta)} = c_k, \end{aligned} \tag{3.16}$$

since $\beta = -\alpha - 3/2$. The theorem is proved. □

REMARK. We could have derived more precise asymptotic estimates by using a contour further to the left which included more residues. For example, this shows that the error term in Theorem 1 can be pushed down to $O(n^{\alpha-1-k/2})$.

COROLLARY 1. *The expected number of intervals parked inside a starting interval* $[0,1]$ *after* n *attempts are made is*

$$c_0 n^{\alpha} - 1 + o(1). \tag{3.17}$$

Note that c_0 *is about* 1.84.

PROOF. The number of intervals parked in $[0,1]$ is one less that the number of gaps, which is $E_n^{(0)}(1)$. □

COROLLARY 2. *For the expected number parked on the circle,*

$$EN_n = (\alpha+1)\left(1 - \left(\tfrac{1}{2}\right)^{\alpha+1/2}\right)c_0 n^{\alpha} + o(1) = (0.98\ldots)n^{\alpha} + o(1).$$

PROOF. Combine Theorem 1 with (2.1) and note that $(n-1)^{\alpha} = n^{\alpha} - o(a)$ since $\alpha < 1$. □

We remark that the asymptotic formulas give excellent estimates, even for relatively small n. For example, numerical calculations show that the relative error in estimating (2.9) with $x = 1$ by the asymptotic formula $c_0 n^{\alpha} - 1$ is 1.1, 0.46, 0.24 and 0.14% for $n = 10$, 20, 30 and 40, respectively.

Interestingly, asymptotics of the form of Corollaries 1 and 2 have arisen in a completely different context. In an analysis of quadtree data structures, Flajolet, Gonnet, Puech and Robson (1991) solved the recurrence

$$Q_0 = 0, \qquad Q_n = 1 + \frac{4}{n(n+1)} \sum_{k=1}^{n-1} (n-k) Q_k, \qquad n \geq 1,$$

and showed that $Q_n \sim cn^{2\alpha}$, $n \to \infty$, where $c = \Gamma(2\alpha+2)/(2\gamma(\alpha+1)^3)$.

4. The limiting gap distribution. Here we find the asymptotic behavior of the distribution of gaps left behind after attempting to park n intervals in $[0,1]$. (For the case of parking arcs on a circle, see the remarks following the proof of Theorem 2 below.) The dependence of $E_n^{(k)}(1)$ on k in Theorem 1 suggests that gap lengths will be on the order of $n^{-1/2}$, so we define a distribution function

$$F_n(t) = \frac{\text{expected number of gaps with lengths} \leq t/\sqrt{n}}{\text{expected total number of gaps}}. \tag{4.1}$$

As in Brennan and Durrett (1987), the limit law to follow uses the method of moments.

THEOREM 2. *For $t > 0$, $\lim_{n\to\infty} F_n(t) = F(t)$, where $F(t)$ has the density*

$$f(x) = 2\pi^{-1/2}\Gamma(\alpha + 3/2)x^{-1/2}e^{-x^2/2}W_{-\alpha-1/4,1/4}(x^2), \qquad x \ge 0, \tag{4.2}$$

with $W_{\kappa,\mu}(x)$ denoting Whittaker's confluent hypergeometric function [*see Erdélyi* (1953)].

REMARK. As we will see below, the density (4.2) is that of a random variable $\sqrt{YZ}$, where Y and Z are independent, with Y having the unit-exponential density e^{-y}, $Y \ge 0$, and Z having the beta density with exponents $1/2$, $\alpha + 1$:

$$\frac{\Gamma(\alpha + 3/2)}{\Gamma(1/2)\Gamma(\alpha + 2)} z^{-1/2}(1 - z)^{\alpha}, \qquad 0 \le z \le 1.$$

PROOF. By (4.1) the expected number of gaps of length less than or equal to x is

$$E_n^{(0)}(1)F_n(x\sqrt{n}), \tag{4.3}$$

so the expected sum of the kth powers of the gap lengths is

$$\begin{aligned}\int_0^\infty x^k d\big(E_n^{(0)}(1)F_n(x\sqrt{n})\big) &= E_n^{(0)}(1)\int_0^\infty \left(\frac{y}{\sqrt{n}}\right)^k dF_n(y)\\ &= \frac{E_n^{(0)}(1)}{n^{k/2}}\int_0^\infty y^k\, dF_n(y).\end{aligned} \tag{4.4}$$

On the other hand, this expected sum of kth powers is $E_n^{(k)}(1)$ by definition, so

$$\int_0^\infty y^k\, dF_n(y) = \frac{E_n^{(k)}(1)n^{k/2}}{E_n^{(0)}(1)}. \tag{4.5}$$

By Theorem 1 this converges to c_k/c_0 as $n \to \infty$. Moreover, $(c_k/c_0)/k!$ tends to zero as $k \to \infty$ by Stirling's formula, so if there exists a distribution with moments c_k/c_0, then it is unique by Theorem 30.1 of Billingsley (1979). After finding such a distribution, we will verify that $\lim_{n\to\infty} F_n(t) = F(t)$, $t \ge 0$, is an easy consequence of Theorem 30.2 of Billingsley (1979).

We will in fact find an $F(t)$ with a density $f(t)$ such that

$$\int_0^\infty t^k f(t)\, dt = \frac{c_k}{c_0} \tag{4.6}$$

for all real $k \ge 0$. [Note that c_k is defined by (3.7) for all real $k \ge 0$.] Substitute $k = 2s$, $t = \sqrt{x}$ and set

$$g(x) = \frac{1}{2\sqrt{x}} f(\sqrt{x}) \tag{4.7}$$

to rewrite (4.6) as

$$\begin{aligned}\int_0^\infty x^s g(x)\,dx = \frac{c_{2s}}{c_0} &= \frac{\Gamma(\alpha+3/2)\Gamma(s+1/2)\Gamma(s+1)}{\Gamma(1/2)\Gamma(s+\alpha+3/2)} \\ &= s!\frac{\Gamma(\alpha+3/2)\Gamma(s+1/2)}{\Gamma(1/2)\Gamma(s+\alpha+3/2)}\end{aligned} \tag{4.8}$$

by (3.7). Now observe that (4.8) simply states that $E(X^s) = E(Y^s)E(Z^s)$, where X is a random variable with density $g(x)$ and Y and Z are the exponential and beta distributed random variables defined in the remark following the theorem statement. Now compute the joint density $g(x, y)$ of X and Y and integrate to obtain

$$\begin{aligned}g(x) &= \int_x^\infty g(x,y)\,dy \\ &= \pi^{-1/2}\frac{\Gamma(\alpha+3/2)}{\Gamma(\alpha+1)}\int_x^\infty e^{-y}(xy)^{-1/2}(1-x/y)^\alpha\,dy\end{aligned} \tag{4.9}$$

and, after the change of variables $y = x(1+u)$,

$$g(x) = \pi^{-1/2}\frac{\Gamma(\alpha+3/2)}{\Gamma(\alpha+1)}e^{-x}\int_0^\infty e^{-xu}u^\alpha(1+u)^{-\alpha-1/2}\,du. \tag{4.10}$$

By an integral representation of Whittaker's confluent hypergeometric function [Erdélyi (1953), (18), page 274], the integral in (4.10) is $\Gamma(\alpha+1)\,x^{-3/4}e^{x/2}W_{-\alpha-1/4,1/4}$, so

$$g(x) = \pi^{-1/2}\Gamma(\alpha+3/2)x^{-3/4}e^{-x/2}W_{-\alpha-1/4,1/4}(x). \tag{4.11}$$

Inverting (4.7) gives $f(x) = 2xg(x^2)$, the density of $\sqrt{X}$. Substituting for $g(x^2)$ from (4.11) then yields (4.2).

Finally, since F is determined by its moments and since the kth moment of F_n converges to the kth moment of F for all $k \geq 0$, $\{F_n\}_{n=1}^\infty$ converges weakly to F by Theorem 30.2 of Billingsley (1979). However, $F(t)$ is continuous on $[0, \infty)$, so by definition of weak convergence, $\lim_{n\to\infty} F_n(t) = F(t)$, as desired. □

Theorem 2 holds without modification for the expected gap length distribution in the circle problem as well. This can be seen by an easy generalization of Corollary 2 which shows that the expected sum of kth powers of gap lengths for the circle problem is asymptotically a constant factor

$$\left(\frac{1+\sqrt{17}}{4}\right)\left[1-\left(\frac{1}{2}\right)^{(\sqrt{17}-1)/2}\right] = 0.18565\ldots$$

times the result $E_n^{(k)}(1)$ for the problem on $[0, 1]$. The analysis in the proof of Theorem 2 shows that such overall scale factors do nothing to the distribution. Thus the same limiting distribution also applies when parking intervals in $[0, x]$ with $x \in (0, 1)$, for in this case, the scale factor is $(E_n^{(k)}(x))/(E_n^{(k)}(1))$, which is asymptotically x^{α}, by Theorem 1. The initial configuration changes the *number* of gaps after n attempted parkings by a constant factor, but has no bearing on the asymptotic distribution of their lengths.

5. Final remarks. We leave as an open problem the inverse of the arc parking problem: Given the number N to be parked, what is the asymptotic distribution of the number n_N of arcs that have to be inspected before exactly N are parked? Our experience with this problem suggests that it is more difficult than the ones solved in Sections 2–4. The same comment applies to calculating asymptotics for the variance given by (2.12) and (2.13), which we also leave as an open problem.

In on-line parking optimization problems, intervals must be selected or rejected in sequence, but an interval can be rejected even if it fits in a gap. One such problem is to determine the acceptance rule which maximizes the expected number of intervals parked. Because of the more elaborate state, this problem appears to be more difficult than stochastic on-line packing problems [e.g., see Coffman, Flatto and Weber (1987)].

Extensions to our parking problem can be formulated from the models studied by Kelly (1985, 1987), Ziedins (1987) and Ziedins and Kelly (1989). For example, a ring consisting of $k > 1$ independent cables is an important generalization; arcs are to be parked on all k cables. In this problem, arcs can overlap, but at most k can cover any point on the ring.

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