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LIMIT THEOREMS AND RATES OF CONVERGENCE FOR EUCLIDEAN FUNCTIONALS[1]

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A Beardwood–Halton–Hammersley type of limit theorem is established for a broad class of Euclidean functionals which arise in stochastic optimization problems on the d-dimensional unit cube. The result, which applies to all functionals having a certain "quasiadditivity" property, involves minimal structural assumptions and holds in the sense of complete convergence. It extends Steele's classic theorem and includes such functionals as the length of the shortest path through a random sample, the minimal length of a tree spanned by a sample, the length of a rectilinear Steiner tree spanned by a sample and the length of a Euclidean matching. A rate of convergence is proved for these functionals.

1. Introduction and statement of results. Let L denote a real-valued function defined on the finite subsets of $[0,1]^d, d \geq 2$. In a classic paper, Steele (1981) showed that under natural conditions on L,

$$\lim_{n \to \infty} L(U_1, \dots, U_n)/n^{(d-1)/d} = \beta(L) \quad \text{a.s.,} \tag{1.1}$$

where here and elsewhere $(U_i)_{i \geq 1}$ denotes an i.i.d. sequence of random variables with the uniform distribution on $[0,1]^d$ and where $\beta(L)$ denotes a constant. The appeal of (1.1) lies in its applications to a variety of functionals L arising in problems of geometric probability. It is known that (1.1) holds whenever L is homogeneous, translation invariant, subadditive and monotonic, as described by conditions A2, A3, A4′ and A7 below.

Under several additional but less natural conditions on L, Steele showed that if $(X_i)_{i \geq 1}$ denotes an i.i.d. sequence with an arbitrary distribution on $[0,1]^d$, then

$$\lim_{n \to \infty} L(X_1, \dots, X_n)/n^{(d-1)/d} = \beta(L) \int f(x)^{(d-1)/d} dx \quad \text{a.s.,} \tag{1.2}$$

where f is the density of the absolutely continuous part of the law of X_1. Steele's result is motivated by the famous Beardwood, Halton and Hammersley (1959) theorem for the traveling salesman problem (TSP), which proves (1.2) when $L(x_1, \dots, x_n)$ denotes the length of a minimum tour through $(x_1, \dots, x_n)$.

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There has been considerable recent work on refinements and extensions of (1.1) and (1.2). Alexander (1994) has determined a rate of convergence in expectation in (1.1), thus resolving a long-standing conjecture of Beardwood, Halton and Hammersley (1959). Rhee (1993) has shown that (1.2) holds in the sense of complete convergence whenever f is the uniform density. Jaillet (1993) has shown that (1.2) holds for i.i.d. random variables on the d-dimensional torus.

The aim of this paper is to provide a natural and simple approach to the limit theory of Euclidean functionals. We will see that many Euclidean functionals have a certain "quasiadditivity" property which only involves natural conditions such as superadditivity, subadditivity and continuity and which leads to proofs which are relatively short and simple. The study of "quasiadditive" functionals provides a simple unifying framework which is not only more general than that described by Steele (1981), but which also allows us to easily recover and, in some cases, extend the work of Alexander (1993), Rhee (1993) and Jaillet (1993).

The contributions of this paper are threefold. First, using the notion of quasiadditivity, we provide a general approach to the limit theory of Euclidean functionals, one which includes the TSP, minimal spanning tree (MST), Steiner tree, and minimal matching functionals as special cases. Since this approach includes the MST, it is thus more general than Steele's approach, which is limited to monotone increasing functionals. Our approach is not confined to functionals defined on the unit cube, but also furnishes asymptotics on the d-dimensional torus.

Second, we show that quasiadditivity insures that (1.2) holds in the sense of complete convergence for any density f, thus extending the work of Rhee (1993).

Third, we show that quasiadditive functionals admit a rate of convergence

$$|\mathbb{E}L(U_1,\dots,U_n) - \beta(L)n^{(d-1)/d}| = O(n^{(d-2)/d}). \tag{1.3}$$

The approach simplifies the rate results of Alexander (1994) and Jaillet (1992).

We remark that the quasiadditivity condition insuring the general asymptotic result (1.2) and the rate (1.3) involve minimal structural conditions, and in this sense, seem to capture the essence of the Beardwood, Halton and Hammersley (1959) theorem for the TSP. The formulation of our conditions is inspired in part by Rhee (1993).

As in Steele (1981), we list assumptions on L:

A1. $L(\phi) = 0$.

A2. *Homogeneity.* For every $\alpha > 0$ and every finite subset F of $\mathbb{R}^d$, $L(\alpha F) = \alpha L(F)$.

A3. *Translation invariance.* For every $x \in \mathbb{R}^d$, $L(x + F) = L(F)$.

A4. *Subadditivity.* There exists a constant C_1 with the following property: If $(Q_i)_{i \le m^d}$ denotes a partition of $[0,1]^d$ into m^d subcubes of edge length m^{-1}, and if $(q_i)_{i \le m^d}$ denotes the unique collection of vectors such that the translated cube $Q_i - q_i$ has the form $m^{-1}[0,1]^d$, then for every finite subset F of $[0,1]^d$,

$$L(F) \le m^{-1} \sum_{i \le m^d} L\big(m[(F \cap Q_i) - q_i]\big) + C_1 m^{d-1}.$$

A5. *Superadditivity.* For the same conditions as above on Q_i, m and q_i we now have

$$L(F) \ge m^{-1} \sum_{i \le m^d} L\big(m[(F \cap Q_i) - q_i]\big) - C_2 m^{d-1}.$$

Clearly, if L satisfies homogeneity (A2), translation invariance (A3) and the usual subadditivity,

A4′. $$L(F) \le \sum_{i \le m^d} L(F \cap Q_i) + C_1 m^{d-1}$$

then subadditivity (A4) follows. Thus, A4 is weaker than A2, A3 and A4′. A similar comment applies to the superadditivity condition. While this distinction may seem at first purely formal, we will see that it allows us the right amount of flexibility in the sequel.

The final assumption on L is:

A6. *Continuity.* There exists a constant C_3 such that for all finite subsets F and G of $[0,1]^d$,

$$|L(F \cup G) - L(F)| \le C_3 (\operatorname{card} G)^{(d-1)/d}.$$

As a simple consequnce of continuity and condition A1, notice that

$$L(G) \le C_3 (\operatorname{card} G)^{(d-1)/d}.$$

Rhee (1993) has shown that continuity is a consequence of A2, A3, A4′, monotonicity of the form

A7. $$L(F) \le L(F \cup \{x\})$$

and simple subadditivity

$$L(F_1 \cup F_2) \le L(F_1) + L(F_2) + C_1'.$$

In general, it is difficult to verify that L simultaneously satisfies subadditivity (A4) and superadditivity (A5). We distinguish between these possibilities and agree to say that if L satisfies assumptions A1, A4 and A6, then it is a *continuous subadditive Euclidean functional*; if L satisfies assumptions A1, A5, and A6, then it is a *continuous superadditive Euclidean functional.* As may be seen from the work of Rhee (1993), if L denotes either a continuous subadditive or superadditive Euclidean functional, then

$$\lim_{n\to\infty} \mathbb{E}L(U_1,\dots,U_n)/n^{(d-1)/d} = \beta(L). \tag{1.4}$$

It turns out that many continuous subadditive Euclidean functionals L on $[0,1]^d$ are naturally related to a dual superadditive Euclidean functional $\hat{L}$, where $1 + L(F) \geq \hat{L}(F)$ for all sets F and where

$$|\mathbb{E}L(U_1,\dots,U_n) - \mathbb{E}\hat{L}(U_1,\dots,U_n)| \leq C_4 n^{(d-2)/d}, \tag{1.5}$$

with C_4 a constant. This fact is central to all that follows and forms the key to simplifying both the asymptotic results of Steele (1981) and the rate results of Alexander (1994). We point out that the dual $\hat{L}$ is not uniquely defined and is any superadditive Euclidean functional satisfying (1.5). Also, it is far from obvious that dual functionals exist. We will see that the *boundary-rooted* version of L, namely, one where points may be connected to the boundary of the unit cube, usually has the requisite property (1.5) of the dual. We will also see that while the boundary-rooted version is neither homogeneous nor translation invariant, it does enjoy the superadditivity property A5.

When the continuous subadditive Euclidean functional L and its dual $\hat{L}$ enjoy the approximation property (1.5), we say that L (and also $\hat{L}$) is a quasiadditive continuous Euclidean functional.

The main results are as follows.

THEOREM 1.1. *If L is a quasiadditive continuous Euclidean functional, then the asymptotic result* (1.2) *holds in the sense of complete convergence.*

The next result shows that the quasiadditive structure yields some remarkable rate of convergence properties.

THEOREM 1.2. (a) $d \geq 3$. *If L is a quasiadditive continuous Euclidean functional, then*

$$|\mathbb{E}L(U_1,\dots,U_n) - \beta(L)n^{(d-1)/d}| = O(n^{(d-2)/d}). \tag{1.6}$$

(b) $d = 2$. *Let L be a quasiadditive continuous Euclidean functional which also satisfies a "weak continuity" assumption: There is a constant C_5 such that for all $n \geq 1$,*

$$|\mathbb{E}L(U_1,\dots,U_n) - \mathbb{E}L(U_1,\dots,U_{n+1})| \leq C_5 n^{-1/2}. \tag{1.7}$$

Then

$$|\mathbb{E}L(U_1,\dots,U_n) - \beta(L)n^{1/2}| = O(1).$$

It turns out that many Euclidean functionals of interest are quasiadditive and also satisfy weak continuity (1.7). This fact forms the main contribution of this paper:

THEOREM 1.3. *The following are quasiadditive: TSP, MST, Steiner tree and Euclidean minimal matching functionals. Moreover, the TSP, MST and Steiner tree functionals satisfy* (1.7).

We conclude the Introduction with two simple remarks.

REMARK 1.4. It follows from Theorems 1.1 and 1.3 that the boundary-rooted duals of the TSP, MST, Steiner tree and Euclidean minimal matching functionals satisfy the asymptotic result (1.2) and the rate result (1.6). This fact, which is a simple consequence of the above results, does not seem to have been noticed before.

REMARK 1.5. Let L denote either the TSP, MST, Steiner tree or minimal matching functional and let L^T denote the corresponding functional on the d-dimensional torus equipped with the flat metric on the d-cube. In the sequel we will see that the boundary-rooted duals $\hat{L}$ satisfy the relation

$$\hat{L}(F) \le L^T(F) + 1 \le L(F) + 1,$$

where $F \subset [0,1]^d$ is arbitrary. Thus, if X_i, $i \ge 1$, are i.i.d. random variables with values in the d-cube, then the asymptotic behavior of L^T coincides with that of L:

$$\lim_{n\to\infty} L^T(X_1, \dots, X_n)/n^{(d-1)/d} = \beta(L)\int f(x)^{(d-1)/d}\,dx \quad \text{c.c.},$$

where f is the density of the absolutely continuous part of the distribution of X_1.

2. Proofs of Theorems 1.1 and 1.2. This section is devoted entirely to the proofs of the first two results. The proof of Theorem 1.1 follows along the lines of Steele (1981); however, the quasiadditive structure of L produces cleaner arguments. Essentially, the subadditive techniques needed to prove the upper bound implicit in (1.2) are *identical* to the superadditive techniques required to prove the lower bound, which is usually less tractable. Quasiadditivity allows us to apply the same approach to both upper and lower bounds.

PROOF OF THEOREM 1.1.

STEP 1. For the sake of completeness, we show that if L denotes a continuous subadditive Euclidean functional, then

$$\lim_{n\to\infty} \mathbb{E}L(U_1, \dots, U_n)/n^{(d-1)/d} = \beta(L).$$

The proof, which is implicit in Rhee (1993), may be modified to show that continuous superadditive Euclidean functionals $\hat{L}$ enjoy the same limit result. Set $\varphi(n) := \mathbb{E}L(U_1,\dots,U_n)$. Observe by the subadditivity of L, continuity and Jensen's inequality that

$$\varphi(n) \le m^{-1} \sum_{i \le m^d} \Big(\varphi(n/m^d) + C_3(n/m^d)^{(d-1)/(2d)}\Big) + C_1 m^{d-1}$$
$$= m^{d-1}\varphi(n/m^d) + C_3 m^{(d-1)/2} n^{(d-1)/(2d)} + C_1 m^{d-1}.$$

Dividing by $n^{(d-1)/d}$ and replacing n by nm^d yields the relation

$$\varphi(nm^d)/(nm^d)^{(d-1)/d} \le \varphi(n)/n^{(d-1)/d} + C_3/n^{(d-1)/(2d)} + C_1/n^{(d-1)/d}.$$

Set $\beta := \liminf_{n\to\infty} \varphi(n)/n^{(d-1)/d}$ and note that $\beta \le C_3$. For all $\varepsilon > 0$, choose n_0 such that for all $n \ge n_0$ we have $C_3/n^{(d-1)/(2d)} + C_1/n^{(d-1)/d} < \varepsilon$ and $\varphi(n_0)/n_0^{(d-1)/d} \le \beta + \varepsilon$. Thus, for all $m = 1, 2, \dots$ it follows that

$$\varphi(n_0 m^d)/(n_0 m^d)^{(d-1)/d} \le \beta + 2\varepsilon.$$

Using the continuity of L, it is a straightforward exercise to verify that

$$\limsup_{k\to\infty} \varphi(k)/k^{(d-1)/d} \le \beta + 2\varepsilon.$$

Let $\varepsilon \downarrow 0$ to complete the proof.

Finally, we note that $\beta(L) = \beta(\hat{L})$. By the approximation (1.5) we have

$$\mathbb{E}L(U_1,\dots,U_n) - C_4 n^{(d-2)/d} \le \mathbb{E}\hat{L}(U_1,\dots,U_n)$$
$$\le \mathbb{E}L(U_1,\dots,U_n) + C_4 n^{(d-2)/d}.$$

Dividing by $n^{(d-1)/d}$, letting $n \to \infty$ and applying (1.4) shows $\beta(L) = \beta(\hat{L})$.

STEP 2. We now prove that (1.2) holds in the sense of complete convergence. Throughout, let μ be the law of X_1 and f the density of its absolutely continuous part.

By the clever isoperimetric arguments of Rhee [(1993), Theorem 1 and its proof], it is enough to show that (1.2) holds in expectation, that is, to show

$$\lim_{n\to\infty} \mathbb{E}L(X_1,\dots,X_n)/n^{(d-1)/d} = \beta(L)\int f(x)^{(d-1)/d}\,dx. \tag{2.1}$$

The proof of (2.1) consists of two parts.

(a) Fix $\varepsilon > 0$ and suppose that the density of the absolutely continuous part of μ has the form

$$\varphi(x) = \sum_{i=1}^{m^d} \varphi_i 1_{Q_i}(x), \tag{2.2}$$

where $\varphi_i \ge 0$ and where $\{Q_i\}_{i \le m^d}$ is the partition of $[0,1]^d$ consisting of subcubes with edges parallel to the axes and with edge length $m^{-1} < \varepsilon$. We show that (2.1) holds in this setting.

To see this, follow an approach which is similar to Steele (1981, 1988). Let E denote the singular support of μ and let λ denote Lebesgue measure on the cube. We may assume that (1) $E \subset A \cup B$, where A and B are disjoint, $\lambda(A) = 0$ and $\mu(A) \le \varepsilon$, and (2) for some $J \subset I := \{1, 2, \dots, m^d\}$, $B = \bigcup_{i\in J} Q_i$ and $\lambda(B) \le \varepsilon$.

By continuity, property (1), Jensen's inequality and subadditivity,

$$
\begin{aligned}
&\mathbb{E}L(X_1,\dots,X_n)\\
&\quad\le EL(\{X_1,\dots,X_n\}\setminus A) + C_3(\varepsilon n)^{(d-1)/d}\\
&\quad\le m^{-1}\sum_{i\in I\setminus J}\mathbb{E}L(m[\{X_1,\dots,X_n\}\setminus A\cap Q_i - q_i])\\
&\qquad + m^{-1}\sum_{i\in J}\mathbb{E}L(m[\{X_1,\dots,X_n\}\setminus A\cap Q_i - q_i])\\
&\qquad + C_1 m^{d-1} + C_3(\varepsilon n)^{(d-1)/d}.
\end{aligned}
\tag{2.3}
$$

Letting $\{U_k\}_{k=1}^{\infty}$ be i.i.d. with the uniform distribution on $[0,1]^d$, it follows by continuity and Jensen's inequality that the first sum is bounded by

$$
\begin{aligned}
&m^{-1}\sum_{i\in I\setminus J}\left\{\mathbb{E}L\left(\{U_k\}_{k=1}^{[\varphi_i m^{-d}n]}\right) + C_3\left(\varphi_i m^{-d}n\right)^{(d-1)/(2d)}\right\}\\
&\qquad\le m^{-1}\sum_{i\in I\setminus J}\left\{\mathbb{E}L\left(\{U_k\}_{k=1}^{[\varphi_i m^{-d}n]}\right) + K(m)n^{(d-1)/(2d)}\right\}
\end{aligned}
$$

since the expected number of points in the subcube Q_i is $\varphi_i m^{-d}n$. Here $K(m)$ is a constant depending only on m. The second sum is bounded by

$$
\begin{aligned}
\sum_{i\in J} C_3 m^{-1}\left(n\mu(Q_i)\right)^{(d-1)/d} &= C_3 n^{(d-1)/d}\left\{\sum_{i\in J}\left(m^{-d}\right)^{1/d}\mu(Q_i)\right\}^{(d-1)/d}\\
&\le C_3 n^{(d-1)/d}\left(\sum_{i\in J} m^{-d}\right)^{1/d}\\
&= C_3 n^{(d-1)/d}\left(\lambda(B)\right)^{1/d}\\
&\le C_3 \varepsilon^{1/d} n^{(d-1)/d}
\end{aligned}
$$

by Hölder's inequality and the estimate $\lambda(B)\le\varepsilon$.

Combining the above estimates and dividing (2.3) by $n^{(d-1)/d}$ yields

$$
\begin{aligned}
&\frac{\mathbb{E}L(X_1,\dots,X_n)}{n^{(d-1)/d}}\\
&\quad\le \sum_{i\in I\setminus J} m^{-1}\left\{\frac{[\varphi_i m^{-d}n]}{n}\right\}^{(d-1)/d}\frac{\mathbb{E}L\left(\{U_k\}_{k=1}^{\{\varphi_i m^{-d}n\}}\right)}{[\varphi_i m^{-d}n]^{(d-1)/d}} + K(m)n^{(1-d)/(2d)}\\
&\qquad + C_3\varepsilon^{1/d} + \frac{C_1 m^{(d-1)}}{n^{(d-1)/d}} + C_3\varepsilon^{(d-1)/d}.
\end{aligned}
$$

Therefore, letting $n \to \infty$ and applying (1.4), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{E} L(X_1,\dots,X_n)/n^{(d-1)/d} &\le \sum_{i\in I\setminus J} \varphi_i^{(d-1)/d} m^{-d}\beta(L) + C_3\varepsilon^{1/d} + C_3\varepsilon^{(d-1)/d}\\ &= \beta(L)\int_{\bigcup_{i\in I\setminus J} Q_i} \varphi(x)^{(d-1)/d}\,dx + C_3\varepsilon^{1/d} + C_3\varepsilon^{(d-1)/d}.\end{aligned}$$

Let ε tend to zero and apply the bounded convergence theorem to conclude

$$\limsup_{n\to\infty} \mathbb{E} L(X_1,\dots,X_n)/n^{(d-1)/d} \le \beta(L)\int \varphi(x)^{(d-1)/d}\,dx. \tag{2.4}$$

Similarly, by the continuity and superadditivity of the dual $\hat{L}$, we obtain

$$\begin{aligned}\mathbb{E}\hat{L}(X_1,\dots,X_n) \ge m^{-1} &\sum_{i\in I\setminus J} \mathbb{E}\hat{L}(m[\{X_1,\dots,X_n\}\setminus A \cap Q_i - q_i])\\ &- C_2 m^{d-1} - C_3(\varepsilon n)^{(d-1)/d}.\end{aligned}$$

As above, we deduce the analogous lower bound

$$\liminf_{n\to\infty} \mathbb{E}\hat{L}(X_1,\dots,X_n)/n^{(d-1)/d} \ge \beta(\hat{L})\int \varphi(x)^{(d-1)/d}\,dx. \tag{2.5}$$

Combining (2.4) with (2.5), using the inequality $1 + L \ge \hat{L}$ and applying the identity $\beta(L) = \beta(\hat{L})$, we see that (2.1) holds when the density of the absolutely continuous part of μ has the form (2.2).

(b) Next, we show that (2.1) holds for an arbitrary density f. We follow a thinning argument similar to that of Steele (1981). Let E denote the singular support of μ. Let $\varphi(x)$ be a function of the form $\sum_{j=1}^{m^d} \alpha_j 1_{Q_j}(x)$ subject to the condition that $\int_{[0,1]^d} \varphi(x)\,dx = 1 - \mu(E)$. Define A by

$$A := \{x: f(x) \le \varphi(x)\}\setminus E.$$

Define a new sequence of random variables: If $X_i \in A \cup E$, set $X_i' = X_i$; if X_i is not an element of $A \cup E$, set X_i' equal to X_i or a fixed $a_0 \in A$ according to an independent randomization with probabilities $\varphi(X_i)/f(X_i)$ and $1 - \varphi(X_i)/f(X_i)$, respectively. On A^c the sequence $(X_i')_{i\ge 1}$ thus represents a thinned version of $(X_i)_{i\ge1}$. Finally, denote by $(X_i'')_{i\ge1}$ a third sequence of i.i.d. random variables distributed on $[0,1]^d$ with absolutely continuous part $\varphi(x)$ and the same singular distribution as $(X_i)_{i\ge1}$.

We make the following observations:

X_1' and X_1'' have the same density φ on A^c; that is, for $B \subset A^c$,

$$\Pr(X_1' \in B) = \mu(E \cap B) + \int_B \varphi(x)\,dx = \Pr(X_1'' \in B), \tag{2.6}$$

$$\begin{aligned}\Pr(X_1' \in A) &= 1 - \Pr(X_1' \in A^c)\\ &= 1 - \mu(E) - \int_{A^c}\varphi(x)\,dx = \int_A \varphi(x)\,dx\end{aligned} \tag{2.7}$$

and

$$\Pr(X_1'' \in A) = \int_A \varphi(x)\,dx. \tag{2.8}$$

Two applications of the continuity property (A6), together with (2.7) and Jensen's inequality imply the upper bound

$$\begin{aligned}
&\mathbb{E}L(X_1,\dots,X_n)\\
&\quad\le \mathbb{E}L(\{X_i\}_{i\le n}\cap A^c) + C_3(n\mu(A))^{(d-1)/d}\\
&\quad\le \mathbb{E}L(\{X_i'\}_{i\le n}\cap A^c) + C_3(n\mu(A))^{(d-1)/d} + C_3\left(n\int_A \varphi(x)\,dx\right)^{(d-1)/d}.
\end{aligned}$$

By (2.6) and (2.8) and another application of continuity, the above equals

$$\begin{aligned}
&\mathbb{E}L(\{X_I''\}_{i\le n}\cap A^c) + C_3(n\mu(A))^{(d-1)/d} + C_3\left(n\int_A \varphi(x)\,dx\right)^{(d-1)/d}\\
&\quad\le \mathbb{E}L(\{X_i''\}_{i\le n}) + C_3(n\mu(A))^{(d-1)/d} + 2C_3\left(n\int_A \varphi(x)\,dx\right)^{(d-1)/d}.
\end{aligned}$$

Dividing by $n^{(d-1)/d}$, letting $n\to\infty$ and applying part (a) of the proof to the sequence $(X_i'')_{i\ge 1}$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\frac{\mathbb{E}L(X_1,\dots,X_n)}{n^{(d-1)/d}} &\le \beta(L)\int_{[0,1]^d}\varphi(x)^{(d-1)/d}dx\\
&\quad + C_3(\mu(A))^{(d-1)/d} + 2C_3\left(\int_A \varphi(x)\,dx\right)^{(d-1)/d}.
\end{aligned}$$

The arbitrariness of φ then implies that

$$\limsup_{n\to\infty}\frac{\mathbb{E}L(X_1,\dots,X_n)}{n^{(d-1)/d}} \le \beta(L)\int_{[0,1]^d} f(x)^{(d-1)/d}\,dx.$$

In exactly the same way, use the continuity of the dual $\hat{L}$ to show the analogous lower estimate

$$\liminf_{n\to\infty}\frac{\mathbb{E}\hat{L}(X_1,\dots,X_n)}{n^{(d-1)/d}} \ge \beta(L)\int_{[0,1]^d} f(x)^{(d-1)/d}\,dx.$$

Combining these estimates, we obtain the desired result (2.1). □

PROOF OF THEOREM 1.2. The proof breaks down into two cases: $d\ge 3$ and $d=2$.

CASE 1 ($d\ge 3$). We begin with a fundamental observation which is a simple consequence of the super- and subadditivity assumptions.

LEMMA 2.1. *Let $(U_i)_{i=1}^n$ be i.i.d. random variables with the uniform distribution on $[0,1]^d$, $d\ge 3$. There exists a constant K such that for all integers n and m, with n/m^d integral,*

$$\mathbb{E}L(U_1,\dots,U_n) \le m^{d-1}\mathbb{E}L(U_1,\dots,U_{n/m^d}) + Kn^{(d-1)/2d}m^{(d-1)/2} \tag{2.9}$$

and

(2.10) $E\hat{L}(U_1,\dots,U_n) \ge m^{d-1}\mathbb{E}\hat{L}(U_1,\dots,U_{n/m^d}) - Kn^{(d-1)/(2d)}m^{(d-1)/2}.$

PROOF OF LEMMA 2.1. The proof of (2.9) has been shown already in Step 1 of the proof of Theorem 1.1; (2.10) may be shown similarly. □

We now deduce the rate (1.6) given by Theorem 1.2. Letting K be a constant whose value may vary from line to line, by (2.9) we have for n/m^d integral,

$$\frac{\mathbb{E}L(U_1,\dots,U_n)}{n^{(d-1)/d}} \le \frac{\mathbb{E}L(U_1,\dots,U_{n/m^d})}{(n/m^d)^{(d-1)/d}} + \frac{Km^{(d-1)/2}}{n^{(d-1)/(2d)}}$$

and therefore, for all n and m,

$$\frac{\mathbb{E}L(U_1,\dots,U_{nm^d})}{(nm^d)^{(d-1)/d}} \le \frac{\mathbb{E}L(U_1,\dots,U_n)}{n^{(d-1)/d}} + \frac{K}{n^{(d-1)/(2d)}}.$$

Letting $m \to \infty$ and applying Theorem 1.1, it follows that

$$\frac{\mathbb{E}L(U_1,\dots,U_n)}{n^{(d-1)/d}} - \beta(L) \ge \frac{-K}{n^{(d-1)/(2d)}}.$$

Therefore,

(2.11) $$\mathbb{E}L(U_1,\dots,U_n) - \beta(L)n^{(d-1)/d} \ge -Kn^{(d-1)/(2d)}.$$

In exactly the same way, use the dual estimate (2.10) to conclude

(2.12) $$\mathbb{E}\hat{L}(U_1,\dots,U_n) - \beta(\hat{L})n^{(d-1)/d} \le Kn^{(d-1)/(2d)}.$$

By the approximation (1.5) it follows that

$$|\mathbb{E}L(U_1,\dots,U_n) - \beta(L)n^{(d-1)/d}| \le Kn^{(d-2)/d}.$$

This completes Case 1.

CASE 2 ($d = 2$). This follows Case 1, with some simple modifications which take into account the weak continuity assumption. The following lemma is based on an observation of Alexander (1994).

LEMMA 2.2. *Let $N(n)$ denote a Poisson random variable with parameter n and which is independent of $U_1,\dots,U_n$. Then the weak continuity hypothesis* (1.7) *implies both*

$$|\mathbb{E}L(U_1,\dots,U_n) - \mathbb{E}L(U_1,\dots,U_{N(n)})| = O(1)$$

and

$$|\mathbb{E}\hat{L}(U_1,\dots,U_n) - \mathbb{E}\hat{L}(U_1,\dots,U_{N(n)})| = O(1).$$

PROOF. By the approximation (1.5), it will suffice to prove the first estimate. Conditioning on $N(n)$ yields

$$
\begin{aligned}
&|\mathbb{E}L(U_1,\dots,U_n) - \mathbb{E}L(U_1,\dots,U_{N(n)})| \\
&\quad\le \sum_{n/2\le k\le 3n/2} C_5|k-n|(n/2)^{-1/2}\Pr\{N(n)=k\} \\
&\qquad + \sum_{0\le k<n/2} C_3|k-n|^{1/2}\Pr\{N(n)=k\} \\
&\qquad + \sum_{k>3n/2} C_3|k-n|^{1/2}\Pr\{N(n)=k\} \\
&\quad\le C_5 + C_3 n^{1/2}\Pr\{N(n)<n/2\} + C_3\mathbb{E}\Big\{|N(n)|^{1/2}1_{N(n)>3n/2}\Big\}.
\end{aligned}
$$

Since $(N(n)-n)/n^{1/2}$ has exponential tails, the last two terms approach zero. □

Adhering to the notation of Lemma 2.2, we may now find the desired rate of convergence. By subadditivity and superadditivity, we obtain

$$\mathbb{E}L(U_1,\dots,U_{N(nm^2)}) \le m\mathbb{E}L(U_1,\dots,U_{N(n)}) + C_1 m \tag{2.9'}$$

and the dual estimate

$$\mathbb{E}\hat{L}(U_1,\dots,U_{N(nm^2)}) \ge m\mathbb{E}L(U_1,\dots,U_{N(n)}) - C_1 m. \tag{2.10'}$$

From (2.9′) we deduce for all n and m, by Lemma 2.2,

$$\mathbb{E}L(U_1,\dots,U_{nm^2}) \le m\mathbb{E}L(U_1,\dots,U_n) + Km.$$

Dividing by $(nm^2)^{1/2}$, letting $m\to\infty$ and applying Theorem 1.1 with $d=2$, it follows that

$$\mathbb{E}L(U_1,\dots,U_n) - \beta(L)n^{1/2} \ge -K.$$

Similarly, the dual estimate (2.10′) implies the estimate

$$\mathbb{E}\hat{L}(U_1,\dots,U_n) - \beta(L)n^{1/2} \le K.$$

By the approximation (1.5), it follows that

$$|\mathbb{E}L(U_1,\dots,U_n) - \beta(L)n^{1/2}| \le K,$$

completing Case 2. This completes the proof of Theorem 1.2. □

3. Proof of Theorem 1.3. We first point out that the TSP, MST and Steiner tree functionals all enjoy the weak continuity property, as noted by Alexander (1994). It now remains to show that the TSP, MST, Steiner tree and minimal matching are all quasiadditive, that is, satisfy the approximation (1.5). We will treat these one at a time.

A. The TSP functional. For $n\ge 2$, let $L(x_1,\dots,x_n)$ denote the length of the shortest tour through the points $\{x_i\}_{i\le n}$. By tour we mean a closed path which visits every vertex exactly once. By convention, $L(x_1, x_2)$ is twice the length of the edge joining x_1 and x_2. Let $L_0(F, \{a, b\})$ be the length of the shortest path through $F \cup (a, b)$ with endpoints a and b, where a and b are not necessarily distinct and lie on the boundary of $[0, 1]^d$. The key to quasiadditivity lies in defining the following dual functional: For all finite subsets $F \subset [0, 1]^d$, set

$$L_r(F) := \min \sum_i L_0(F_i, \{a_i, b_i\}),$$

where the minimum ranges over all partitions $(F_i)_{i \ge 1}$ of F and all sequences of pairs of points $\{a_i, b_i\}_{i \ge 1}$ lying on the boundary of $[0, 1]^d$. Thus $L_r(F)$ denotes the minimum over all partitions $(F_i)_{i \ge 1}$ and all sequences of pairs $\{a_i, b_i\}_{i \ge 1}$ of the sum of the lengths of the shortest paths through $F_i \cup \{a_i, b_i\}$ with endpoints a_i and b_i. Each path is rooted to the boundary of the unit cube. The functional $L_r(F)$ may be interpreted as the cost of an optimal tour through the points of F which occasionally exits to the boundary at one point and reenters at another, incurring no cost when moving along the boundary.

It is known that L is a continuous subadditive functional and, therefore, to show quasiadditivity, we only need to verify that the dual L_r is a continuous superadditive functional satisfying the approximation (1.5). Superadditivity may be established in a straightforward way: indeed, if E is the graph which realizes $L_r(F)$, then the restriction of E to a typical subcube Q_j, $j \le m^d$, has a sum of edge lengths which exceeds the optimal rooted tour on $F \cap Q_j$, with rooting to the boundary of Q_j.

The next lemma shows continuity.

LEMMA 3.1. *L_r is continuous, that is, satisfies assumption* A6.

PROOF. By optimality and the bound

$$L_r(G) \le L(G) + 1 \le K(\operatorname{card} G)^{(d-1)/d},$$

we get

$$L_r(F \cup G) \le L_r(F) + L_r(G) \le L_r(F) + K(\operatorname{card} G)^{(d-1)/d}.$$

The lower bound

$$L_r(F \cup G) \ge L_r(F)$$

follows from simple monotonicity arguments involving the triangle inequality. □

Thus L_r is a continuous superadditive functional and it only remains to show that the dual functional L_r satisfies the approximation property (1.5). This is accomplished with the following lemmas, the first of which follows from simple geometric considerations.

LEMMA 3.2. *Let Q be a d-dimensional cube of edge length s and centered inside a cube Q' of edge length $s + 2\varepsilon$, where s/ε is integral. Then $Q' \setminus Q$ may be partitioned into at most $K\varepsilon^{1-d}$ subcubes of edge length ε, where K depends upon d and s.*

LEMMA 3.3. *Let $\{U_i\}_{i \le n}$ be i.i.d. random variables with the uniform distribution on the unit cube $[0,1]^d$. Then the expected number of points rooted to the boundary in the optimal rooted tour $L_r(U_1, \dots, U_n)$ is bounded above by $Kn^{(d-1)/d}$, where K depends only on d.*

PROOF. The proof depends upon a dyadic subdivision of $[0,1]^d$. Let Q_0 be the cube of edge length $1/3$ and centered within $[0,1]^d$. Let Q_1 be the cube of edge length $2/3$ and centered within $[0,1]^d$. Partition $Q_1 \setminus Q_0$ into subcubes of edge length $1/6$. By Lemma 3.2 there are at most $K6^{d-1}$ such subcubes.

Continue with the partitioning scheme in this way. At the jth stage, define the cube Q_j of edge length $1 - 2(3 \cdot 2^j)^{-1}$ and partition $Q_j \setminus Q_{j-1}$ into subcubes of edge length $(3 \cdot 2^j)^{-1}$. By Lemma 3.2 there are at most $K3^{d-1}(2^j)^{d-1}$ such cubes.

Carry out k stages, where k is the unique integer chosen such that $2^{d(k-1)} \le n < 2^{dk}$. The cube Q_k is partitioned into at most

$$\sum_{j=0}^{k} K3^{d-1}2^{j(d-1)} \le Kn^{(d-1)/d}$$

subcubes with the property that each subcube has an edge length which is smaller than the distance between the subcube and the boundary of $[0,1]^d$. Finally, by further partitioning each subcube of this partition into 2^{ld} congruent subcubes, where l is the least integer satisfying $2^l > d^{1/2}$, we obtain a partition $\mathscr{Q}$ of Q_k into at most $Kn^{(d-1)/d}$ subcubes, with the property that the diameter of each subcube is less than the distance between it and the boundary of $[0,1]^d$.

Now make the simple but fundamental observation that in an optimal rooted tour on $\{U_i\}_{i \le n}$, each subcube in $\mathscr{Q}$ contains at most two sample points which are rooted to the boundary. Indeed, as the diameter of the subcube is less than the distance to the boundary of $[0,1]^d$, optimality implies that at most one path from each subcube of $\mathscr{Q}$ may be joined to the boundary.

Finally, as the Lebesgue measure of $[0,1]^d \setminus Q_k$ is $O(n^{-1/d})$, the expected number of points in $[0,1]^d \setminus Q_k$ which are rooted to the boundary is also at most $Kn^{(d-1)/d}$. Combining this with number of points in Q_k which are rooted to the boundary gives the desired conclusion. □

In the sequel, given a graph E, let $l(E)$ denote the sum of the edge lengths in E. The next lemma is straightforward and follows from the triangle inequality.

LEMMA 3.4. *Let $A_1, \dots, A_k$ be disjoint finite sets of points in $[0,1]^d$, let G_i be a tour on A_i and let $a_i \in A_i$, $1 \le i \le k$. Let $A := \{a_i\}_{i \le k}$ and let G be a tour on A. Then*

$$L\left(\bigcup_{i=1}^{k} A_i\right) \le l(G) + \sum_{i=1}^{k} l(G_i).$$

Equipped with the above lemmas, it is now possible to prove that the functional L is quasiadditive:

LEMMA 3.5. *Let $\{U_i\}_{i \le n}$ be i.i.d. with the uniform distribution on $[0,1]^d$. Then*

$$|\mathbb{E}L(U_1, \dots, U_n) - \mathbb{E}L_r(U_1, \dots, U_n)| \le Kn^{(d-2)/d}.$$

PROOF. Consider a rooted tour G which achieves $L_r(U_1, \dots, U_n)$. Call the points where G is rooted to the boundary "marks." Each mark is thus the endpoint of a rooted path through a subset of $\{U_i\}_{i \le n}$; possibly the subset is a singleton, in which case the path contains two copies of a single edge. Construct an optimal matching E on the set of marks (if the number of marks has odd parity, connect one point to itself). As the marks form a subset of the boundary of $[0,1]^d$, it follows that

$$l(E) \le L(\{\text{marks}\}) \le K(\text{card}\{\text{marks}\})^{(d-2)/(d-1)}.$$

By Lemma 3.3 and Jensen's inequality

$$\mathbb{E}l(E) \le K \cdot \mathbb{E}(\text{card}\{\text{marks}\})^{(d-2)/(d-1)} \le Kn^{(d-2)/d}. \tag{3.1}$$

The matching given by E takes paths rooted to the boundary and forms a collection of tours $\{G_i\}$ from them. The sum of the lengths of the tours satisfies

$$\sum_i l(G_i) = l(G) + l(E).$$

It follows from the definition of G and from (3.1) that

$$\mathbb{E}\sum_i l(G_i) \le \mathbb{E}L_r(U_1, \dots, U_n) + Kn^{(d-2)/d}. \tag{3.2}$$

Next, choose one representative mark from each tour G_i and connect these representatives with an optimal tour G' on the boundary. As in (3.1), we deduce

$$\mathbb{E}l(G') \le Kn^{(d-2)/d}. \tag{3.3}$$

Finally, by the monotonicity of L, Lemma 3.4, (3.2) and (3.3), in this order, it follows that

$$\begin{aligned} \mathbb{E}L(\{U_i\}_{i \le n}) &\le \mathbb{E}L(\{U_i\}_{i \le n} \cup \{\text{marks}\}) \\ &\le \mathbb{E}l(G') + \mathbb{E}\sum_i l(G_i) \\ &\le \mathbb{E}L_r(\{U_i\}_{i \le n}) + Kn^{(d-2)/d}. \end{aligned}$$

Since we trivially have

$$\mathbb{E}L_r(\{U_i\}_{i \le n}) \le \mathbb{E}L(\{U_i\}_{i \le n}) + 1,$$

Lemma 3.5 follows immediately. □

We have thus showed that the TSP functional is quasiadditive. It is a relatively simple matter to see that the MST, the Steiner tree and the Euclidean matching functionals are also quasiadditive. Indeed, the methods are very similar and, in some cases, easier. We sketch the arguments as follows.

B. The MST functional. Denote the MST functional by $T(x_1, \dots, x_n)$; it is the weight of the minimal spanning tree of $V := \{x_1, \dots, x_n\}$, where the weight assigned to edge e is its length $|e|$. More precisely,

$$T(x_1, \dots, x_n) = \min_{T} \sum_{e \in T} |e|,$$

where the minimum is over all connected graphs T with vertex set V. It is well known that T is continuous and subadditive. To see that it is quasiadditive, define the dual functional

$$T_r(F) := \min \sum_i T(F_i \cup \{a_i\}),$$

where the minimum ranges over all partitions $(F_i)_{i \geq 1}$ of F and all sequences of points $\{a_i\}_{i \geq 1}$ on the boundary of $[0, 1]^d$. Thus, $T_r(F)$ denotes the minimum over all partitions $(F_i)_{i \geq 1}$ and all points $\{a_i\}_{i \geq 1}$ of the sum of the lengths of the trees through disjoint subsets F_i of F, where the ith tree is rooted to the boundary point a_i. The graph realizing $T_r(F)$ may be interpreted as a collection of small trees connected via the boundary into one large tree, where the connections along the boundary incur no cost. To show that T is quasiadditive, we only need to verify that the dual T_r is a continuous superadditive functional satisfying (1.5).

It is clear that T_r is superadditive; to see that it is continuous, we may modify the arguments of Steele (1988).

To see that approximation (1.5) is satisfied, we use a modification of the approach for the TSP. Using the notation of subsection A, note that

$$(3.4) \quad T(U_1, \dots, U_n) \leq T_r(U_1, \dots, U_n) + L(\{\text{marks}\}) + \Sigma(U_1, \dots, U_n),$$

where $\Sigma := \Sigma(U_1, \dots, U_n)$ denotes the sum of the lengths of the edges of $T_r(U_1, \dots, U_n)$ which are rooted to the boundary of $S_1 := [0, 1]^d$. We will sketch the proof that $\mathbb{E}\Sigma = O(n^{(d-2)/d})$ when $d = 2$. Let S_2 be the square of edge length $1 - 2n^{-1/2}$ centered within S_1. Edges which root vertices in $S_1 \setminus S_2$ contribute on the average $O(1)$ to Σ. Edges which root vertices in S_2 also contribute on the average $O(1)$. To see this, subdivide S_2 into parallel horizontal strips of width $n^{-1/2}$. By optimality, at most only the left and right most points in any such strip are rooted to the sides of S_1. On the average, these points are a distance $n^{-1/2}$ from the boundary. Repeating this argument for vertical strips gives $\mathbb{E}\Sigma = O(1)$ and also shows that the expected number of "marks" is $O(n^{(d-1)/d})$.

Notice also that

$$(3.5) \qquad T_r(U_1, \dots, U_n) \leq T(U_1, \dots, U_n) + 1/2.$$

The last inequality follows since any MST can be made into a rooted MST by simply connecting one vertex to the boundary. Thus, the MST functional is quasiadditive.

C. The Steiner tree functional. Denote the Steiner tree functional by $S(x_1, \dots, x_n)$. It is a connected graph containing $\{x_1, \dots, x_n\}$ and which has the least total sum of edge lengths among all such graphs. It is well known that S is continuous and subadditive. To see that S is quasiadditive, define the dual Steiner tree functional S_r by

$$S_r(F) := \min \sum_i S(F_i \cup \{a_i\}).$$

The functional S_r is the boundary-rooted version of S and is the analog of T_r. To see that S_r is continuous, follow the proof of Lemma 3.1 verbatim. To see that S_r satisfies the approximation (1.5), simply follow the approach of Section B, noting that S_r satisfies the inequalities (3.4) and (3.5).

D. The Euclidean minimal matching functional. Denote the Euclidean minimal matching functional by $M(x_1, \dots, x_n)$. Thus $M(x_1, \dots, x_n)$ denotes the length of the least Euclidean matching of the points $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$, that is,

$$M(x_1, \dots, x_n) = \min_{\sigma} \sum_{i=1}^{[n/2]} \|x_{\sigma(2i-1)} - x_{\sigma(2i)}\|,$$

where the minimum is over all permutations σ of $\{1, 2, \dots, n\}$. Simple and standard arguments show that M is subadditive and continuous. To show quasiadditivity, define the dual Euclidean matching functional $M_r(F)$ as the length of the least Euclidean matching of points in F; matching to boundary points is permitted. In other words, each point in F is paired with either a boundary point or another point in F; M_r minimizes the sum of the edge lengths over all such pairings. The functional M_r is the boundary-rooted version of M. It is clearly superadditive. Continuity of M_r may be established precisely in the same way as the continuity of M.

To establish (1.5), observe that for essentially the same reasons as in the previous examples,

$$M(U_1, \dots, U_n) \le M_r(U_1, \dots, U_n) + M(\{\text{marks}\})$$

and

$$M_r(U_1, \dots, U_n) \le M(U_1, \dots, U_n).$$

Thus, the Euclidean matching functional M is quasiadditive.

This completes the proof of Theorem 1.3. □

4. Concluding remarks. We have seen that the notion of quasiadditivity provides a general approach to the limit theory of Euclidean functionals L. Central to the theory is the use of a boundary-rooted dual functional $\hat{L}$ which approximates L and which has a useful superadditivity property. The subadditivity of L, together with the superadditivity of its dual $\hat{L}$, leads to a natural simplification and extension of the asymptotic theory as developed by Steele (1981). It also leads to complete convergence results as well as simple rates of convergence.

While we have seen that the TSP, MST, Steiner tree and Euclidean minimal matching functionals are all quasiadditive, we have not attempted to show that other functionals are quasiadditive. In Yukich (1994), it is shown, for example, that the Euclidean semimatching functional is quasiadditive. We expect that the theory of quasiadditive functionals includes other well-known functionals.

The theory of quasiadditive functionals may enjoy other benefits. In Yukich (1994) it is shown that the quasiadditivity of L leads in a natural way to the construction of a functional L^A which closely approximates L and which has an expected polynomial execution time. The functional L^A may be interpreted as an analog of Karp's heuristic for the TSP. Finally, it appears that the methods developed here may be appropriately modified to treat Euclidean functionals with power-weighted edges as well as those defined on bipartite samples.

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