NON-UNIVERSAL FLUCTUATIONS OF THE EMPIRICAL MEASURE FOR ISOTROPIC STATIONARY FIELDS ON $\mathbb{S}^2 \times \mathbb{R}$

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In this paper, we consider isotropic and stationary real Gaussian random fields defined on $\mathbb{S}^2 \times \mathbb{R}$ and we investigate the asymptotic behavior, as $T \to +\infty$, of the empirical measure (excursion area) in $\mathbb{S}^2 \times [0, T]$ at any threshold, covering both cases when the field exhibits short and long memory, that is, integrable and nonintegrable temporal covariance. It turns out that the limiting distribution is not universal, depending both on the memory parameters and the threshold. In particular, in the long memory case a form of Berry's cancellation phenomenon occurs at zero-level, inducing phase transitions for both variance rates and limiting laws.

1. Introduction.

1.1. *Background and motivations.* In recent years, special interest has been devoted to the study of random fields $Z = \{Z(x), x \in \mathbb{S}^2\}$ defined on the two-dimensional unit sphere $\mathbb{S}^2$, finding applications in several areas such as medical imaging, atmospheric sciences, geophysics, solar physics and cosmology (see, e.g., [10, 11, 23, 30]). In particular, considerable attention has been drawn by the investigation of geometric functionals of Gaussian excursion sets on manifolds (see, e.g., [1, 2]). Indeed, aiming to study the geometry of a random field Z, it is natural to introduce the family of excursion sets

$$\{x \in \mathbb{S}^2 : Z(x) \geq u\}$$

indexed by the threshold $u \in \mathbb{R}$; under Gaussianity and isotropy, the expected value of their Lipschitz–Killing curvatures (i.e., area, boundary length and Euler–Poincaré characteristic), is easily obtained as a special case of the celebrated Gaussian kinematic formula, see for example, [1], Ch. 13. However, what is more challenging is to investigate fluctuations around these expected values and for this purpose, asymptotic methods must be exploited, considering sequences of random fields. In particular, a number of recent papers has focussed on the asymptotic behavior of sequences of Gaussian Laplace eigenfunctions (random spherical harmonics), in the *high-energy* limit, that is, as the eigenvalues diverge. Several results have been given concerning the asymptotic variance, the limiting distribution and the correlation for different values of the thresholding parameter $u \in \mathbb{R}$ of Lipschitz–Killing curvatures of their excursion sets, see for example, [8, 9, 25, 26, 29, 39, 42] and the references therein; see also [7, 17, 24, 32, 34, 41] for related results on the standard flat torus and on the Euclidean plane. Some of these results entail rather surprising issues, for instance the cancellation of the leading variance terms for specific threshold values (typically in the *nodal* case $u = 0$) and the possibility to express wide classes of functionals as simple polynomial integrals on $\mathbb{S}^2$ of the underlying fields, up to lower order terms.

The purpose of this paper is to begin the investigation of these same issues for a different class of fields, that is, isotropic and stationary Gaussian fields on $\mathbb{S}^2 \times \mathbb{R}$, which can be immediately interpreted as spherical random fields evolving over time (see, e.g., [3, 5, 22] and the references therein). Although the present manuscript is mainly of theoretical nature, it is very easy to figure out several areas of applications where such random fields emerge most naturally, including the scientific research streams mentioned above. In the next subsection, we introduce our setting in more detail.

1.2. *Sphere-cross-time random fields.* Let us fix a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. We denote by $\mathbb{S}^2$ the two-dimensional unit sphere with the round metric. A space–time real-valued spherical random field

$$(1) \qquad Z = \{Z(x, t), x \in \mathbb{S}^2, t \in \mathbb{R}\}$$

is a collection, indexed by $\mathbb{S}^2 \times \mathbb{R}$, of real random variables such that the map

$$Z : \Omega \times \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$$

is $\mathfrak{F} \otimes \mathfrak{B}(\mathbb{S}^2 \times \mathbb{R})$-measurable, where $\mathfrak{B}(\mathbb{S}^2 \times \mathbb{R})$ stands for the Borel σ-field of $\mathbb{S}^2 \times \mathbb{R}$. We say that Z is Gaussian if for every $n \geq 1$, $x_1, \ldots, x_n \in \mathbb{S}^2$, $t_1, \ldots t_n \in \mathbb{R}$, the random vector $(Z(x_1, t_1), \ldots, Z(x_n, t_n))$ is Gaussian.

CONDITION 1.1.   The space–time real-valued spherical real random field Z in (1) is Gaussian and:

- zero-mean, that is, $\mathbb{E}[Z(x, t)] = 0$ for every $x \in \mathbb{S}^2$, $t \in \mathbb{R}$;
- stationary and isotropic, that is,

$$(2) \qquad \mathbb{E}[Z(x, t)Z(y, s)] = \Gamma(\langle x, y \rangle, t - s)$$

  for every $x, y \in \mathbb{S}^2$, $t, s \in \mathbb{R}$, where $\Gamma : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ is a positive semidefinite function and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^3$;
- mean square continuous, that is, Γ is continuous.

The assumption of zero-mean is of course just a convenient normalization with no mathematical impact. The assumption of Gaussianity ensures that we need to make no distinction between so-called weak and strong stationarity, see for example, [23], Definition 5.9, and it simplifies some of our proofs to follow; moreover, it is the common background with basically all the previous literature on the geometry of excursion sets (starting from [1]), likewise the assumption of mean square-continuity, see for example, [3, 18] and the references therein.

From now on we assume that Z in (1) satisfies Condition 1.1.

1.2.1. *Karhunen–Loève expansions.* It is well known (see, e.g., [3], Theorem 3.3, or [22], Theorem 3) that the following expansion for the covariance function Γ in (2) holds:

$$(3) \qquad \Gamma(\theta, \tau) = \sum_{\ell=0}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell(\tau) P_\ell(\theta), \quad (\theta, \tau) \in [-1, 1] \times \mathbb{R},$$

where $\{C_\ell, \ell \geq 0\}$ is a sequence of continuous positive semidefinite functions on $\mathbb{R}$, P_ℓ denotes the ℓ-th Legendre polynomial [36], Section 4.7, and the series is uniformly convergent, which is equivalent to

$$(4) \qquad \sum_{\ell=0}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell(0) < +\infty.$$

Obviously $C_\ell(0) \geq 0$ for every $\ell = 0, 1, 2, \ldots$. Let $T > 0$, it is straightforward (see, e.g., [5]) to prove that the following Karhunen–Loève expansion for Z holds in $L^2(\Omega \times \mathbb{S}^2 \times [0, T])$:

$$(5) \qquad Z(x, t) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m}(t) Y_{\ell,m}(x),$$

where $\{Y_{\ell,m}, \ell \geq 0, m = -\ell, \ldots, \ell\}$ is the standard real orthonormal basis of spherical harmonics [23], Section 3.4, for $L^2(\mathbb{S}^2)$, and

$$(6) \qquad a_{\ell,m}(t) = \int_{\mathbb{S}^2} Z(x, t) Y_{\ell,m}(x) \, dx,$$

so that $\{a_{\ell,m}, \ell \geq 0, m = -\ell, \ldots, \ell\}$ is a family of independent, stationary, centered, Gaussian processes on $\mathbb{R}$ such that for every $t, s \in \mathbb{R}$

$$\mathbb{E}[a_{\ell,m}(t) a_{\ell,m}(s)] = C_\ell(t - s).$$

REMARK 1.2. We will investigate the asymptotic behavior of Z in $\mathbb{S}^2 \times [0, T]$ as $T \to +\infty$, in particular the set of attainable laws of its geometrical functionals. Our proofs will rely on $L^2(\mathbb{P})$-bounds, *uniform* with respect to the parameter T, thus allowing a fruitful use of the expansion in (5) which actually holds true for every *fixed* $T > 0$.

Now let

$$\widetilde{\mathbb{N}} := \{\ell \geq 0 : C_\ell(0) \neq 0\}.$$

From now on, we will consider only $\ell \in \widetilde{\mathbb{N}}$ unless otherwise specified. Let us define

$$(7) \qquad Z_\ell(x, t) := \sum_{m=-\ell}^{\ell} a_{\ell,m}(t) Y_{\ell,m}(x), \quad (x, t) \in \mathbb{S}^2 \times \mathbb{R}.$$

By construction, $\{Z_\ell, \ell \in \widetilde{\mathbb{N}}\}$ is a sequence of independent random fields and each $Z_\ell(\cdot, t)$ almost surely solves the Helmholtz equation

$$\Delta_{\mathbb{S}^2} Z_\ell(\cdot, t) + \ell(\ell + 1) Z_\ell(\cdot, t) = 0,$$

where $\Delta_{\mathbb{S}^2}$ denotes the spherical Laplacian. For notational convenience and without loss of generality we also assume that

$$(8) \qquad \mathbb{E}[Z^2(x, t)] = \sum_{\ell \in \widetilde{\mathbb{N}}} \sigma_\ell^2 = 1, \qquad \sigma_\ell^2 := \mathbb{E}[Z_\ell^2(x, t)] = \frac{2\ell + 1}{4\pi} C_\ell(0).$$

1.2.2. *Long and short range dependence.* For $\ell \in \widetilde{\mathbb{N}}$, Bochner Theorem ensures that there exists a probability measure μ_ℓ on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ such that

$$\frac{C_\ell(\tau)}{C_\ell(0)} = \int_{\mathbb{R}} e^{i\lambda\tau} \, d\mu_\ell(\lambda), \quad \tau \in \mathbb{R}.$$

If μ_ℓ is absolutely continuous with respect to the Lebesgue measure, then we may introduce the normalized spectral density as the function $f_\ell : \mathbb{R} \to \mathbb{R}^+$ such that

$$(9) \qquad \frac{C_\ell(\tau)}{C_\ell(0)} = \int_{\mathbb{R}} e^{i\lambda\tau} f_\ell(\lambda) \, d\lambda, \quad \tau \in \mathbb{R};$$

we have of course

$$\int_{\mathbb{R}} f_\ell(\lambda) \, d\lambda = 1.$$

If C_ℓ is integrable on $\mathbb{R}$, then clearly f_ℓ exists.

Let us now define the family of symmetric real-valued functions $\{g_\beta, \beta \in (0, 1]\}$ as follows:

$$(10) \qquad g_\beta(\tau) = \begin{cases} (1 + |\tau|)^{-\beta} & \text{if } \beta \in (0, 1), \\ (1 + |\tau|)^{-\alpha} & \text{if } \beta = 1, \end{cases}$$

for some $\alpha \in [2, +\infty)$. We believe that the assumption $= \alpha \in [2, +\infty)$ is not essential for the validity of our main findings; indeed it seems likely that it can be replaced with $\alpha \in (1, +\infty)$. Nevertheless, the current formulation is instrumental to obtain neat proofs of some technical results, in particular Lemma 4.13 in Appendix 5.5.

CONDITION 1.3. There exists a sequence $\{\beta_\ell \in (0, 1], \ell \in \widetilde{\mathbb{N}}\}$ such that

$$C_\ell(\tau) = G_\ell(\tau) \cdot g_{\beta_\ell}(\tau), \quad \ell \in \widetilde{\mathbb{N}},$$

where g_{β_ℓ} is as in (10) and

$$\sup_{\ell \in \widetilde{\mathbb{N}}} \left| \frac{G_\ell(\tau)}{C_\ell(0)} - 1 \right| = o(1) \quad \text{as } \tau \to +\infty.$$

Moreover $0 \in \widetilde{\mathbb{N}}$ (i.e., $C_0(0) \neq 0$) and if $\beta_0 = 1$ then

$$\int_{\mathbb{R}} C_0(\tau) \, d\tau > 0.$$

From now on we assume that Condition 1.3 holds for the sequence $\{C_\ell, \ell \in \widetilde{\mathbb{N}}\}$. Note that $G_\ell(0) = C_\ell(0)$ for every $\ell \in \widetilde{\mathbb{N}}$.

REMARK 1.4 (Abelian/Tauberian type results). Let $\ell \in \widetilde{\mathbb{N}}$. The coefficient β_ℓ in Condition 1.3 can be interpreted as a "memory" parameter; in particular, for $\beta_\ell = 1$ (resp. $\beta_\ell \in (0, 1)$) the covariance function C_ℓ is integrable on $\mathbb{R}$ (resp. $\int_{\mathbb{R}} |C_\ell(\tau)| \, d\tau = +\infty$) and the corresponding process has so-called short (resp. long) memory behavior (note that $C_\ell(0)$ is always nonnegative but $C_\ell(\tau)$ need not be, for $\tau > 0$). Under some regularity assumptions, an equivalent characterization could be given in terms of the behavior at the origin of the spectral density f_ℓ in (9): long-memory entailing divergence to infinity, whereas in the short-memory/integrable case f_ℓ is immediately seen to be bounded in 0. Clearly one could choose alternative parametrizations for $g_\beta(\tau)$, such as for instance

$$g_\beta(\tau) = (1 + |\tau|^2)^{-\beta/2} \quad \text{or} \quad g_\beta(\tau) = (1 + |\tau|^\gamma)^{-\beta};$$

these choices would not alter by any means the substance of our results, as our condition is basically requiring that, for all ℓ,

$$\lim_{\tau \to \infty} \frac{C_\ell(\tau)}{C_\ell(0)\tau^{-\beta_\ell}} = 1.$$

Possible generalizations would be to allow for the possibility of slowly-varying factors, that is, to allow for autocorrelations of the form $L(|\tau|)\tau^{-\beta}$, where $L(\cdot)$ is such that $\lim_{\tau \to \infty} L(|\tau|)/L(c|\tau|) = 1$ for all $c > 0$. These generalizations are common in the long memory literature but would not alter by any means the substance of our results, so we avoid to consider them for brevity's sake.

Some conventions. From now on, $c \in (0, +\infty)$ will stand for a universal constant which may change from line to line. Let $\{a_n, n \geq 0\}$, $\{b_n, n \geq 0\}$ be two sequences of positive numbers: we will write $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to +\infty$, $a_n \approx b_n$ whenever $a_n/b_n \to c$, $a_n = o(b_n)$ if $a_n/b_n \to 0$, and finally $a_n = O(b_n)$ if eventually $a_n/b_n \leq c$.

2. Main results. Let $u \in \mathbb{R}$. We consider the random process $\mathcal{A}_u$ on $\mathbb{R}$ defined as

$$(11) \qquad \mathcal{A}_u(t) := \mathrm{area}\big(Z(\cdot, t)^{-1}([u, \infty))\big) = \int_{\mathbb{S}^2} \mathbf{1}_{Z(x,t) \geq u} \, dx, \quad t \in \mathbb{R},$$

where dx is an element of Lebesgue measure on the sphere. In words, $\mathcal{A}_u(t)$ represents the empirical measure (i.e., the excursion area) of $Z(\cdot, t)$ corresponding to the level u; its expected value is immediately seen to be given by $\mathbb{E}[\mathcal{A}_u(t)] = 4\pi(1 - \Phi(u))$, where

$$\Phi(u) := \int_u^{+\infty} \phi(t) \, dt, \qquad \phi(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

Φ (resp. ϕ) denoting the tail distribution (resp. probability density) function of a standard Gaussian random variable. This area functional for spherical random fields has been considered by many authors, starting for instance from [19], see also Chapter 2 in [16].

We are interested in the fluctuations of $\mathcal{A}_u(t)$ around its expected value, and we hence introduce the following statistics: for $T > 0$

$$(12) \qquad \mathcal{M}_T(u) := \int_{[0,T]} \big(\mathcal{A}_u(t) - \mathbb{E}[\mathcal{A}_u(t)]\big) \, dt$$

and its normalized version

$$(13) \qquad \widetilde{\mathcal{M}}_T(u) := \frac{\mathcal{M}_T(u)}{\sqrt{\mathrm{Var}\,\mathcal{M}_T(u)}}.$$

CONDITION 2.1. Let $\{\beta_\ell, \ell \in \widetilde{\mathbb{N}}\}$ be the sequence defined in Condition 1.3.

- The sequence $\{\beta_\ell, \ell \in \widetilde{\mathbb{N}}, \ell \geq 1\}$ admits minimum. Let us set

$$(14) \qquad \beta_{\ell^\star} := \min\{\beta_\ell, \ell \in \widetilde{\mathbb{N}}, \ell \geq 1\}, \qquad \mathcal{I}^\star := \{\ell \in \widetilde{\mathbb{N}} : \beta_\ell = \beta_{\ell^\star}\}.$$

- If $\mathcal{I}^\star \neq \widetilde{\mathbb{N}}$, then the sequence $\{\beta_\ell, \ell \in \widetilde{\mathbb{N}} \setminus \mathcal{I}^\star, \ell \geq 1\}$ admits minimum. Let us set

$$(15) \qquad \beta_{\ell^{\star\star}} := \min\{\beta_\ell, \ell \in \mathbb{N} \setminus \mathcal{I}^\star, \ell \geq 1\}.$$

Note that $\beta_{\ell^\star}, \beta_{\ell^{\star\star}} \in (0, 1]$ and for $\ell \in \mathcal{I}^\star$, obviously $C_\ell(0) > 0$. In words, $\beta_{\ell^\star}$ represents the smallest exponent corresponding to the largest memory, $\mathcal{I}^\star$ the set of multipoles where this minimum is achieved, and $\beta_{\ell^{\star\star}}$ the second smallest exponent β_ℓ governing the time decay of the autocovariance C_ℓ at some given multipole ℓ. Note that we are *excluding* the multipole $\ell = 0$ by the definition of $\beta_{\ell^\star}$ and $\beta_{\ell^{\star\star}}$ in (14) and (15), on the other hand $\ell = 0$ may belong to $\mathcal{I}^\star$. We assume that Condition 2.1 holds from now on.

As we shall see below, the asymptotic behavior of $\mathcal{M}_T(u)$ in (12), as $T \to +\infty$, is governed by a subtle interplay between the value of the parameters $\beta_{\ell^\star}$, $\beta_{\ell^{\star\star}}$ and the threshold level u.

2.1. *Long memory behavior.* We start investigating the case of long-range dependence.

THEOREM 2.2. *If either $u \neq 0$ and $\beta_0 < \min(2\beta_{\ell^\star}, 1)$ or $u = 0$ and $\beta_0 < \min(3\beta_{\ell^\star}, 1)$, then*

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T^{2-\beta_0}} = \frac{2\phi^2(u)C_0(0)}{(1-\beta_0)(2-\beta_0)},$$

and

$$\widetilde{\mathcal{M}}_T(u) \xrightarrow[T \to +\infty]{d} Z,$$

where $Z \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable and $\to^d$ denotes convergence in distribution.

Recall that by assumption $C_0(0) > 0$ (see Condition 1.3), hence the limiting variance constant in Theorem 2.2 is strictly positive.

REMARK 2.3.   In words, Theorem 2.2 holds when the zero-th order multipole component $\{a_{00}(t), t \in \mathbb{R}\}$ is long memory ($\beta_0 < 1$) and all the other multipoles have asymptotically smaller variance (a consequence of either $\beta_0 < 2\beta_{\ell^\star}$, when $u \neq 0$, or $\beta_0 < 3\beta_{\ell^\star}$, when $u = 0$, as we will show below). It should be recalled that, by (6),

$$a_{00}(t) = \int_{\mathbb{S}^2} Z(x,t) Y_{00}(x) \, dx = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{S}^2} Z(x,t) \, dx,$$

that is, $a_{00}(t)$ corresponds to the sample mean of the random field $Z(\cdot, t)$ at the instant $t \in \mathbb{R}$.

REMARK 2.4.   It seems possible to extend Theorem 2.2 both in the direction of functional convergence, that is, uniform convergence with respect to u (see [12, 28]) and to quantitative versions of the central limit theorem in Kolmogorov or Wassestein distances, see [31] for the discussion of the celebrated Stein–Malliavin approach, as well as Chapter 2 of [16] for early results in the case of the excursion area. For brevity's sake, these extensions are not investigated here and are left as possible topics for future research.

The limiting distribution in Theorem 2.2 is universal; this is not the case for the theorems to follow. We need first to recall one more definition.

DEFINITION 2.5.   The random variable X_β has the standard Rosenblatt distribution (see, e.g., [37] and also [13, 38]) with parameter $\beta \in (0, \frac{1}{2})$ if it can be written as

$$(16) \qquad X_\beta = a(\beta) \int_{(\mathbb{R}^2)'} \frac{e^{i(\lambda_1 + \lambda_2)} - 1}{i(\lambda_1 + \lambda_2)} \frac{W(d\lambda_1) W(d\lambda_2)}{|\lambda_1 \lambda_2|^{(1-\beta)/2}},$$

where W is the white noise Gaussian measure on $\mathbb{R}$, the stochastic integral is defined in the Ito's sense (excluding the diagonals: as usual, $(\mathbb{R}^2)'$ stands for the set $\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 \neq \lambda_2\}$), and

$$(17) \qquad a(\beta) := \frac{\sigma(\beta)}{2\Gamma(\beta)\sin((1-\beta)\pi/2)},$$

with

$$\sigma(\beta) := \sqrt{\frac{1}{2}(1-2\beta)(1-\beta)}.$$

We say the random vector V satisfies a composite Rosenblatt distribution of degree $N \in \mathbb{N}$ with parameters $c_1, \ldots, c_N \in \mathbb{R}$, if

$$(18) \qquad V = V_N(c_1, \ldots, c_N; \beta) \overset{d}{=} \sum_{k=1}^{N} c_k X_{k;\beta},$$

where $\{X_{k;\beta}\}_{k=1,\ldots,N}$ is a collection of i.i.d. standard Rosenblatt random variables of parameter β.

Note that indeed $\mathbb{E}[X_\beta] = 0$ and $\mathrm{Var}(X_\beta) = 1$. The Rosenblatt distribution was first introduced in [37] and has already appeared in the context of spherical isotropic Gaussian random fields as the exact distribution of the correlogramme, see [21].

REMARK 2.6.   The characteristic function Ξ_V of $V = V_N(c_1, \ldots, c_N; \beta)$ in (18) is given by (see, e.g., [40])

$$\Xi_V(\theta) = \prod_{k=1}^{N} \xi_\beta(c_k \theta), \qquad \xi_\beta(\theta) = \exp\left(\frac{1}{2} \sum_{j=2}^{+\infty} (2i\theta\sigma(\beta))^j \frac{a_j}{j}\right),$$

where ξ_β is the characteristic function of X_β in (16), the series is only convergent near the origin and

$$a_j := \int_{[0,1]^j} |x_1 - x_2|^{-\beta} |x_2 - x_3|^{-\beta} \cdots |x_{j-1} - x_j|^{-\beta} |x_j - x_1|^{-\beta} \, dx_1 \, dx_2 \cdots dx_j.$$

Note that when $\beta \to 0^+$ then ξ_β approaches the characteristic function of $\frac{1}{\sqrt{2}}(Z^2 - 1)$, where $Z \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable. As $\beta \to \frac{1}{2}^-$ the limit is the characteristic function of Z. Many more characterizations of the Rosenblatt distribution have been given in the literature: for instance its infinite divisibility properties and Levy–Khinchin representation are discussed in [20] and the references therein.

THEOREM 2.7.   *Assume that $u \neq 0$. If $2\beta_{\ell^\star} < \min(\beta_0, 1)$ we have*

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T^{2-2\beta_{\ell^\star}}} = \frac{u^2 \phi(u)^2}{2(1 - 2\beta_{\ell^\star})(1 - \beta_{\ell^\star})} \sum_{\ell \in \mathcal{I}^\star} (2\ell + 1)C_\ell(0)^2.$$

If $\beta_0 = 1$ and $2\beta_{\ell^\star} = 1$ we have

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T \log T} = u^2 \phi(u)^2 \sum_{\ell \in \mathcal{I}^\star} (2\ell + 1)C_\ell(0)^2.$$

Assume in addition that $\#\mathcal{I}^\star$ is finite, then as $T \to +\infty$

$$(19) \qquad \widetilde{\mathcal{M}}_T(u) \xrightarrow{d} \sum_{\ell \in \mathcal{I}^\star} \frac{C_\ell(0)}{\sqrt{v^\star}} V_{2\ell+1}(1, \ldots, 1; \beta_{\ell^\star}),$$

where $\{V_{2\ell+1}(1, \ldots, 1; \beta_{\ell^\star}), \ell \in \mathcal{I}^\star\}$ is a family of independent composite Rosenblatt random variables as in (18) and

$$v^\star = a(\beta_{\ell^\star})^2 \sum_{\ell \in \mathcal{I}^\star} \frac{2(2\ell + 1)C_\ell(0)^2}{(1 - \beta_{\ell^\star})(1 - 2\beta_{\ell^\star})},$$

where $a(\beta_{\ell^\star})$ is as in (17).

Recall that for $\ell \in \mathcal{I}^\star$ we have $C_\ell(0) > 0$ (see (14)) hence the limiting variance constants in Theorem 2.7 are strictly positive. For the limiting random variable in (19), note that

$$\sum_{\ell \in \mathcal{I}^\star} \frac{C_\ell(0)}{\sqrt{v^\star}} V_{2\ell+1}(1, \ldots, 1; \beta_{\ell^\star}) \overset{d}{=} V_{N^\star}(c_1, \ldots, c_{N^\star}; \beta_{\ell^\star}),$$

where $N^\star := \sum_{\ell \in \mathcal{I}^\star}(2\ell + 1)$ and

$$(c_1, \ldots, c_{N^\star}) = \frac{1}{\sqrt{v^\star}} (\underbrace{C_{\ell_1}(0), \ldots, C_{\ell_1}(0)}_{(2\ell_1+1) \text{ times}}, \ldots, \underbrace{C_{\ell_k}(0), \ldots, C_{\ell_k}(0)}_{(2\ell_k+1) \text{ times}}),$$

with $\mathcal{I}^\star = \{\ell_1, \ldots, \ell_k\}$.

REMARK 2.8 (Normal approximation of Rosenblatt distributions). The distribution of the random variable in (18) is, of course, non-Gaussian. However, in some circumstances it can be closely approximated by a Normal law. Indeed, consider for simplicity the case where the minimum for $\{\beta_\ell, \ell \in \widetilde{\mathbb{N}}\}$ is attained in a single multipole that we call $\ell^\star$, that is, $\mathcal{I}^\star = \{\ell^\star\}$. Then the limiting distribution in (19) is

$$\frac{C_{\ell^\star}(0)}{\sqrt{v^\star}} V_{2\ell^\star+1}(1, \ldots, 1; \beta_{\ell^*}) = \frac{1}{\sqrt{2\ell^\star+1}} V_{2\ell^\star+1}(1, \ldots, 1; \beta_{\ell^*})$$

and by an immediate application of the classical Berry–Esseen Theorem (see, e.g., [14]) one has that

$$d_{\mathrm{Kol}}\left(\frac{1}{\sqrt{2\ell^\star+1}} V_{2\ell^\star+1}(1, \ldots, 1, \beta_{\ell^\star}), Z\right) \leq c \cdot \frac{\mathbb{E}[|X_{\beta_{\ell^\star}}|^3]}{\sqrt{2\ell^\star+1}},$$

where $Z \sim \mathcal{N}(0, 1)$ and d_{Kol} denotes Kolmogorov distance, see for example, [31], Section C.2. The value of $\ell^\star$ for a given random field is fixed, so no central limit theorem occurs; however for $\ell^\star$ large enough the resulting composite Rosenblatt distribution can become arbitrary close to a standard Gaussian variable. Recall here that components at high multipoles $\ell^\star \gg 0$ correspond to small-scale features, so these cases would correspond to random fields which are dominated by a "local" behavior with high-persistence over time.

THEOREM 2.9. *Assume that $u = 0$ and that there exists an even[1] multipole $\ell \in \mathcal{I}^\star$. If $3\beta_{\ell^\star} < \min(1, \beta_0)$, then*

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T^{2-3\beta_{\ell^\star}}}$$

$$= \frac{2}{3!(1-3\beta_{\ell^\star})(2-3\beta_{\ell^\star})} \sum_{\ell_1, \ell_2, \ell_3 \in \mathcal{I}^\star} \mathcal{G}^{000}_{\ell_1 \ell_2 \ell_3} \prod_{i=1}^{3} \sqrt{\frac{2\ell_i+1}{4\pi}} C_{\ell_i}(0),$$

where

$$(20) \qquad \mathcal{G}^{000}_{\ell_1 \ell_2 \ell_3} := \int_{\mathbb{S}^2} Y_{\ell_1, 0}(x) Y_{\ell_2, 0}(x) Y_{\ell_3, 0}(x)\, dx$$

is a so-called Gaunt integral (cf. (22)). If $\beta_0 = 1$ and $3\beta_{\ell^\star} = 1$ then

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T \log T} = \frac{8\pi H_{q-1}(u)^2 \phi(u)^2}{3!} \sum_{\ell_1, \ell_2, \ell_3 \in \mathcal{I}^\star} \mathcal{G}^{000}_{\ell_1, \ell_2, \ell_3} \prod_{i=1}^{3} \sqrt{\frac{2\ell_i+1}{4\pi}} C_{\ell_i}(0).$$

Moreover we have, as $T \to +\infty$,

$$(21) \qquad \widetilde{\mathcal{M}}_T(u) = -\frac{\mathrm{Var}(\mathcal{M}_T(u))^{-1/2}}{3!\sqrt{2\pi}} \int_{\mathbb{S}^2 \times [0, T]} H_3\big(Z(x, t)\big)\, dx\, dt + o_{\mathbb{P}}(1),$$

where $H_3(t) := t^3 - 3t$, $t \in \mathbb{R}$ is the third Hermite polynomial (cf. (24)) and $o_{\mathbb{P}}(1)$ is a family of random variables converging to zero in probability.

Recall that for $\ell \in \mathcal{I}^\star$ we have $C_\ell(0) > 0$ (see (14)) and note that $\mathcal{G}^{000}_{\ell_1 \ell_2 \ell_3}$ in (20) is non-negative (see, e.g., [23], Remark 3.45); moreover if ℓ is even, then $\mathcal{G}^{000}_{\ell\ell\ell} > 0$, see [23], in particular Proposition 3.43 and (3.61). Hence under the assumptions of Theorem 2.9 the limiting variance constants are strictly positive.

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[1]The motivation for this assumption is described just after the statement of Theorem 2.9.

REMARK 2.10. Using the same steps as in the classical papers [13, 38], it seems possible to prove that the right hand side of (21) (and hence $\widetilde{\mathcal{M}}_T(u)$), under the setting of Theorem 2.9, converges in distribution to a weighted sum of higher order Rosenblatt random variables (more precisely, of order 3). However, because for the probability laws of the latter very little is known, and even less so for their linear combinations, we refrain from rigorously investigating this issue here.

REMARK 2.11. For simplicity of presentation, we are ruling out some boundary cases (such as $\beta_0 = 2\beta_{\ell^\star}$), which could be dealt with the same techniques as we shall exploit below: the limit distributions would just correspond to linear combinations of the asymptotic random variables that we obtained above.

2.2. *Short memory behavior.* Theorems 2.2, 2.7 and 2.9 have all considered cases where some form of long-memory behavior is present on the temporal side, meaning that $\beta_\ell < 1$ for at least one instance of the multipole ℓ. In this section we investigate the case where on all scales no form of long-range dependence occurs.

We first need to introduce some more notation: for $q \geq 3$, let $\ell_1, \ldots, \ell_q \geq 0$ and $m_i \in \{-\ell_i, \ldots, \ell_i\}$ for $i = 1, \ldots, q$. The *generalized Gaunt integral* ([23], page 82) of parameters $q, \ell_1, \ldots, \ell_q, m_1, \ldots, m_q$ is defined as (cf. (20))

$$(22) \qquad \mathcal{G}^{m_1 \cdots m_q}_{\ell_1 \cdots \ell_q} := \int_{\mathbb{S}^2} Y_{\ell_1, m_1}(x) \cdots Y_{\ell_q, m_q}(x) \, dx,$$

where $\{Y_{\ell,m}, \ell \geq 0, m = -\ell, \ldots, \ell\}$ still denotes the family of spherical harmonics introduced in Section 1.2.1.

THEOREM 2.12. *Assume $\beta_0 = 1$. If either $u \neq 0$ and $2\beta_{\ell^\star} > 1$ or $u = 0$ and $3\beta_{\ell^\star} > 1$, we have*

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T} = \sum_{q=1}^{+\infty} s_q^2,$$

where

$$s_1^2 := \phi(u)^2 \int_{\mathbb{R}} C_0(\tau) \, d\tau,$$

$$(23) \quad s_2^2 := \frac{u^2 \phi(u)^2}{2} \sum_{\ell=0}^{+\infty} (2\ell + 1) \int_{\mathbb{R}} C_\ell(\tau)^2 \, d\tau,$$

$$s_q^2 := \frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \sum_{\ell_1, \ldots, \ell_q = 0}^{+\infty} \mathcal{G}^{0 \ldots 0}_{\ell_1 \ldots \ell_q} \int_{\mathbb{R}} \prod_{i=1}^q \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(\tau) \, d\tau, \quad q \geq 3.$$

Moreover, as $T \to +\infty$,

$$\widetilde{\mathcal{M}}_T(u) \xrightarrow{d} Z,$$

$Z \sim \mathcal{N}(0, 1)$ *being a standard Gaussian random variable.*

Recall that for $\beta_0 = 1$ we have $\int_{\mathbb{R}} C_0(\tau) \, d\tau \in (0, +\infty)$ (see Condition 1.3) so that $s_1^2 > 0$ yielding $\sum_{q \geq 1} s_q^2 > 0$ (the limiting variance constant is strictly positive).

REMARK 2.13 (On Berry's cancellation). It is interesting to note that a phase transition occurs at $u = 0$. Indeed, for $2\beta_{\ell^\star} < 1$ one observes a form of *Berry's cancellation phenomenon* (see, e.g., [4, 42]), in the sense that the variance diverges with a smaller order rate. More precisely, there are two possibilities:

- for $3\beta_{\ell^\star} < 1$ (resp. $3\beta_{\ell^\star} = 1$), the rate of the variance changes from $T^{2-2\beta_{\ell^\star}}$ to $T^{2-3\beta_{\ell^\star}}$, (resp. $T \log T$) and the limiting distribution is non-Gaussian (Theorem 2.9 and Remark 2.10);
- for $3\beta_{\ell^\star} > 1$, the rate of the variance changes from $T^{2-2\beta_{\ell^\star}}$ to T, and the limiting distribution is Gaussian (Theorem 2.12).

3. Outline of the paper. The results in Theorems 2.2, 2.7, 2.9 and 2.12 fully characterize the behavior of the empirical measure for sphere-cross-time random fields. The resulting scheme is, in the end, rather simple and can be summarized as follows.

- *Short Memory Behavior*: this setting corresponds to integrable covariance functions and occurs either when $\beta_0 = 1$ and $2\beta_{\ell^\star} > 1$, for $u \neq 0$, or when $\beta_0 = 1$ and $3\beta_{\ell^\star} > 1$, for $u = 0$. In such circumstances, the limiting distribution is always Gaussian and the variance, as $T \to +\infty$, is asymptotic to T, for all values of the threshold parameter u. Hence, no form of Berry's cancellation, as in Remark 2.13, can occur.
- *Long Memory Behavior*: this setting corresponds to nonintegrable temporal autocovariance and in this case the picture is more complicated:
  - for $\beta_0 < \min(2\beta_{\ell^\star}, 1)$, the variance grows as $T^{2-\beta_0}$ and the limiting distribution is Gaussian, for all values of u;
  - for $2\beta_{\ell^\star} < \min(\beta_0, 1)$, the variance grows as $T^{2-2\beta_{\ell^\star}}$ and the limiting distribution is non-Gaussian (we denote it as composite Rosenblatt), for $u \neq 0$; however, for $u = 0$, a form of Berry's cancellation occurs, the variance is of order $T^{\max(2-3\beta_{\ell^\star}, 1)}$, the limiting distribution being non-Gaussian for $2 - 3\beta_{\ell^\star} > 1$ and Gaussian for $2 - 3\beta_{\ell^\star} < 1$.

3.1. *Overview of the proofs.* The rationale behind these results can be more easily understood if we review the main ideas behind the proofs.

3.1.1. *Chaotic expansions.* The main technical tool that we are going to exploit is the possibility to expand our area functional $\mathcal{M}_T(u)$ in (12) into so-called Wiener chaoses, by means of the Stroock–Varadhan decomposition, see [31], Section 2.2, as well as our Section 4. Briefly, the latter is based on the fact that the sequence of (normalized) Hermite polynomials $\{H_q/\sqrt{q!}\}_{q \geq 0}$

$$(24) \qquad H_0 \equiv 1, \qquad H_q(u) := (-1)^q \phi(u)^{-1} \frac{d^q}{du^q} \phi(u), \quad q \geq 1$$

(where ϕ still denotes the probability density function of a standard Gaussian random variable) is a complete orthonormal basis of the space of square integrable functions on the real line with respect to the Gaussian measure. The first three polynomials are $H_0(u) = 1$, $H_1(u) = u$, $H_2(u) = u^2 - 1$, $H_3(u) = u^3 - 3u$.

From (12) we have the following *orthogonal* expansion:

$$(25) \qquad \mathcal{M}_T(u) = \sum_{q=0}^{\infty} \mathcal{M}_T(u)[q],$$

the series converging in $L^2(\Omega)$, where (see Lemma 4.1)

$$(26) \qquad \mathcal{M}_T(u)[q] = \frac{H_{q-1}(u)\phi(u)}{q!} \int_{[0,T]} \int_{\mathbb{S}^2} H_q(Z(x,t)) \, dx \, dt$$

is the orthogonal projection of $\mathcal{M}_T(u)$ onto the so-called qth Wiener chaos. Note that if $u = 0$, then $\mathcal{M}_T(u)[q] = 0$ whenever q is even.

3.1.2. *Sharp asymptotics.* The crucial step behind our arguments is to investigate the *sharp* asymptotic behavior, as $T \to +\infty$, of the variances for these chaotic projections. In order to simplify this discussion we assume here that $\beta_{\ell^\star} \leq \beta_0$, see the next sections for a complete analysis. For every $u \in \mathbb{R}$,

$$(27) \qquad \mathrm{Var}(\mathcal{M}_T(u)[1]) = c_1 \cdot T^{\max(2-\beta_0,1)}(1 + o(1)).$$

For $q \geq 2$ and either $u \neq 0$ or $u = 0$ and q odd (recall that for $u = 0$ the projections onto even order chaoses vanish), we have

$$(28) \qquad \mathrm{Var}(\mathcal{M}_T(u)[q]) = c_q \cdot T^{\max(2-q\beta_{\ell^\star},1)}(1 + \mathbf{1}_{q\beta_{\ell^\star}=1} \cdot \log T)(1 + o(1)).$$

Here, for $q \geq 1$, $c_q = c_q(u, \beta_{\ell^\star}, \mathcal{I}^\star)$ is a finite and positive constant depending in particular on q, the level u and the coefficient $\beta_{\ell^\star}$. Thanks to (25) and (31),

$$\mathrm{Var}(\mathcal{M}_T(u)) = \sum_{q=1}^{\infty} \mathrm{Var}(\mathcal{M}_T(u)[q])$$

and hence, up to controlling the sequence $\{c_q, q \geq 1\}$, from (27) and (28) we have that, as $T \to \infty$,

$$\widetilde{\mathcal{M}}_T(u) = \frac{\mathcal{M}_T(u)[1]}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u)[1])}} + o_{\mathbb{P}}(1) \quad \text{for } \beta_0 < \min(2\beta_{\ell^\star}, 1), u \neq 0,$$

$$\widetilde{\mathcal{M}}_T(u) = \frac{\mathcal{M}_T(u)[1]}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u)[1])}} + o_{\mathbb{P}}(1) \quad \text{for } \beta_0 < \min(3\beta_{\ell^\star}, 1), u = 0,$$

$$\widetilde{\mathcal{M}}_T(u) = \frac{\mathcal{M}_T(u)[2]}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u)[2])}} + o_{\mathbb{P}}(1) \quad \text{for } 2\beta_{\ell^\star} < \min(\beta_0, 1), u \neq 0,$$

$$\widetilde{\mathcal{M}}_T(u) = \frac{\mathcal{M}_T(u)[3]}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u)[3])}} + o_{\mathbb{P}}(1) \quad \text{for } 3\beta_{\ell^\star} < \min(\beta_0, 1), u = 0,$$

where $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to zero in probability. The asymptotic distribution of (13) can then be derived in the cases considered just above by a careful analysis of these single components: the first chaotic term is Gaussian for every $T > 0$, the second one asymptotically follows a composite Rosenblatt distribution. On the other hand, in the remaining cases, (e.g., $u \neq 0$, $\beta_0 = 1$ and $2\beta_{\ell^\star} > 1$ or $u = 0$, $\beta_0 = 1$ and $3\beta_{\ell^\star} > 1$) it is not possible to identify a single dominating component; indeed, all the chaotic projections contribute with a variance of the same rate T, and the Gaussian limiting behavior will follow from a Breuer–Major type argument [6], [31], Section 5.3, Section 7.

3.2. *Discussion.* We can further summarize our results as follows:

	$u \neq 0$	$u = 0$	Asymptotic distribution
First chaos dominates if	$\beta_0 < \min(2\beta_{\ell^\star}, 1)$ (Var $\approx T^{2-\beta_0}$)	$\beta_0 < \min(3\beta_{\ell^\star}, 1)$ (Var $\approx T^{2-\beta_0}$)	Gaussian
Second chaos dominates if	$2\beta_{\ell^\star} < \min(\beta_0, 1)$ (Var $\approx T^{2-2\beta_{\ell^\star}}$)	never (it vanishes)	non-Gaussian (composite Rosenblatt 2)
Third chaos dominates if	never	$3\beta_{\ell^\star} < \min(\beta_0, 1)$ (Var $\approx T^{2-3\beta_{\ell^\star}}$)	non-Gaussian
All chaoses dominate if	$\beta_0 = 1, 2\beta_{\ell^\star} > 1$ (Var $\approx T$)	$\beta_0 = 1, 3\beta_{\ell^\star} > 1$ (Var $\approx T$)	Gaussian

These findings should be compared with a rapidly growing literature devoted to the investigation of geometric functionals over spherical random fields in a different regime; in particular, a number of papers (see, e.g., [8, 25, 26, 29, 39, 42]) have considered the high-frequency behavior (e.g., when the eigenvalues diverge) for spherical random eigenfunctions with no form of temporal dependence. The results we exhibited here have some analogies, but also important differences, with this stream of literature. In particular

- for the excursion area of random spherical harmonics at $u \neq 0$ [25, 29] it is indeed the case that the high-energy behavior is dominated by the second-order chaotic projection, whose asymptotic distribution is, however, Gaussian. The same asymptotic behavior occurs for other geometric functionals, such as the boundary length of excursion sets and their Euler–Poincaré characteristic, see [8, 35];
- for $u = 0$, the limiting variance is always of smaller-order, and asymptotic Gaussianity holds [26, 29].

These differences can be explained as follows. Because in the case of high-frequency asymptotics one deals with sequences of eigenspaces of growing dimensions, the second chaotic components correspond to a sum of a growing number of i.i.d. coefficients, whence a standard central limit theorem holds. In our case here, the dimension of the sum of the eigenspaces which correspond to the *strongest memory* does not diverge in general, and hence asymptotic Gaussianity need not to hold. Moreover, in the case of high-frequency asymptotics the *linear* projection term a_{00} is dropped by construction: on the contrary, for the random fields we investigate here this term can be dominant for instance when $\beta_0 < \min(2\beta_{\ell^\star}, 1)$, in which case Gaussianity follows trivially.

As far as Berry's cancellation is concerned, this can occur in the present circumstances only when $H_2(Z(\cdot, \cdot))$ exhibits long memory behavior, that is, nonintegrable temporal autocovariance: this is indeed the case for $2\beta_{\ell^\star} < 1$. If this condition is not met, all chaotic components have integrable temporal autocovariance, none of them dominates and a central limit theorem is established by means of a Breuer–Major Theorem. Note that the presence of long memory behavior in the field Z is a necessary, but not sufficient condition for the covariance of $H_2(Z(\cdot, \cdot))$ to be nonintegrable.

As a final analogy, a remarkable feature of high-frequency asymptotics for random eigenfunctions is the fact that geometric functionals turn out to be asymptotically fully correlated over different levels, and even among themselves, see [9] and the references therein. It is then of interest to investigate if similar features appear in the present framework. We present here a small result that highlights this point.

PROPOSITION 3.1. *Assume that $u \neq 0$, $2\beta_{\ell^\star} < \min(\beta_0, 1)$ and that there exists a unique*

$$\ell^\star := \arg \min_{\ell \in \widetilde{\mathbb{N}}, \ell \geq 1} \beta_\ell,$$

then, as $T \to \infty$,

$$\mathrm{Corr}(\mathcal{M}_T(u), m_{T;\ell^\star}(u)) \to 1,$$

where

$$m_{T;\ell^\star}(u) := \frac{u}{2\sigma_{\ell^\star}} \phi\left(\frac{u}{\sigma_{\ell^\star}}\right) \int_{\mathbb{S}^2} \int_0^T H_2\left(\frac{Z_{\ell^\star}(x, t)}{\sigma_{\ell^\star}}\right) dx \, dt.$$

REMARK 3.2. Note that, if we introduce the process

$$\mathcal{M}_{T;\ell}(u) = \int_0^T \left(\mathcal{A}_{u;\ell}(t) - \mathbb{E}[\mathcal{A}_{u;\ell}(t)]\right) dt,$$

where

$$\mathcal{A}_{u;\ell}(t) := \int_{\mathbb{S}^2} \mathbf{1}_{Z_\ell(x,t) \geq u} \, dx,$$

(cf. (7) and (11)) then $m_{T;\ell^\star}(u) = \mathcal{M}_{T;\ell^\star}(u)[2]$, the second order chaotic component of the functional of the monochromatic field $Z_{\ell^\star}$.

4. Stroock–Varadhan decompositions. The first tool that is needed in order to establish our asymptotic results is the derivation of the analytic form for the chaotic expansion (25) of the empirical measure. The result is very close to analogous findings given by [12, 29]. For a complete discussion on Wiener chaos and related topics see for example, [31], Section 2.2.

LEMMA 4.1. *For every $T > 0$ we have that*

$$(29) \qquad \mathcal{M}_T(u) = \sum_{q \geq 1} \frac{J_q(u)}{q!} \int_{[0,T]} \int_{\mathbb{S}^2} H_q(Z(x,t)) \, dx \, dt,$$

where $J_q(u) = H_{q-1}(u)\phi(u)$, ϕ still being the density function of a standard Gaussian random variable and H_q the Hermite polynomial (24) of order q. The convergence of the series in (29) is in the $L^2(\Omega)$-sense.

PROOF. Let $Z \sim \mathcal{N}(0,1)$, then

$$\mathbf{1}_{Z \geq u} = \sum_{q=0}^{\infty} \frac{J_q(u)}{q!} H_q(Z),$$

where the right hand side converges in the $L^2(\Omega)$-sense and the coefficients $J_q(u)$ are given by

$$J_q(u) := \mathbb{E}[\mathbf{1}_{Z \geq u} H_q(Z)] = \int_{\mathbb{R}} \mathbf{1}_{x \geq u} H_q(x)\phi(x) \, dx$$

$$= (-1)^q \int_u^{+\infty} \frac{d^q}{dx^q} \phi(x) \, dx = H_{q-1}(u)\phi(u)$$

(note that for fixed $x \in \mathbb{S}^2$, $t \in \mathbb{R}$, $Z(x,t)$ is standard Gaussian). Now consider the sequence of random variables

$$\left\{ \sum_{q=0}^{Q} \frac{J_q(u)}{q!} \int_{[0,T]} \int_{\mathbb{S}^2} H_q(Z(x,t)) \, dx \, dt, Q \geq 1 \right\};$$

let us prove that

$$(30) \qquad \sum_{q=0}^{Q} \frac{J_q(u)}{q!} \int_{[0,T]} \int_{\mathbb{S}^2} H_q(Z(x,t)) \, dx \, dt \xrightarrow{Q \to +\infty} \mathcal{M}_T(u)$$

in the $L^2(\Omega)$-sense. We have, thanks to Jensen inequality and Fubini–Tonelli Theorem,

$$\mathbb{E}\left[\left(\mathcal{M}_T(u) - \sum_{q=0}^{Q} \frac{J_q(u)}{q!} \int_{[0,T]} \int_{\mathbb{S}^2} H_q(Z(x,t)) \, dx \, dt \right)^2 \right]$$

$$= \mathbb{E}\left[\left(\int_{[0,T]} \int_{\mathbb{S}^2} \left(\mathbf{1}_{Z(x,t) \geq u} - \sum_{q=0}^{Q} \frac{J_q(u)}{q!} H_q(Z(x,t)) \right) dx \, dt \right)^2 \right]$$

$$\leq 4\pi T \int_{[0,T]} \int_{\mathbb{S}^2} \mathbb{E}\left[\left(\mathbf{1}_{Z(x,t)\geq u} - \sum_{q=0}^{Q} \frac{J_q(u)}{q!} H_q(Z(x,t))\right)^2\right] dx\, dt$$

$$= (4\pi)^2 T^2 \mathbb{E}\left[\left(\mathbf{1}_{Z\geq u} - \sum_{q=0}^{Q} \frac{J_q(u)}{q!} H_q(Z)\right)^2\right] \xrightarrow[Q\to+\infty]{} 0,$$

hence (30) holds and the proof is concluded. $\square$

In particular, the zeroth projection is

$$\mathcal{M}_T(u)[0] = \mathbb{E}[\mathcal{M}_T(u)] = 0; \tag{31}$$

for the first one we have, recalling (5) and (7),

$$\mathcal{M}_T(u)[1] = \phi(u) \int_{[0,T]} \int_{\mathbb{S}^2} Z(x,t)\, dx\, dt = \phi(u) \lim_{L\to\infty} \int_{[0,T]} \int_{\mathbb{S}^2} \sum_{\ell=0}^{L} Z_\ell(x,t)\, dx\, dt,$$

where the limit is in the $L^2(\Omega)$-sense. Hence

$$\mathcal{M}_T(u)[1] = \phi(u) \int_{[0,T]} \frac{a_{00}(t)}{\sqrt{4\pi}}\, dt, \tag{32}$$

the spherical harmonics of degree $\ell \geq 1$ having zero mean on the sphere. Furthermore

$$\mathcal{M}_T(u)[2] = \frac{u\phi(u)}{2} \int_{[0,T]} \int_{\mathbb{S}^2} (Z^2(x,t) - 1)\, dx\, dt,$$

$$\mathcal{M}_T(u)[3] = \frac{(u^2 - 1)\phi(u)}{2} \int_{[0,T]} \int_{\mathbb{S}^2} (Z^3(x,t) - 3Z(x,t))\, dx\, dt.$$

Thanks to orthogonality of the chaotic components, from Lemma 4.1 we get

$$\begin{aligned}
\mathrm{Var}(\mathcal{M}_T(u)) &= \sum_{q=1}^{\infty} \mathrm{Var}(\mathcal{M}_T(u)[q]) \\
&= \sum_{q=1}^{\infty} \frac{J_q(u)^2}{q!} \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \Gamma(\langle x, y\rangle, t-s)^q\, dx\, dy\, dt\, ds,
\end{aligned} \tag{33}$$

where Γ is the covariance function in (2).

4.1. *First order chaotic projections.* In this subsection we investigate the variance behavior of the first chaotic component (32).

LEMMA 4.2. *We have, as $T \to +\infty$,*

$$\lim_{T\to\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[1])}{T^{2-\beta_0}} = \frac{2\phi(u)^2 C_0(0)}{(1-\beta_0)(2-\beta_0)} \quad \text{if } \beta_0 \in (0,1)$$

and

$$\lim_{T\to\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[1])}{T} = \phi(u)^2 \int_{\mathbb{R}} C_0(\tau)\, d\tau \quad \text{if } \beta_0 = 1.$$

Recall that Condition 1.3 ensures that $C_0(0) > 0$ and that for $\beta_0 = 1$

$$\int_{\mathbb{R}} C_0(\tau) \, d\tau \in (0, +\infty),$$

hence Lemma 4.2 gives the exact rate for the variance, the limiting constants being strictly positive. From (32) we can write

$$(34) \qquad \operatorname{Var}(\mathcal{M}_T(u)[1]) = \phi(u)^2 \int_{[0,T]^2} C_0(t - s) \, dt \, ds.$$

REMARK 4.3. We will often make use of the following standard computation, that we report in this remark and that are taken for granted in the rest of the article. Making the change of variable $\tau = t - s$ for the double integral on the right hand side of (34), it is well known that (see, e.g., [16], page 25):

$$\int_{[0,T]^2} C_0(t - s) \, dt \, ds = \int_0^T ds \int_{-s}^{T-s} C_0(\tau) \, d\tau = T \int_{-T}^T \left(1 - \frac{|\tau|}{T}\right) C_0(\tau) \, d\tau.$$

It is now easy to investigate the asymptotic behavior, as $T \to +\infty$, of the variance of the first order chaotic component.

PROOF OF LEMMA 4.2. From Remark 4.3 and (34) we can write

$$(35) \qquad \operatorname{Var}(\mathcal{M}_T(u)[1]) = 2T\phi(u)^2 \int_0^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau.$$

If $\beta_0 = 1$, recalling from Condition 1.3 that in this case the covariance C_0 is integrable on $\mathbb{R}$, we immediately have (thanks to dominated convergence theorem) the exact asymptotic behavior of the variance

$$\lim_{T \to \infty} \frac{\operatorname{Var}(\mathcal{M}_T(u)[1])}{T} = 2\phi(u)^2 \int_0^{+\infty} C_0(\tau) \, d\tau = \phi(u)^2 \int_{\mathbb{R}} C_0(\tau) \, d\tau.$$

Now assume $\beta_0 < 1$. Let $\varepsilon > 0$, thanks to Condition 1.3, there exists $M > 0$ such that, for $\tau > M$,

$$(36) \qquad \sup_{\ell \in \widetilde{\mathbb{N}}} \left| \frac{G_\ell(\tau)}{C_\ell(0)} - 1 \right| < \varepsilon,$$

and we can write (from (35))

$$\operatorname{Var}(\mathcal{M}_T(u)[1])$$

$$(37) \qquad = 2T\phi(u)^2 \int_0^M \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau + 2T\phi(u)^2 \int_M^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau$$

$$= O(1) + 2T\phi(u)^2 \int_0^M C_0(\tau) \, d\tau + 2T\phi(u)^2 \int_M^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau.$$

Consider the last integral on the right hand side of (37) and write

$$\frac{1}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) C_0(\tau) \, d\tau = \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-\beta_0} \, d\tau$$

$$+ \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left(\frac{G_0(\tau)}{C_0(0)} - 1\right) (1 + \tau)^{-\beta_0} \, d\tau.$$

We have

$$(38) \qquad \lim_{T \to \infty} \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-\beta_0} \, d\tau = \frac{C_0(0)}{(1 - \beta_0)(2 - \beta_0)}$$

and

$$\lim_{T \to \infty} \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left(\frac{G_0(\tau)}{C_0(0)} - 1\right) (1 + \tau)^{-\beta_0} \, d\tau = 0. \tag{39}$$

The proof of (38) is straightforward; recall that by assumption $C_0(0) > 0$. It remains to prove (39). For $T > M$

$$\frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left|\frac{G_0(\tau)}{C_0(0)} - 1\right| (1 + \tau)^{-\beta_0} \, d\tau$$

$$\leq \varepsilon \frac{C_0(0)}{T^{1-\beta_0}} \int_M^T (1 + \tau)^{-\beta_0} \, d\tau$$

$$= \varepsilon \frac{C_0(0)}{1 - \beta_0} \left(\left(1 + \frac{1}{T}\right)^{1-\beta_0} - \left(\frac{M+1}{T}\right)^{1-\beta_0}\right) \leq \varepsilon \frac{C_0(0)}{1 - \beta_0} \left(1 + \frac{1}{T}\right)^{1-\beta_0}.$$

Hence

$$\limsup_{T \to +\infty} \left|\frac{C_0(0)}{T^{1-\beta_0}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left(\frac{G_0(\tau)}{C_0(0)} - 1\right) (1 + \tau)^{-\beta_0} \, d\tau\right| \leq \varepsilon \frac{C_0(0)}{1 - \beta_0}$$

and the result follows, ε being arbitrary. Plugging (38) and (39) into (37) we find

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[1])}{T^{2-\beta_0}} = \frac{2\phi(u)^2 C_0(0)}{(1 - \beta_0)(2 - \beta_0)}, \quad \beta_0 \in (0, 1)$$

and this concludes the proof. $\square$

4.2. *Second order chaotic projections.* Our next step is a careful analysis for the variance of the second order chaotic component $\mathcal{M}_T(u)[2]$, which will play a dominating role in most long memory scenarios (see Section 2.1). For $q = 2$ we have

$$\mathrm{Var}(\mathcal{M}_T(u)[2]) = \frac{u^2 \phi(u)^2}{2} \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \Gamma(\langle x, y \rangle, t - s)^2 \, dx \, dy \, dt \, ds.$$

Now, thanks to (3) and (4),

$$\int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \Gamma(\langle x, y \rangle, t - s)^2 \, dx \, dy \, dt \, ds$$

$$= \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \left(\sum_{\ell=0}^{\infty} C_\ell(t - s) \frac{(2\ell + 1)}{4\pi} P_\ell(\langle x, y \rangle)\right)^2 dx \, dy \, dt \, ds$$

$$= \sum_{\ell_1, \ell_2=0}^{\infty} \int_{[0,T]^2} C_{\ell_1}(t - s) C_{\ell_2}(t - s) \frac{(2\ell_1 + 1)(2\ell_2 + 1)}{(4\pi)^2}$$

$$\times \int_{\mathbb{S}^2 \times \mathbb{S}^2} P_{\ell_1}(\langle x, y \rangle) P_{\ell_2}(\langle x, y \rangle) \, dx \, dy \, dt \, ds$$

$$= \sum_{\ell_1, \ell_2=0}^{\infty} \int_{[0,T]^2} C_{\ell_1}(t - s) C_{\ell_2}(t - s) \frac{(2\ell_1 + 1)(2\ell_2 + 1)}{(4\pi)^2} \frac{(4\pi)^2}{2\ell_1 + 1} \mathbf{1}_{\ell_1 = \ell_2} \, dt \, ds$$

$$= \sum_{\ell_1=0}^{\infty} (2\ell_1 + 1) \int_{[0,T]^2} C_{\ell_1}(t - s)^2 \, dt \, ds.$$

Hence

$$\mathrm{Var}(\mathcal{M}_T(u)[2]) = \frac{u^2 \phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) \int_{[0,T]^2} C_\ell(t - s)^2 \, dt \, ds.$$

The next result is of fundamental importance for the study of the asymptotic behavior of $\mathrm{Var}(\mathcal{M}_T(u)[2])$; its proof will be given in the Appendix.

LEMMA 4.4. *Fix $\ell \in \widetilde{\mathbb{N}}$. If $2\beta_\ell < 1$, then*

$$\lim_{T \to \infty} \frac{1}{T^{2-2\beta_\ell}} \int_{[0,T]^2} C_\ell^2(t-s) \, dt \, ds = \frac{C_\ell(0)^2}{(1-\beta_\ell)(1-2\beta_\ell)}.$$

If $2\beta_\ell = 1$, then

$$\lim_{T \to \infty} \frac{1}{T \log T} \int_{[0,T]^2} C_\ell^2(t-s) \, dt \, ds = 2C_\ell(0)^2.$$

If $2\beta_\ell > 1$, then

$$\lim_{T \to \infty} \frac{1}{T} \int_{[0,T]^2} C_\ell^2(t-s) \, dt \, ds = \int_{\mathbb{R}} C_\ell(\tau)^2 \, d\tau.$$

Let us write

(40)
$$\mathrm{Var}(\mathcal{M}_T(u)[2]) = \frac{u^2 \phi(u)^2}{2} \int_{[0,T]^2} C_0(t-s)^2 \, dt \, ds$$
$$+ \frac{u^2 \phi(u)^2}{2} \sum_{\ell=1}^{\infty} (2\ell+1) \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds.$$

Now recall the definition of $\beta_{\ell^\star}$ in (14).

PROPOSITION 4.5. *Assume $u \neq 0$. For $2\beta_{\ell^\star} < 1$ and $\beta_{\ell^\star} \leq \beta_0$, we have that*

(41)
$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2])}{T^{2-2\beta_{\ell^\star}}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_{\ell^\star})(1-\beta_{\ell^\star})} \sum_{\ell \in \mathcal{I}^\star} (2\ell+1) C_\ell(0)^2;$$

for $2\beta_{\ell^\star} < 1$ and $\beta_0 < \beta_{\ell^\star}$,

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2])}{T^{2-2\beta_0}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_0)(1-\beta_0)} C_0(0)^2;$$

for $2\beta_{\ell^\star} = 1$ and $\beta_{\ell^\star} \leq \beta_0$,

(42)
$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2])}{T \log T} = u^2 \phi(u)^2 \sum_{\ell \in \mathcal{I}^\star} (2\ell+1) C_\ell(0)^2;$$

for $\beta_{\ell^\star} = \frac{1}{2}$ and $\beta_0 < \beta_{\ell^\star}$,

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2])}{T^{2-2\beta_0}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_0)(1-\beta_0)} C_0(0)^2;$$

for $2\beta_{\ell^\star} > 1$ and $2\beta_0 > 1$,

(43)
$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2])}{T} = \frac{u^2 \phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell+1) \int_{\mathbb{R}} C_\ell(\tau)^2 \, d\tau;$$

for $2\beta_{\ell^\star} > 1$ and $2\beta_0 = 1$,

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2])}{T \log T} = u^2 \phi(u)^2 \int_{\mathbb{R}} C_0(\tau)^2;$$

finally, for $2\beta_{\ell^\star} > 1$ and $2\beta_0 < 1$,

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2])}{T^{2-2\beta_0}} = \frac{u^2 \phi(u)^2}{2(1-2\beta_0)(1-\beta_0)} C_0(0)^2.$$

Recall that by assumption $C_0(0) > 0$, and for $\ell \in \mathcal{I}^\star$ we have $C_\ell(0) > 0$ (see (14)); as a consequence, Proposition 4.5 gives the exact rate for the variance, the limiting constants being strictly positive.

REMARK 4.6. In words, for $\beta_{\ell^\star} \leq \beta_0$, when $2\beta_{\ell^\star} < 1$ (resp. $2\beta_{\ell^\star} = 1$), we have a form of long-range dependence and the second order chaotic component of the functional $\mathcal{M}_T(u)$ is dominated by a subset of the multipoles; the variance scales as order $T^{2-2\beta_{\ell^\star}}$ (resp. $T \log T$). On the contrary, when $2\beta_{\ell^\star} > 1$, a form of short-range dependence holds and all frequencies contribute with variance terms of order T.

In order to prove Proposition 4.5 we will also need the following technical results. The proofs of Lemma 4.7 and 4.8 are given in the Appendix Section 5.5, the proofs of Lemma 4.9 and Lemma 4.10 are indeed similar and we omit the details for brevity.

LEMMA 4.7. Let $\varepsilon, M > 0$ be as in (36). For ℓ such that $\beta_\ell = 1$ and $T > \max(1, M)$

$$(44) \qquad \frac{1}{T} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds \leq 2C_\ell(0)^2 \left(M + 2\frac{(\varepsilon+1)^2}{\alpha-1} \right),$$

where $\alpha \geq 2$ comes from the definition in (10).

LEMMA 4.8. Let $\varepsilon, M > 0$ be as in (36) and $2\beta_{\ell^\star} < 1$.

- For $\ell \in \mathcal{I}^\star$, $\beta_\ell < 1$ and $T > \max(1, M)$

$$(45) \qquad \frac{1}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds \leq 2C_\ell(0)^2 \left(M + \frac{(\varepsilon+1)^2}{-2\beta_{\ell^\star}+1} \left(1 + \frac{1}{M}\right)^{1-2\beta_{\ell^\star}} \right).$$

- Let $m(\beta_{\ell^\star}) := \max_{x>0} \frac{\log(1+x)}{x^{1-2\beta_{\ell^\star}}}$ and T_m the corresponding arg max. For $\ell \notin \mathcal{I}^\star$, $\ell \geq 1$, $\beta_\ell < 1$ and $T > \max(1, M, T_m)$ we have

$$\frac{1}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds$$

$$\leq 2C_\ell(0)^2 \left(M + 2m(\beta_{\ell^\star})(\varepsilon+1)^2 \mathbf{1}_{2\beta_{\ell^{\star\star}}=1} \right.$$

$$(46) \qquad\qquad + (\varepsilon+1)^2 \frac{1}{-2\beta_{\ell^{\star\star}}+1} \left(1 + \frac{1}{M}\right)^{-2\beta_{\ell^{\star\star}}+1} \mathbf{1}_{2\beta_{\ell^{\star\star}}<1}$$

$$\left. + (\varepsilon+1)^2 \frac{1}{2\beta_{\ell^{\star\star}}-1} \left(\frac{1}{1+M}\right)^{2\beta_{\ell^{\star\star}}-1} \mathbf{1}_{2\beta_{\ell^{\star\star}}>1} \right).$$

LEMMA 4.9. Let $\varepsilon, M > 0$ be as in (36) and $2\beta_{\ell^\star} = 1$.

- For $\ell \in \mathcal{I}^\star$, $\beta_\ell < 1$ and $T > \max(1, M, e)$

$$\frac{1}{T \log T} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds \leq 2C_\ell(0)^2 (M + \log(e+1)).$$

- For $\ell \notin \mathcal{I}^\star$, $\ell \geq 1$, $\beta_\ell < 1$ and $T > \max(1, M)$

$$\frac{1}{T \log T} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds \leq 2C_\ell(0)^2 \left(M + \int_{\mathbb{R}} (1+|\tau|)^{-2\beta_{\ell^{\star\star}}} \, d\tau \right).$$

LEMMA 4.10. *Let ε, $M > 0$ be as in (36). If $2\beta_{\ell^\star} > 1$, for $\ell \geq 1$ and $\beta_\ell < 1$ we have*

$$\frac{1}{T} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds \leq 2C_\ell(0)\Big(M + (\varepsilon+1)^2 \int_{\mathbb{R}} (1+|\tau|)^{-2\beta_{\ell^\star}} \, d\tau\Big)$$

whenever $T > \max(1, M)$.

We are now in the position to prove Proposition 4.5.

PROOF OF PROPOSITION 4.5. Assume first that $2\beta_{\ell^\star} < 1$ and $\beta_{\ell^\star} \leq \beta_0$. For the asymptotic behavior of the first term on the right hand side of (40)) we refer to Lemma 4.4:

$$(47) \qquad \lim_{T\to\infty} \frac{\int_{[0,T]^2} C_0(t-s)^2 \, dt \, ds}{T^{2-2\beta_{\ell^\star}}} = \begin{cases} 0 & \text{if } \beta_{\ell^\star} < \beta_0, \\ \dfrac{C_0(0)^2}{(1-\beta_{\ell^\star})(1-2\beta_{\ell^\star})} & \text{if } \beta_{\ell^\star} = \beta_0. \end{cases}$$

Now, since from (4) we have

$$(48) \qquad \sum_{\ell=0}^{+\infty} (2\ell+1)C_\ell(0)^2 < +\infty,$$

thanks to Lemma 4.7 and Lemma 4.8 we can apply dominate convergence theorem and then Lemma 4.4 to get

$$\lim_{T\to\infty} \sum_{\ell\in\mathcal{I}^\star} \frac{(2\ell+1)}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds$$

$$(49)$$

$$= \sum_{\ell\in\mathcal{I}^\star} \lim_{T\to\infty} \frac{(2\ell+1)}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds = \frac{\sum_{\ell\in\mathcal{I}^\star}(2\ell+1)C_\ell(0)^2}{(1-\beta_{\ell^\star})(1-2\beta_{\ell^\star})}.$$

Let us now prove that

$$(50) \qquad \lim_{T\to\infty} \sum_{\ell\notin\mathcal{I}^\star,\ell\geq 1} \frac{(2\ell+1)}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds = 0.$$

Thanks again to Lemma 4.7 and Lemma 4.8, since (48) holds, we can apply dominated convergence theorem to obtain (50). More precisely, we have to distinguish between some different cases: from Lemma 4.4

$$\lim_{T\to\infty} \sum_{\ell\geq 1:\beta_{\ell^\star}<\beta_\ell<\frac{1}{2}} \frac{(2\ell+1)}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds$$

$$= \sum_{\ell\geq 1:\beta_{\ell^\star}<\beta_\ell<\frac{1}{2}} (2\ell+1) \lim_{T\to\infty} \underbrace{\frac{T^{2-2\beta_\ell}}{T^{2-2\beta_{\ell^\star}}}}_{\to 0} \underbrace{\frac{1}{T^{2-2\beta_\ell}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds}_{\to C_\ell(0)^2(1-\beta_\ell)^{-1}(1-2\beta_\ell)^{-1}} = 0;$$

moreover

$$\lim_{T\to\infty} \sum_{\ell\geq 1:\beta_\ell=\frac{1}{2}} \frac{(2\ell+1)}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds$$

$$= \sum_{\ell\geq 1:\beta_\ell=\frac{1}{2}} (2\ell+1) \lim_{T\to\infty} \underbrace{\frac{T\log T}{T^{2-2\beta_{\ell^\star}}}}_{\to 0} \underbrace{\frac{1}{T\log T} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds}_{\to 2C_\ell(0)^2} = 0;$$

and

$$\lim_{T\to\infty} \sum_{\ell\geq 1:\beta_\ell>\frac{1}{2}} \frac{2\ell+1}{T^{2-2\beta_{\ell\star}}} \int_{[0,T]^2} C_\ell(t-s)^2\, dt\, ds$$

$$= \sum_{\ell\geq 1:\beta_\ell>\frac{1}{2}} (2\ell+1) \lim_{T\to\infty} \underbrace{\frac{T}{T^{2-2\beta_{\ell\star}}}}_{\to 0} \underbrace{\frac{1}{T}\int_{[0,T]^2} C_\ell(t-s)^2\, dt\, ds}_{\to \int_{\mathbb{R}} C_\ell(\tau)^2\, d\tau} = 0.$$

Putting together (47), (49) and (50) we immediately get (41) in Proposition 4.5.

Now assume $2\beta_0 > 1$ and $2\beta_{\ell\star} > 1$. Then obviously $2\beta_\ell > 1$ for each $\ell \in \widetilde{\mathbb{N}}$ and hence, using Lemma 4.7 and 4.10 and Lemma 4.4 as before, we have

$$\lim_{T\to\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2])}{T} = \lim_{T\to\infty} \frac{u^2\phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell+1)\frac{1}{T}\int_{[0,T]^2} C_\ell(t-s)^2\, dt\, ds$$

$$= \frac{u^2\phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell+1) \lim_{T\to\infty} 2\int_0^T \left(1-\frac{\tau}{T}\right)C_\ell^2(\tau)\, d\tau$$

$$= \frac{u^2\phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell+1) \int_{-\infty}^{+\infty} C_\ell(\tau)^2\, d\tau,$$

which is (43). Note that we automatically get

$$\sum_{\ell=0}^{\infty} (2\ell+1) \int_{-\infty}^{+\infty} C_\ell(\tau)^2\, d\tau < +\infty,$$

and the proof is concluded, the remaining cases requiring analogous proofs. $\quad\square$

4.3. *Higher order chaotic projections.* In this subsection we want to investigate the behavior of higher order chaotic components. Let $q \geq 3$, from (29) we can write

$$(51) \qquad \mathrm{Var}\big(\mathcal{M}_T(u)[q]\big) = \frac{\phi(u)^2 H_{q-1}(u)^2}{q!} \int_{[0,T]^2} \int_{\mathbb{S}^2\times\mathbb{S}^2} \Gamma\big(\langle x, y\rangle, t-s\big)^q\, dx\, dy\, dt\, ds.$$

Thanks to (3) we have that

$$\int_{[0,T]^2} \int_{\mathbb{S}^2\times\mathbb{S}^2} \Gamma\big(\langle x, y\rangle, t-s\big)^q\, dx\, dy\, dt\, ds$$

$$= \int_{[0,T]^2} \int_{\mathbb{S}^2\times\mathbb{S}^2} \left(\sum_{\ell=0}^{\infty} C_\ell(t-s)\frac{(2\ell+1)}{4\pi} P_\ell\big(\langle x, y\rangle\big)\right)^q dx\, dy\, dt\, ds$$

$$= \sum_{\ell_1,\ell_2,\ldots,\ell_q=0}^{\infty} \int_{[0,T]^2} \int_{\mathbb{S}^2\times\mathbb{S}^2} C_{\ell_1}(t-s)C_{\ell_2}(t-s)\cdots C_{\ell_q}(t-s)$$

$$\times \frac{2\ell_1+1}{4\pi} P_{\ell_1}\big(\langle x, y\rangle\big)\frac{2\ell_2+1}{4\pi} P_{\ell_2}\big(\langle x, y\rangle\big)\cdots \frac{2\ell_q+1}{4\pi} P_{\ell_q}\big(\langle x, y\rangle\big) dx\, dy\, dt\, ds.$$

Recall the addition formula for spherical harmonics [23], (3.42),

$$\frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell,m}(x)Y_{\ell,m}(y) = P_\ell\big(\langle x, y\rangle\big), \quad x, y \in \mathbb{S}^2,$$

and the definition of generalized Gaunt integral in (22) to write

(52)
$$\int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{2\ell_1 + 1}{4\pi} P_{\ell_1}(\langle x, y \rangle) \frac{2\ell_2 + 1}{4\pi} P_{\ell_2}(\langle x, y \rangle) \cdots \frac{2\ell_q + 1}{4\pi} P_{\ell_q}(\langle x, y \rangle) \, dx \, dy$$
$$= \sum_{m_1 = -\ell_1}^{\ell_1} \cdots \sum_{m_q = -\ell_q}^{\ell_q} (\mathcal{G}_{\ell_1 \cdots \ell_q}^{m_1 \cdots m_q})^2.$$

On the other hand, exploiting again (22)

(53)
$$\int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{2\ell_1 + 1}{4\pi} P_{\ell_1}(\langle x, y \rangle) \frac{2\ell_2 + 1}{4\pi} P_{\ell_2}(\langle x, y \rangle) \cdots \frac{2\ell_q + 1}{4\pi} P_{\ell_q}(\langle x, y \rangle) \, dx \, dy$$
$$= 4\pi \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) \mathcal{G}_{\ell_1 \ell_2 \cdots \ell_q}^{0 \cdots 0}.$$

In particular $\mathcal{G}_{\ell_1 \ldots \ell_q}^{0 \cdots 0} \geq 0$. In order to check (53) recall that

$$Y_{\ell,0}(\theta_x, \varphi_x) = \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell(\cos \theta_x),$$

where (θ_x, φ_x) are the angular coordinates of the point $x \in \mathbb{S}^2$; then, letting o be the *north pole* of the sphere, we have

$$\int_{\mathbb{S}^2 \times \mathbb{S}^2} \sqrt{\frac{2\ell_1 + 1}{4\pi}} P_{\ell_1}(\langle x, y \rangle) \cdots \sqrt{\frac{2\ell_q + 1}{4\pi}} P_{\ell_q}(\langle x, y \rangle) \, dx \, dy$$
$$= 4\pi \int_{\mathbb{S}^2} \sqrt{\frac{2\ell_1 + 1}{4\pi}} P_{\ell_1}(\langle x, o \rangle) \cdots \sqrt{\frac{2\ell_q + 1}{4\pi}} P_{\ell_q}(\langle x, o \rangle) \, dx$$
$$= 4\pi \int_{\mathbb{S}^2} \sqrt{\frac{2\ell_1 + 1}{4\pi}} P_{\ell_1}(\cos \theta_x) \cdots \sqrt{\frac{2\ell_q + 1}{4\pi}} P_{\ell_q}(\cos \theta_x) \, dx = 4\pi \mathcal{G}_{\ell_1 \ldots \ell_q}^{0 \cdots 0}.$$

As a consequence, from (51) we can write

(54) $\quad \mathrm{Var}(\mathcal{M}_T(u)[q]) = \frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \sum_{\ell_1, \ldots, \ell_q = 0}^{\infty} k_{\ell_1 \ldots \ell_q}(T) \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) \mathcal{G}_{\ell_1 \ldots \ell_q}^{0 \cdots 0},$

where

(55) $\quad\quad\quad k_{\ell_1, \ldots, \ell_q}(T) := \int_{[0,T]^2} C_{\ell_1}(t - s) C_{\ell_2}(t - s) \cdots C_{\ell_q}(t - s) \, dt \, ds.$

Note that

$$k_{\ell_1, \ldots, \ell_q}(T) = \mathbb{E}\left[\left(\int_{[0,T]} a_{\ell_1, 0}(t) \cdots a_{\ell_q, 0}(t) \, dt \right)^2 \right].$$

In order to study the asymptotic behavior, as $T \to +\infty$, of (54) we will need the following result whose proof is given in the Appendix.

LEMMA 4.11. *Let $\ell_1, \ldots, \ell_q$ be such that $\beta_{\ell_1} + \cdots + \beta_{\ell_q} < 1$, then*

(56) $\quad \lim_{T \to \infty} \frac{k_{\ell_1 \ldots \ell_q}(T)}{T^{2 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q})}} = \frac{C_{\ell_1}(0) \cdots C_{\ell_q}(0)}{(1 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q}))(2 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q}))}.$

and if $\beta_{\ell_1} + \cdots + \beta_{\ell_q} = 1$

$$\lim_{T \to \infty} \frac{k_{\ell_1 \ldots \ell_q}(T)}{T \log T} = 2 C_{\ell_1}(0) \cdots C_{\ell_q}(0).$$

On the contrary, let $\ell_1, \ldots, \ell_q$ be such that $\beta_{\ell_1} + \cdots + \beta_{\ell_q} > 1$, then

$$(57) \qquad \lim_{T \to \infty} \frac{k_{\ell_1 \ldots \ell_q}(T)}{T} = \int_{-\infty}^{+\infty} C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) \, d\tau.$$

Recall (54).

PROPOSITION 4.12.   *Let $q \geq 3$. If $q\beta_{\ell^\star} < 1$ and $\beta_{\ell^\star} \leq \beta_0$ then*

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T^{2 - q\beta_{\ell^\star}}}$$

$$= \frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!(1 - q\beta_{\ell^\star})(2 - q\beta_{\ell^\star})} \sum_{\ell_1, \ell_2, \ldots, \ell_q \in \mathcal{I}^\star} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0) \right) \mathcal{G}^{0 \cdots 0}_{\ell_1 \ldots \ell_q};$$

on the other hand, if $q\beta_{\ell^\star} < 1$ and $\beta_{\ell^\star} > \beta_0$ then

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T^{2 - q\beta_0}} = \frac{H_{q-1}(u)^2 \phi(u)^2}{(4\pi)^{q-2} q!} \frac{C_0(0)^q}{(1 - q\beta_0)(2 - q\beta_0)}.$$

If $q\beta_{\ell^\star} = 1$ and $\beta_{\ell^\star} \leq \beta_0$ then

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T \log T} = \frac{8\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \sum_{\ell_1, \ell_2, \ldots, \ell_q \in \mathcal{I}^\star} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0) \right) \mathcal{G}^{0 \cdots 0}_{\ell_1 \ldots \ell_q};$$

if $q\beta_{\ell^\star} = 1$ and $\beta_{\ell^\star} > \beta_0$ then

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T^{2 - q\beta_0}} = \frac{H_{q-1}(u)^2 \phi(u)^2}{(4\pi)^{q-2} q!} \frac{C_0(0)^q}{(1 - q\beta_0)(2 - q\beta_0)}.$$

On the other hand, if $q\beta_{\ell^\star} > 1$ and $q\beta_0 > 1$, then

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T} = s_q^2,$$

where

$$s_q^2 := \frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \sum_{\ell_1, \ell_2, \ldots, \ell_q = 0}^{\infty} \mathcal{G}^{0 \cdots 0}_{\ell_1 \ldots \ell_q} \int_{-\infty}^{+\infty} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(\tau) \right) d\tau;$$

moreover if $q\beta_{\ell^\star} > 1$ and $q\beta_0 = 1$, then

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T \log T} = \frac{H_{q-1}(u)^2 \phi(u)^2}{(4\pi)^{q-2} q!} 2 C_0(0)^q;$$

finally if $q\beta_{\ell^\star} > 1$ and $q\beta_0 < 1$, then

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T^{2 - q\beta_0}} = \frac{H_{q-1}(u)^2 \phi(u)^2}{(4\pi)^{q-2} q!} \frac{C_0(0)^q}{(1 - q\beta_0)(2 - q\beta_0)}.$$

In order to prove Proposition 4.12 we will also need the following technical results; the proofs of Lemma 4.13 and Lemma 4.14 are postponed to the Appendix Section 5.5, the proofs of the remaining lemmas are very similar and we omit the details.

LEMMA 4.13. *Let $\varepsilon, M > 0$ be as in* (36). *If there is at least one index $j \in \{1, \ldots, q\}$ such that $\beta_{\ell_j} = 1$ we have for $T > \max(1, M)$,*

$$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T} \le 2C_{\ell_1}(0) \ldots C_{\ell_q}(0)\left(M + \frac{1}{q \min(\beta_0, \beta_{\ell^\star})}\left(\frac{\varepsilon + 1}{(1 + M)^{\min(\beta_0, \beta_{\ell^\star})}}\right)^q\right).$$

LEMMA 4.14. *Let $\varepsilon, M > 0$ be as in* (36) *and $q\beta_{\ell^\star} < 1$.*

• *For $\ell_1, \ldots, \ell_q \in \mathcal{I}^\star$, $\beta_{\ell_j} < 1$ for every j and $T > \max(1, M)$ we have*

$$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T^{2-q\beta_{\ell^\star}}} \le 2C_{\ell_1}(0) \cdots C_{\ell_q}(0)\left(M + \frac{(\varepsilon + 1)^q}{1 - q\beta_{\ell^\star}}\left(1 + \frac{1}{M}\right)^{1-q\beta_{\ell^\star}}\right).$$

• *For $(\ell_1, \ldots, \ell_q) \notin (\mathcal{I}^\star)^q$, $\ell_j \ge 1$, $\beta_{\ell_j} < 1$ for every j and $T > \max(1, M, T_m)$*

$$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T^{2-q\beta_{\ell^\star}}}$$

$$\le 2C_{\ell_1}(0) \ldots C_{\ell_q}(0)$$

$$\times \left(M + \frac{(\varepsilon + 1)^q}{-(\beta_{\ell^{\star\star}} + (q - 1)\beta_{\ell^\star}) + 1}\left(1 + \frac{1}{M}\right)^{-(\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star}) + 1}\mathbf{1}_{\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} < 1}\right.$$

$$+ 2(\varepsilon + 1)^q m(\beta_{\ell^\star})\mathbf{1}_{\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} = 1}$$

$$\left.+ \frac{(\varepsilon + 1)^q}{\beta_{\ell^{\star\star}} + (q - 1)\beta_{\ell^\star} - 1}\left(\frac{1}{M + 1}\right)^{\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} - 1}\mathbf{1}_{\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} > 1}\right).$$

LEMMA 4.15. *Let $\varepsilon, M > 0$ be as in* (36) *and $q\beta_{\ell^\star} = 1$.*

• *For $\ell_1, \ldots, \ell_q \in \mathcal{I}^\star$, $\beta_{\ell_j} < 1$ for every j and $T > \max(1, M, e)$*

$$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T \log T} \le 2C_{\ell_1}(0) \cdots C_{\ell_q}(0)\left(M + \log(e + 1)\right).$$

• *For $(\ell_1, \ldots, \ell_q) \notin (\mathcal{I}^\star)^q$, $\ell_j \ge 1$, $\beta_{\ell_j} < 1$ for every j and $T > \max(1, M, e)$*

$$\frac{k_{\ell_1 \ldots \ell_q}(T)}{T \log T} \le 2C_{\ell_1}(0) \cdots C_{\ell_q}(0)\left(M + (\varepsilon + 1)^q \int_{\mathbb{R}} \left(1 + |\tau|\right)^{-(\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star})} d\tau\right).$$

LEMMA 4.16. *Let $\varepsilon, M > 0$ be as in* (36) *and $q\beta_{\ell^\star} > 1$. Then for $T > \max(1, M)$*

$$\frac{k_{\ell_1 \ell_2 \ldots \ell_q}(T)}{T} \le 2C_{\ell_1}(0) \cdots C_{\ell_q}(0)\left(M + \frac{(1 + M)}{q\beta_{\ell^\star} - 1}\left(\frac{1 + \varepsilon}{(1 + M)^{\beta_{\ell^\star}}}\right)^q\right)$$

for any $\ell_1, \ldots, \ell_q$ such that $\beta_{\ell_j} < 1$, $\ell_j \ge 1$ for every j.

LEMMA 4.17. *Let $\varepsilon, M > 0$ be as in* (36), *and set $U := U(\ell_1, \ldots, \ell_q) = \{j \in \{1, \ldots, q\} : \ell_j = 0\}$. If $\beta_{\ell^\star} \le \beta_0$*

$$k_{\ell_1 \ldots \ell_{q-\#U} 0 \ldots 0}(T) \le 2T C_{\ell_1}(0) \cdots C_{\ell_{q-\#U}}(0) C_0(0)^{\#U}$$

$$\times \left(M + (\varepsilon + 1)^q \int_{[M,T]} \left(1 + |\tau|\right)^{-((q-1)\beta_{\ell^\star} + \beta_0)} d\tau\right),$$

otherwise if $\beta_{\ell^\star} > \beta_0$

$$k_{\ell_1 \ldots \ell_{q-\#U} 0 \ldots 0}(T) \le 2T C_{\ell_1}(0) \cdots C_{\ell_{q-\#U}}(0) C_0(0)^{\#U}\left(M + (\varepsilon + 1)^q \int_{[M,T]} \left(1 + |\tau|\right)^{-q\beta_0} d\tau\right).$$

We are now in the position to prove Proposition 4.12.

PROOF OF PROPOSITION 4.12. Note first that

$$(58) \qquad \sum_{\ell_1,\ldots,\ell_q=0}^{\infty} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} C_{\ell_i}(0) \right) \mathcal{G}_{\ell_1\ldots\ell_q}^{0\cdots0} < +\infty,$$

since the following estimate (see [27], Section 4.2.1):

$$(59) \qquad \mathcal{G}_{\ell_1\cdots\ell_q}^{0\cdots0} \le \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)\cdots(2\ell_{q-1}+1)}{(4\pi)^{q-2}(2\ell_q+1)}}$$

and (4) hold. Now, assume $q\beta_{\ell^\star} < 1$ and $\beta_{\ell^\star} \le \beta_0$ and recall equation (54). Let $J := \{(\ell_1,\ldots,\ell_q) : \text{there is at least one index } j \in \{1,\ldots q\} \text{ such that } \ell_j = 0\}$. Then, thanks to Lemma 4.17, we can apply dominated convergence theorem and then Lemma 4.11 to get

$$\lim_{T\to\infty} \sum_{(\ell_1,\ldots\ell_q)\in J} \frac{k_{\ell_1\ldots\ell_q}(T)}{T^{2-q\beta_{\ell^\star}}} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \right) \mathcal{G}_{\ell_1\ell_2\cdots\ell_q}^{0\cdots0}$$

$$(60)$$
$$= \begin{cases} 0 & \text{if } \beta_{\ell^\star} < \beta_0, \\ \displaystyle\sum_{(\ell_1,\ldots,\ell_q)\in(\mathcal{I}^\star)^q\cap J} \frac{C_{\ell_1}(0)\cdots C_{\ell_q}(0)}{(1-q\beta_{\ell^\star})(2-q\beta_{\ell^\star})} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \right) \mathcal{G}_{\ell_1\cdots\ell_q}^{0\cdots0} & \text{if } \beta_{\ell^\star} = \beta_0. \end{cases}$$

Now, thanks to Lemma 4.13, Lemma 4.14, Lemma 4.17 and (58) together with Lemma 4.11 we have

$$\lim_{T\to\infty} \sum_{(\ell_1,\ldots,\ell_q)\in(\mathcal{I}^\star)^q} \frac{k_{\ell_1\ell_2\cdots\ell_q}(T)}{T^{2-q\beta_{\ell^\star}}} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \right) \mathcal{G}_{\ell_1\ell_2\cdots\ell_q}^{0\cdots0}$$

$$(61)$$
$$= \sum_{(\ell_1,\ldots,\ell_q)\in(\mathcal{I}^\star)^q} \lim_{T\to\infty} \frac{k_{\ell_1\ell_2\cdots\ell_q}(T)}{T^{2-q\beta_{\ell^\star}}} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \right) \mathcal{G}_{\ell_1\ell_2\cdots\ell_q}^{0\cdots0}$$

$$= \sum_{(\ell_1,\ldots,\ell_q)\in(\mathcal{I}^\star)^q} \frac{C_{\ell_1}(0)\cdots C_{\ell_q}(0)}{(1-q\beta_{\ell^\star})(2-q\beta_{\ell^\star})} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \right) \mathcal{G}_{\ell_1\ell_2\cdots\ell_q}^{0\cdots0}.$$

Analogously

$$(62) \qquad \lim_{T\to\infty} \sum_{(\ell_1,\ldots,\ell_q)\notin(\mathcal{I}^\star)^q} \frac{k_{\ell_1\ell_2\cdots\ell_q}(T)}{T^{2-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \right) \mathcal{G}_{\ell_1\ell_2\cdots\ell_q}^{0\cdots0} = 0.$$

Le us check (62). We have

$$\lim_{T\to\infty} \sum_{\substack{(\ell_1,\ell_2,\ldots,\ell_q)\notin\mathcal{I}^\star: \\ \beta_{\ell_1}+\cdots+\beta_{\ell_q}<1,\ell_j\ge1}} \frac{k_{\ell_1\ell_2\cdots\ell_q}(T)}{T^{2-q\beta_{\ell^\star}}} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \right) \mathcal{G}_{\ell_1\ell_2\cdots\ell_q}^{0\cdots0}$$

$$= \sum_{\substack{(\ell_1,\ell_2,\ldots,\ell_q)\notin\mathcal{I}^\star: \\ \beta_{\ell_1}+\cdots+\beta_{\ell_q}<1,\ell_j\ge1}} \lim_{T\to\infty} \underbrace{\frac{k_{\ell_1\ell_2\cdots\ell_q}(T)}{T^{2-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}}}_{\to \frac{C_{\ell_1}(0)\cdots C_{\ell_q}(0)}{(1-(\beta_{\ell_1}+\cdots+\beta_{\ell_q}))(2-(\beta_{\ell_1}+\cdots+\beta_{\ell_q}))}}$$

$$\times \underbrace{\frac{T^{2-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}}{T^{2-q\beta_{\ell^\star}}}}_{\to 0} \left(\prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \right) \mathcal{G}_{\ell_1\ell_2\cdots\ell_q}^{0\cdots0} = 0.$$

Analogously

$$\lim_{T \to \infty} \sum_{\substack{(\ell_1, \ell_2, \ldots, \ell_q) \notin \mathcal{I}^\star: \\ \beta_{\ell_1} + \cdots + \beta_{\ell_q} = 1, \ell_j \geq 1}} \frac{k_{\ell_1 \ell_2 \cdots \ell_q}(T)}{T^{2 - q\beta_{\ell\star}}} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) \mathcal{G}_{\ell_1 \ell_2 \cdots \ell_q}^{0 \cdots 0} = 0$$

and finally

$$\lim_{T \to \infty} \sum_{\substack{(\ell_1, \ell_2, \ldots, \ell_q) \notin \mathcal{I}^\star: \\ \beta_{\ell_1} + \cdots + \beta_{\ell_q} > 1, \ell_j \geq 1}} \frac{k_{\ell_1 \ell_2 \cdots \ell_q}(T)}{T^{2 - q\beta_{\ell\star}}} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) \mathcal{G}_{\ell_1 \ell_2 \cdots \ell_q}^{0 \cdots 0} = 0,$$

so that (62) is proved.

On the other hand, if we assume $q\beta_0 > 1$ and $q\beta_{\ell\star} > 1$, then obviously $q\beta_\ell > 1$ for all $\ell \in \widetilde{\mathbb{N}}$, and $\beta_{\ell_1} + \cdots + \beta_{\ell_q} > 1$ for all $\ell_1, \ldots, \ell_q \in \widetilde{\mathbb{N}}$. Then, thanks to Lemma 4.11, Lemma 4.13, Lemma 4.16 and Lemma 4.17,

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[q]))}{T} = 4\pi \sum_{\ell_1, \ell_2, \ldots, \ell_q = 0}^{\infty} \lim_{T \to \infty} \frac{k_{\ell_1 \ell_2 \cdots \ell_q}(T)}{T} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) \mathcal{G}_{\ell_1 \ell_2 \cdots \ell_q}^{0 \cdots 0}$$

$$= 4\pi \sum_{\ell_1, \ell_2, \ldots, \ell_q = 0}^{\infty} \mathcal{G}_{\ell_1 \cdots \ell_q}^{0 \cdots 0} \int_{-\infty}^{+\infty} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(\tau) \right) d\tau,$$

which concludes the proof. In particular, we have proved that the series on the right hand side of the previous formula converges. The remaining cases can be treated analogously. $\square$

5. Proofs of the main results.

5.1. *Proof of Theorem* 2.2. PROOF. Recall (33). Assume first that $u \neq 0$ and $\beta_0 < \min(2\beta_{\ell\star}, 1)$. For the first chaotic projection, since $\beta_0 < 1$, from Lemma 4.2 we have

$$(63) \qquad \lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T^{2 - \beta_0}} = \frac{2\phi(u)^2 C_0(0)}{(1 - \beta_0)(2 - \beta_0)}.$$

Let $Q \in \{2, 3, \ldots\}$ be such that

$$Q\beta_{\ell\star} > 1.$$

For $q \in \{2, 3, \ldots, Q - 1\}$ we have, from Proposition 4.5 and Proposition 4.12, since $\beta_0 < 2\beta_{\ell\star}$,

$$(64) \qquad \lim_{T \to \infty} \sum_{q=2}^{Q-1} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{T^{2 - \beta_0}} = \sum_{q=2}^{Q-1} \lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{T^{2 - \beta_0}} = 0.$$

Let us now prove that

$$(65) \qquad \lim_{T \to \infty} \sum_{q=Q}^{+\infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{T^{2 - \beta_0}} = 0.$$

Recall (54); thanks to (59) we can write for any $\ell_1, \ldots, \ell_q \geq 0$

$$\frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \left(\prod_{i=1}^{q} \sqrt{\frac{2\ell_i + 1}{4\pi}} \right) \mathcal{G}_{\ell_1 \cdots \ell_q}^{0 \cdots 0}$$

$$\leq \frac{(4\pi)^2 H_{q-1}(u)^2 \phi(u)^2}{q!} \left(\prod_{i=1}^{q} \frac{2\ell_i + 1}{4\pi} \right) =: b_q(\ell_1, \ldots, \ell_q; u).$$

For $q \geq Q$ we have of course $q\beta_{\ell^\star} > 1$. Let $\varepsilon, M > 0$ be as in (36). From Lemma 4.16 we have for $T > \max(1, M)$

$$\sum_{\substack{\ell_1, \ell_2, \ldots, \ell_q \geq 1 \\ \beta_{\ell_1}, \ldots, \beta_{\ell_q} < 1}} b_q(\ell_1, \ldots, \ell_q; u) \frac{k_{\ell_1, \ldots, \ell_q}(T)}{T^{2-\beta_0}}$$

$$\leq \sum_{\substack{\ell_1, \ell_2, \ldots, \ell_q \geq 1 \\ \beta_{\ell_1}, \ldots, \beta_{\ell_q} < 1}} b_q(\ell_1, \ldots, \ell_q; u) \frac{k_{\ell_1, \ldots, \ell_q}(T)}{T}$$

$$(66) \quad \leq 2 \sum_{\substack{\ell_1, \ell_2, \ldots, \ell_q \geq 1 \\ \beta_{\ell_1}, \ldots, \beta_{\ell_q} < 1}} b_q(\ell_1, \ldots, \ell_q; u) C_{\ell_1}(0) \cdots C_{\ell_q}(0) \Big(M + \frac{(1+M)}{q\beta_{\ell^\star} - 1} \Big(\frac{1+\varepsilon}{(1+M)^{\beta_{\ell^\star}}} \Big)^q \Big)$$

$$\leq 2 \sum_{\ell_1, \ell_2, \ldots, \ell_q \geq 0} b_q(\ell_1, \ldots, \ell_q; u) C_{\ell_1}(0) \cdots C_{\ell_q}(0) \Big(M + \frac{(1+M)}{q\beta_{\ell^\star} - 1} \Big(\frac{1+\varepsilon}{(1+M)^{\beta_{\ell^\star}}} \Big)^q \Big)$$

$$= 2 \frac{(4\pi)^2 H_{q-1}(u)^2 \phi(u)^2}{q!} \Big(M + \frac{(1+M)}{q\beta_{\ell^\star} - 1} \Big(\frac{1+\varepsilon}{(1+M)^{\beta_{\ell^\star}}} \Big)^q \Big),$$

recalling (8). The following estimate holds (see, e.g., [15], Proposition 3): for every $q \geq 0$ and $x \in \mathbb{R}$

$$\big| e^{-x^2/4} H_q(x) \big| \leq c \sqrt{q!} q^{-1/12},$$

hence the series whose term is the right hand side of (66) is finite, that is,

$$(67) \qquad \sum_{q=Q}^{+\infty} \frac{H_{q-1}(u)^2 \phi(u)^2}{q!} \Big(M + \frac{(1+M)}{q\beta_{\ell^\star} - 1} \Big(\frac{1+\varepsilon}{(1+M)^{\beta_{\ell^\star}}} \Big)^q \Big) < +\infty,$$

as soon as M is sufficiently large. Repeating the same argument as for (66), using Lemma 4.13 and Lemma 4.17, and thanks to (67), we can apply dominated convergence theorem and then Proposition 4.5 and Proposition 4.12 to get

$$\lim_{T \to \infty} \sum_{q=Q}^{+\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T^{2-\beta_0}} = \sum_{q=Q}^{+\infty} \lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[q])}{T^{2-\beta_0}} = 0,$$

which is (65). Putting together (63), (64) and (65) we finally find that

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T^{2-\beta_0}} = \frac{2\phi(u)^2 C_0(0)}{(1-\beta_0)(2-\beta_0)} =: K_0(u).$$

Note that, if $u = 0$, then $\mathcal{M}_T(u)[2] \equiv 0$ and the sufficient condition in order to have (65) is $\beta_0 < \min(3\beta_{\ell^\star}, 1)$.

This implies that, if either $u \neq 0$ and $\beta_0 < \min(2\beta_{\ell^\star}, 1)$ or $u = 0$ and $\beta_0 < \min(3\beta_{\ell^\star}, 1)$, then

$$\widetilde{\mathcal{M}}_T(u) = \frac{\mathcal{M}_T(u)[1]}{\sqrt{K_0(u)} T^{1-\beta_0/2}} + o_{\mathbb{P}}(1).$$

Consequently, since $\mathcal{M}_T(u)[1]$ is Gaussian for any $T > 0$, it is clear that the asymptotic distribution of $\widetilde{\mathcal{M}}_T(u)$ is standard Gaussian. $\quad \square$

5.2. *Proof of Theorem* 2.7. We will need the following well-known result.

THEOREM 5.1 ([13, 38]). *Let $\xi(t)$, $t \in \mathbb{R}$, be a real measurable mean-square continuous stationary Gaussian process with mean $\mathbb{E}[\xi(t)]$ and covariance function $\rho(t - s) = \rho(|t - s|) = \mathrm{Cov}(\xi(t), \xi(s))$. Moreover, assume that*

$$(68) \qquad \rho(t - s) = \frac{L(|t - s|)}{|t - s|^\beta} \quad \text{with } 0 < \beta < 1,$$

where L is a slowly varying function. Let $F : \mathbb{R} \to \mathbb{R}$ be a Borel function such that $\mathbb{E}[F(N)^2] < +\infty$, where N is a standard Gaussian random variable. Then it is a well-known fact that can be expanded as follows

$$F(\xi) = \sum_{k=0}^{\infty} \frac{b_k}{k!} H_k(\xi) \quad \text{where } b_k = \int_{\mathbb{R}} F(\xi) H_k(\xi) \phi(\xi) \, d\xi.$$

Assume there exists an integer r, the so-called Hermitian rank, such that $b_0 = b_1 = \cdots = b_{r-1} = 0$ and $b_r \neq 0$. Then, if $\beta \in (0, 1/r)$, we have that the finite-dimensional distributions of the random process

$$X_T(s) = \frac{1}{T^{1-\beta r/2} L(T)^{r/2}} \int_0^{Ts} \left[F\big(\xi(t)\big) - b_0 \right] dt, \quad 0 \leq s \leq 1,$$

converge weakly, as $T \to \infty$, to the ones of the Rosenblatt process of order r, that is

$$X_\beta(s) := \frac{b_r}{r!} \int_{(\mathbb{R}^r)'} \frac{e^{i(\lambda_1 + \cdots + \lambda_r)s} - 1}{i(\lambda_1 + \cdots + \lambda_r)} \frac{W(d\lambda_1) \cdots W(d\lambda_r)}{|\lambda_1 \cdots \lambda_r|^{(1-\beta)/2}} \, dt, \quad 0 \leq s \leq 1,$$

where W is a complex Gaussian white noise.

PROOF OF THEOREM 2.7. Recall that $u \neq 0$ and $2\beta_{\ell^\star} < \min(\beta_0, 1)$. From Lemma 4.2 we have

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[1])}{T^{2-2\beta_{\ell^\star}}} = 0.$$

Moreover, thanks to Proposition 4.12, as for the proof of Theorem 2.2 (in particular (65)), we have, as $T \to +\infty$,

$$\lim_{T \to \infty} \frac{\sum_{q \geq 3} \mathrm{Var}(\mathcal{M}_T(u)[q])}{T^{2-2\beta_{\ell^\star}}} = 0,$$

so that, recalling also Proposition 4.5,

$$(69) \qquad \frac{\mathcal{M}_T(u)}{T^{1-\beta_{\ell^\star}}} = \frac{\mathcal{M}_T(u)[2]}{T^{1-\beta_{\ell^\star}}} + o_{\mathbb{P}}(1).$$

Moreover, since in $L^2(\Omega)$ we have the following equality:

$$(70) \qquad \mathcal{M}_T(u)[2] = \frac{J_2(u)}{2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^T H_2\big(a_{\ell m}(t)\big) \, dt,$$

it holds that

$$(71) \qquad \frac{\mathcal{M}_T(u)[2]}{T^{1-\beta_{\ell^\star}}} = \frac{1}{T^{1-\beta_{\ell^\star}}} \sum_{\ell \in \mathcal{I}^\star} \sum_{m=-\ell}^{\ell} \frac{J_2(u)}{2} \int_0^T H_2\big(a_{\ell,m}(t)\big) \, dt + o_{\mathbb{P}}(1).$$

Indeed, arguing exactly as in the Proposition 4.5, we have that

$$\lim_{T \to \infty} \mathbb{E}\left[\left(\frac{\mathcal{M}_T(u)[2]}{T^{1-\beta_{\ell\star}}} - \frac{1}{T^{1-\beta_{\ell\star}}} \sum_{\ell \in \mathcal{I}^\star} \sum_{m=-\ell}^{\ell} \frac{J_2(u)}{2} \int_0^T H_2(a_{\ell,m}(t)) \, dt \right)^2 \right]$$

$$= \lim_{T \to \infty} \frac{J_2(u)^2}{2T^{2-2\beta_{\ell\star}}} \sum_{\ell \notin \mathcal{I}^\star} (2\ell+1) \int_0^T \int_0^T C_\ell(t-s)^2 \, dt \, ds = 0.$$

From (69) and (71), in order to understand the asymptotic distribution of $\mathcal{M}_T(u)$, it suffices to investigate the leading term on the right hand side of (71). Recall Condition 1.3, for $\ell \in \mathcal{I}^\star$ we have that

$$C_\ell(\tau) = \frac{G_\ell(\tau)}{(1+|\tau|)^{\beta_{\ell\star}}},$$

where in particular G_ℓ is a slowly varying function. Hence, setting $\xi(t) = a_{\ell,m}(t)$, we automatically have that $\rho = \rho_\ell = C_\ell$, $L = L_\ell = G_\ell$ and, as a consequence, that

$$X_T^{\ell,m} := \frac{1}{C_\ell(0)T^{1-\beta_{\ell\star}}} \int_0^T \frac{J_2(u)}{2} H_2(a_{\ell,m}(t)) \, dt \xrightarrow{d} \frac{J_2(u)}{2a(\beta_{\ell\star})} X_{m;\beta_{\ell\star}} \quad \text{as } T \to \infty,$$

for all $m = -\ell, \dots, \ell$, where, for each m, $X_{m;\beta_{\ell\star}}$ is a standard Rosenblatt random variable (16) of parameter $\beta_{\ell\star}$. Moreover, since the $X_T^{\ell,m}$ are all independent for each T we have that

$$\widetilde{\mathcal{M}}_T(u) = \sqrt{\frac{T^{1-\beta_{\ell\star}}}{\mathrm{Var}(\mathcal{M}_T(u)[2])}} \sum_{\ell \in \mathcal{I}^\star} C_\ell(0) \sum_{m=-\ell}^{\ell} \frac{\int_0^T \frac{J_2(u)}{2} H_2(a_{\ell,m}(t)) \, dt}{C_\ell(0)T^{1-\beta_{\ell\star}}} + o_{\mathbb{P}}(1)$$

$$\xrightarrow{d} \left(\frac{J_2(u)^2}{2} \sum_{\ell \in \mathcal{I}^\star} \frac{(2\ell+1)C_\ell(0)^2}{(1-\beta_{\ell\star})(1-2\beta_{\ell\star})} \right)^{-1/2} \sum_{\ell \in \mathcal{I}^\star} C_\ell(0) \sum_{m=-\ell}^{\ell} \frac{J_2(u)}{2a(\beta_{\ell\star})} X_{m;\beta_\ell^\star}$$

$$= \sum_{\ell \in \mathcal{I}^\star} \frac{C_\ell(0)}{\sqrt{v^\star}} \sum_{m=-\ell}^{\ell} X_{m;\beta_{\ell\star}},$$

where

$$v^\star = a(\beta_{\ell\star})^2 \sum_{\ell \in \mathcal{I}^\star} \frac{2(2\ell+1)C_\ell(0)^2}{(1-\beta_{\ell\star})(1-2\beta_{\ell\star})},$$

and the proof is concluded. $\square$

5.3. *Proof of Theorem* 2.9. First of all assume that $3\beta_{\ell\star} < \min(1, \beta_0)$. Since we are in the case where $u = 0$, we have that all even chaotic projections vanish and hence that

$$\mathrm{Var}(\mathcal{M}_T) = \mathrm{Var}(\mathcal{M}_T[1]) + \mathrm{Var}(\mathcal{M}_T[3]) + \sum_{q \geq 2} \mathrm{Var}(\mathcal{M}_T[2q+1]),$$

where we used the notation $\mathcal{M}_T(0) =: \mathcal{M}_T$. As a consequence, as in the proof of Theorem 2.2, we have

$$\lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T)}{T^{2-3\beta_{\ell\star}}}$$

$$= \lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T[1])}{T^{2-3\beta_{\ell\star}}} + \lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T[3])}{T^{2-3\beta_{\ell\star}}} + \sum_{q \geq 2} \lim_{T \to \infty} \frac{\mathrm{Var}(\mathcal{M}_T[2q+1])}{T^{2-3\beta_{\ell\star}}}.$$

Now,

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T[1])}{T^{2-3\beta_{\ell\star}}} = 0;$$

while from Proposition 4.12 we know that

$$\lim_{T \to \infty} \frac{\text{Var}(\mathcal{M}_T[3])}{T^{2-3\beta_{\ell\star}}} = \frac{2}{3!(1 - 3\beta_{\ell\star})(2 - 3\beta_{\ell\star})} \sum_{\ell_1,\ell_2,\ell_3 \in \mathcal{I}^\star} \left(\prod_{i=1}^{3} \sqrt{\frac{2\ell_i + 1}{4\pi}} C_{\ell_i}(0) \right) \mathcal{G}^{000}_{\ell_1 \ell_2 \ell_3}$$

$$=: K_3.$$

Moreover

$$\lim_{T \to \infty} \frac{\sum_{q \geq 2} \text{Var}(\mathcal{M}_T[2q + 1])}{T^{2-3\beta_{\ell\star}}} = 0,$$

which of course implies that

$$\widetilde{\mathcal{M}}_T(u) = \frac{\mathcal{M}_T(u)[3]}{\sqrt{K_3} T^{1 - \frac{3}{2}\beta_{\ell\star}}} + o_{\mathbb{P}}(1),$$

as claimed.

5.4. *Proof of Theorem* 2.12. The result below is just [31], Theorem 6.3.1, restated for our framework as a lemma. Recall the definition of cumulants for a random variable [23], Section 4.3.

LEMMA 5.2. *Assume that the functional $\widetilde{\mathcal{M}}_T(u)$ in* (13) *satisfies the following conditions*:

(a) *For each $q \geq 1$, $\text{Var}(\widetilde{\mathcal{M}}_T(u)[q]) \to \sigma_q^2$, as $T \to \infty$ and for some $\sigma_q^2 \geq 0$;*
(b) $\sigma^2 := \sum_{q=1}^{\infty} \sigma_q^2 < +\infty$;
(c) *For each $q \geq 2$, $\text{Cum}_4(\widetilde{\mathcal{M}}_T(u)[q]) \to 0$, as $T \to \infty$;*
(d) $\lim_{Q \to \infty} \sup_{T > 0} \sum_{q=Q+1}^{\infty} \text{Var}(\widetilde{\mathcal{M}}_T(u)[q]) = 0$.

Then $\widetilde{\mathcal{M}}_T(u) \overset{d}{\to} Z$, as $T \to \infty$, where $Z \sim \mathcal{N}(0, \sigma^2)$.

We will use Lemma 5.2 to prove Theorem 2.12. Let us first focus on Condition (c).

PROPOSITION 5.3. *Assume $\beta_0 = 1$. If either $u \neq 0$ and $2\beta_{\ell\star} > 1$ or $u = 0$ and $3\beta_{\ell\star} > 1$ we have*

$$\widetilde{\mathcal{M}}_T(u)[q] \overset{d}{\to} Z \quad as \ T \to \infty,$$

where $Z \sim \mathcal{N}(0, \sigma_q^2)$ is a Gaussian random variable whose variance is given by

$$\sigma_q^2 := \frac{s_q^2}{\sum_{k=1}^{+\infty} s_k^2} \in [0, +\infty),$$

where the sequence $\{s_k^2, k \geq 1\}$ is defined in Theorem 2.12.

REMARK 5.4. Note that some of the chaoses might converge to a degenerate Gaussian (that is, with zero expected value and variance).

PROOF OF PROPOSITION 5.3. It suffices to check [31], Theorem 5.2.7, that the fourth cumulant goes to zero as $T \to +\infty$, that is,

$$\lim_{T \to \infty} \mathrm{Cum}_4(\widetilde{\mathcal{M}}_T(u)[q]) = 0.$$

Recall (26). For any $1 \leq \alpha \leq q - 1$, we have (see, e.g., [23], Section 4.3, in particular Section 4.3.1)

$$\mathrm{Cum}_4\left(\int_0^T \int_{\mathbb{S}^2} H_q(Z(x,t)) \, dx \, dt\right)$$

$$= \int_{[0,T]^4} \int_{(\mathbb{S}^2)^4} dx_1 \, dx_2 \, dx_3 \, dx_4 \, dt_1 \, dt_2 \, dt_3 \, dt_4$$

$$\times \mathrm{Cum}(H_q(Z(x_1,t_1)), H_q(Z(x_2,t_2)) H_q(Z(x_3,t_3)) H_q(Z(x_4,t_4)))$$

$$\leq c \int_{[0,T]^4} \int_{(\mathbb{S}^2)^4} |\mathbb{E}[Z(x_1,t_1)Z(x_2,t_2)]|^{q-\alpha} |\mathbb{E}[Z(x_2,t_2)Z(x_3,t_3)]|^{\alpha}$$

$$\times |\mathbb{E}[Z(x_3,t_3)Z(x_4,t_4)]|^{q-\alpha} |\mathbb{E}[Z(x_4,t_4)Z(x_1,t_1)]|^{\alpha} \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dt_1 \, dt_2 \, dt_3 \, dt_4.$$

For x, y positive numbers, it holds that

$$x^{\alpha} y^{\beta} \leq x^{\alpha+\beta} + y^{\alpha+\beta},$$

as a consequence,

$$\mathrm{Cum}_4\left(\int_0^T \int_{\mathbb{S}^2} H_q(Z(x,t)) \, dx \, dt\right)$$

$$\leq c \int_{[0,T]^4} \int_{(\mathbb{S}^2)^4} |\mathbb{E}[Z(x_1,t_1)Z(x_2,t_2)]|^{q-\alpha} |\mathbb{E}[Z(x_2,t_2)Z(x_3,t_3)]|^{\alpha}$$

$$\times |\mathbb{E}[Z(x_3,t_3)Z(x_4,t_4)]|^q \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dt_1 \ldots dt_4$$

$$= c \int_{[0,T]^4} \int_{(\mathbb{S}^2)^4} |\Gamma(\langle x_1, x_2 \rangle, t_2 - t_1)|^{q-\alpha} |\Gamma(\langle x_2, x_3 \rangle, t_3 - t_2)|^{\alpha}$$

$$\times |\Gamma(\langle x_3, x_4 \rangle, t_4 - t_3)|^q \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dt_1 \ldots dt_4$$

$$\leq cT \int_{(\mathbb{S}^2)^4} \int_{[-T,T]} |\Gamma(\langle x_1, x_2 \rangle, s_1)|^{q-\alpha} \, ds_1 \int_{[-T,T]} |\Gamma(\langle x_2, x_3 \rangle, s_2)|^{\alpha} \, ds_2$$

$$\times \int_{[-T,T]} |\Gamma(\langle x_3, x_4 \rangle, s_3)|^q \, ds_3 \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

$$\leq cT \int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{(2\ell+1)}{4\pi} C_\ell(0) \left|\frac{C_\ell(s_1)}{C_\ell(0)}\right|^{q-\alpha} ds_1 \int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{(2\ell+1)}{4\pi} C_\ell(0) \left|\frac{C_\ell(s_2)}{C_\ell(0)}\right|^{\alpha} ds_2$$

$$\times \int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{(2\ell+1)}{4\pi} C_\ell(0) \left|\frac{C_\ell(s_3)}{C_\ell(0)}\right|^q ds_3,$$

where for the last inequality we used Jensen inequality, recalling (8). For $k = 1, \ldots, q-1$ we have that, as $T \to +\infty$,

$$\int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{(2\ell+1)}{4\pi} C_\ell(0) \left|\frac{C_\ell(\tau)}{C_\ell(0)}\right|^k d\tau = O\left(T^{1-k\beta_{\ell^\star}}(1 + \mathbf{1}_{k\beta_{\ell^\star}=1} \log T)\right)$$

whereas for $k = q$ (since $q\beta_{\ell^\star} > 1$)

$$\int_{[-T,T]} \sum_{\ell=0}^{+\infty} \frac{(2\ell+1)}{4\pi} C_\ell(0) \left|\frac{C_\ell(\tau)}{C_\ell(0)}\right|^q d\tau = O(1).$$

Hence, as $T \to +\infty$,

$$\text{Cum}_4\left(\int_0^T \int_{\mathbb{S}^2} H_q(Z(x,t))\, dx\, dt\right) = O\big(T^{3-q\beta_{\ell^\star}}(1 + \delta^1_{(q-\alpha)\beta_{\ell^\star}} \log T)(1 + \delta^1_{\alpha\beta_{\ell^\star}} \log T)\big).$$

From Proposition 4.12 we know that $\text{Var}(\mathcal{M}_T(u)) \sim T \sum_{k=1}^{+\infty} s_k^2$ thus as $T \to +\infty$

$$\text{Cum}_4\left(\frac{\mathcal{M}_T(u)[q]}{\sqrt{\text{Var}(\mathcal{M}_T(u))}}\right) = O\big(T^{1-q\beta_{\ell^\star}}(1 + \mathbf{1}_{(q-\alpha)\beta_{\ell^\star}=1} \log T)(1 + \mathbf{1}_{\alpha\beta_{\ell^\star}=1} \log T)\big)$$

so that

$$\lim_{T\to\infty} \text{Cum}_4\left(\frac{\mathcal{M}_T(u)[q]}{\sqrt{\text{Var}(\mathcal{M}_u(T)[q])}}\right) = 0$$

and the proof is concluded. $\square$

We are now in the position to prove Theorem 2.12.

PROOF OF THEOREM 2.12. Here we have $\beta_0 = 1$. From Lemma 4.2 and Condition 1.3 we know that

$$\lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} = \phi(u)^2 \int_{-\infty}^{+\infty} C_0(\tau)\, d\tau > 0.$$

Assume first that $u \neq 0$ and $2\beta_{\ell^\star} > 1$, then, using Propositions 4.5 and 4.12 we have that

$$\lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u)[2])}{T} = \frac{u^2 \phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell+1) F_\ell,$$

and for $q \geq 3$, since of course $q\beta_{\ell^\star} > 1$,

$$\lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{T} = s_q^2,$$

where we recall that

$$s_q^2 = \frac{4\pi H_{q-1}(u)^2 \phi(u)^2}{q!} \sum_{\ell_1, \ell_2, \ldots, \ell_q=0}^{\infty} \mathcal{G}^{0\cdots0}_{\ell_1\cdots\ell_q} \prod_{i=1}^q \sqrt{\frac{2\ell_i+1}{4\pi}} \int_{-\infty}^{+\infty} C_{\ell_1}(\tau) C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau)\, d\tau.$$

As in the proof of (65), thanks to dominated convergence theorem, we can write

$$\lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u))}{T} = \lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} + \lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u)[2])}{T}$$

$$+ \sum_{q\geq 3} \lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u)[q])}{T}$$

$$= \phi(u)^2 \int_{-\infty}^{+\infty} C_0(\tau)\, d\tau + \frac{u^2 \phi(u)^2}{2} \sum_{\ell=0}^{\infty} (2\ell+1) F_\ell + \sum_{q\geq 3} s_q^2.$$

Now assume that $u = 0$ and $3\beta_{\ell^\star} > 1$, then analogously

$$\lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u))}{T} = \lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u)[1])}{T} + \sum_{q\geq 1} \lim_{T\to\infty} \frac{\text{Var}(\mathcal{M}_T(u)[2q+1])}{T}$$

$$= \phi(u)^2 \int_{-\infty}^{+\infty} C_0(\tau)\, d\tau + \sum_{q\geq 1} s_{2q+1}^2.$$

In order to prove convergence in distribution to a Gaussian random variable, we are going to check the four conditions of Lemma 5.2. Conditions (a) and (b) are verified thanks to Theorem 2.12 and Proposition 4.12. Condition (c) of Lemma 5.2 is immediately verified by Proposition 5.3, thanks to the fourth moment theorem [33], Theorem 1. Let us now check Condition (d). Recall that

$$\phi(u)^2 \int_{\mathbb{R}} C_0(\tau)\, d\tau > 0.$$

Fix $\varepsilon > 0$ such that

$$\phi(u)^2 \int_{\mathbb{R}} C_0(\tau)\, d\tau - \varepsilon > 0.$$

Then we have for some $T_\varepsilon > 0$

$$\frac{\operatorname{Var}(\mathcal{M}_T(u)[1])}{T} \geq \phi(u)^2 \int_{\mathbb{R}} C_0(\tau)\, d\tau - \varepsilon,$$

for every $T > T_\varepsilon$. Hence

$$\sup_{T > T_\varepsilon} \sum_{q=Q}^{\infty} \frac{\operatorname{Var}(\mathcal{M}_T(u)[q])}{\operatorname{Var}(\mathcal{M}_T(u))} \leq \frac{\sup_{T > T_\varepsilon} \sum_{q=Q}^{\infty} \frac{\operatorname{Var}(\mathcal{M}_T(u)[q])}{T}}{\phi(u)^2 \int_{\mathbb{R}} C_0(\tau)\, d\tau - \varepsilon} \to 0,$$

as $Q \to \infty$. As a consequence, Condition (d) of Lemma 5.2 is satisfied and the proof is concluded. $\square$

5.5. *Proof of Proposition* 3.1.   PROOF.   From Theorem 2.7 we have

$$(72) \qquad \lim_{T \to \infty} \frac{\operatorname{Var}(\mathcal{M}_T(u))}{T^{2-2\beta_{\ell^\star}}} = \frac{u^2 \phi(u)^2}{2(1 - 2\beta_{\ell^\star})(1 - \beta_{\ell^\star})} (2\ell^\star + 1) C_{\ell^\star}(0)^2.$$

Let us study the variance of $m_{T;\ell^\star}(u)$.

$$\begin{aligned}
\operatorname{Var}(m_{T;\ell^\star}(u)) &= \frac{u^2 \phi(u/\sigma_{\ell^\star})^2}{2\sigma_{\ell^\star}^2} \int_{(\mathbb{S}^2)^2 \times [0,T]^2} \frac{C_{\ell^\star}(t-s)^2}{C_{\ell^\star}(0)^2} P_{\ell^\star}(\langle x, y \rangle)^2\, dx\, dy\, dt\, ds \\
&= \frac{u^2 \phi(u/\sigma_{\ell^\star})^2}{2\sigma_{\ell^\star}^2} \int_{[0,T]^2} \frac{(4\pi)^2}{2\ell^\star + 1} \frac{C_{\ell^\star}(t-s)^2}{C_{\ell^\star}(0)^2}\, dt\, ds \\
&= \frac{u^2 \phi(u/\sigma_{\ell^\star})^2}{2\sigma_{\ell^\star}^2} 2T \int_{[0,T]} \frac{(4\pi)^2}{2\ell^\star + 1} \left(1 - \frac{\tau}{T}\right) \frac{C_{\ell^\star}(\tau)^2}{C_{\ell^\star}(0)^2}\, d\tau.
\end{aligned}$$

From Proposition 4.4

$$\lim_{T \to \infty} \frac{2T}{T^{2-2\beta_{\ell^\star}}} \int_0^T \left(1 - \frac{\tau}{T}\right) C_{\ell^\star}^2(\tau)\, d\tau = \frac{C_{\ell^\star}(0)^2}{(1 - \beta_{\ell^\star})(1 - 2\beta_{\ell^\star})}$$

so that

$$(73) \qquad \lim_{T \to \infty} \frac{\operatorname{Var}(m_{T;\ell^\star}(u))}{T^{2-2\beta_{\ell^\star}}} = \frac{u^2 \phi(u/\sigma_{\ell^\star})^2}{2\sigma_{\ell^\star}^2} \frac{(4\pi)^2}{2\ell^\star + 1} \frac{1}{(1 - \beta_{\ell^\star})(1 - 2\beta_{\ell^\star})}.$$

Let us now compute the covariance between $\mathcal{M}_T(u)$ and $m_{T;\ell^\star}(u)$: by orthogonality of Wiener chaoses

$$\begin{aligned}
&\operatorname{Cov}(\mathcal{M}_T(u), m_{T;\ell^\star}(u)) \\
&= \operatorname{Cov}(\mathcal{M}_T(u)[2], m_{T;\ell^\star}(u))
\end{aligned}$$

$$= \frac{J_2(u) J_2(\frac{u}{\sigma_{\ell^\star}})}{4} \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \mathbb{E}\left[H_2(Z(x,t)) H_2\left(\frac{Z_{\ell^\star}(y,s)}{\sigma_{\ell^\star}} \right) \right] dx\, dy\, dt\, ds$$

$$= \frac{J_2(u) J_2(\frac{u}{\sigma_{\ell^\star}})}{2} \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \mathbb{E}\left[Z(x,t) \frac{Z_{\ell^\star}(y,s)}{\sigma_{\ell^\star}} \right]^2 dx\, dy\, dt\, ds$$

$$= \frac{J_2(u) J_2(\frac{u}{\sigma_{\ell^\star}})}{2} \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} \mathbb{E}\left[Z_{\ell^\star}(x,t) \frac{Z_{\ell^\star}(y,s)}{\sigma_{\ell^\star}} \right]^2 dx\, dy\, dt\, ds$$

$$= \frac{J_2(u) J_2(\frac{u}{\sigma_{\ell^\star}})}{2\sigma_{\ell^\star}^2} \int_{[0,T]^2} \int_{\mathbb{S}^2 \times \mathbb{S}^2} C_{\ell^\star}(t-s)^2 \left(\frac{2\ell^\star + 1}{4\pi} \right)^2 P_{\ell^\star}(\langle x, y \rangle)^2 \, dx\, dy\, dt\, ds$$

$$= (2\ell^\star + 1) \frac{J_2(u) J_2(\frac{u}{\sigma_{\ell^\star}})}{2\sigma_{\ell^\star}^2} \int_{[0,T]^2} C_{\ell^\star}(t-s)^2 \, dt\, ds.$$

As before

$$(74) \qquad \lim_{T \to \infty} \frac{\mathrm{Cov}(\mathcal{M}_T(u), m_{T;\ell^\star}(u))}{T^{2-2\beta_{\ell^\star}}} = (2\ell^\star + 1) \frac{J_2(u) J_2(\frac{u}{\sigma_{\ell^\star}})}{2\sigma_{\ell^\star}^2} \frac{C_{\ell^\star}(0)^2}{(1 - \beta_{\ell^\star})(1 - 2\beta_{\ell^\star})}.$$

Plugging (72) and (73) into (74) we get

$$\lim_{T \to \infty} \mathrm{Corr}(\mathcal{M}_T(u), m_{T;\ell^\star}(u)) = \lim_{T \to \infty} \frac{\mathrm{Cov}(\mathcal{M}_T(u), m_{T;\ell^\star}(u))}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u))\, \mathrm{Var}(m_{T;\ell^\star}(u))}} = 1,$$

that concludes the proof. $\quad\square$

APPENDIX: PROOFS OF TECHNICAL LEMMAS

From Remark 4.3

$$(75) \qquad \begin{aligned} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt\, ds &= 2T \int_{[0,T]} \left(1 - \frac{\tau}{T} \right) C_\ell(\tau)^2 \, d\tau \\ &\leq 2T \left(\int_{[0,M]} C_\ell(\tau)^2 \, d\tau + \int_{[M,T]} C_\ell(\tau)^2 \, d\tau \right) \\ &\leq 2T C_\ell(0)^2 \left(M + (\varepsilon + 1)^2 \int_{[M,T]} g_{\beta_\ell}(\tau)^2 \, d\tau \right). \end{aligned}$$

PROOF OF LEMMA 4.4. If $\beta_\ell \in (1/2, 1]$ then from Remark 4.3, thanks to dominated convergence theorem,

$$\lim_{T \to \infty} \frac{1}{T} \int_{[0,T]^2} C_\ell^2(t-s) \, dt\, ds = \lim_{T \to \infty} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T} \right) C_\ell(\tau)^2 \, d\tau = \int_{\mathbb{R}} C_\ell(\tau)^2 \, d\tau.$$

Now assume that $2\beta_\ell < 1$ and recall Condition 1.3, then as in Remark 4.3 and the proof of Lemma 4.2 we fix $\varepsilon > 0$ and we know there exists $M > 0$ such that, for $\tau > M$,

$$\sup_\ell \left| \frac{G_\ell(\tau)}{C_\ell(0)} - 1 \right| < \varepsilon$$

(as in (36)), so that

$$\int_{[0,T]^2} C_\ell^2(t-s) \, dt\, ds$$

$$= 2T \int_0^T \left(1 - \frac{\tau}{T} \right) C_\ell^2(\tau) \, d\tau$$

$$(76) \quad = 2T \int_0^M \left(1 - \frac{\tau}{T}\right) C_\ell^2(\tau) \, d\tau + 2T \int_M^T \left(1 - \frac{\tau}{T}\right) C_\ell^2(\tau) \, d\tau.$$

$$= O(1) + 2T \int_0^M C_\ell^2(\tau) \, d\tau + 2T C_\ell(0)^2 \int_M^T \left(1 - \frac{\tau}{T}\right) \left(\frac{G_\ell(\tau)}{C_\ell(0)} - 1\right)^2 (1 + \tau)^{-2\beta_\ell} \, d\tau$$

$$+ 4T C_\ell(0)^2 \int_M^T \left(1 - \frac{\tau}{T}\right) \left(\frac{G_\ell(\tau)}{C_\ell(0)} - 1\right) (1 + \tau)^{-2\beta_\ell} \, d\tau$$

$$+ 2T C_\ell(0)^2 \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-2\beta_\ell} \, d\tau.$$

For the second and the last summands of (76) it is straightforward to check that

$$\lim_{T \to \infty} \frac{2T}{T^{2-2\beta_\ell}} \int_0^M C_\ell^2(\tau) \, d\tau = 0,$$

$$\lim_{T \to \infty} \frac{2T C_\ell(0)^2}{T^{2-2\beta_\ell}} \int_M^T \left(1 - \frac{\tau}{T}\right) (1 + \tau)^{-2\beta_\ell} \, d\tau = \frac{C_\ell(0)^2}{(1 - \beta_\ell)(1 - 2\beta_\ell)}.$$

On the other hand, for the third and the fourth summands

$$(77) \quad \lim_{T \to \infty} \frac{2T C_\ell(0)}{T^{2-2\beta_\ell}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left(\frac{G_\ell(\tau)}{C_\ell(0)} - 1\right)^2 (1 + \tau)^{-2\beta_\ell} \, d\tau = 0,$$

$$(78) \quad \lim_{T \to \infty} \frac{4T C_\ell(0)}{T^{2-2\beta_\ell}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left(\frac{G_\ell(\tau)}{C_\ell(0)} - 1\right) (1 + \tau)^{-2\beta_\ell} \, d\tau = 0.$$

Indeed, we have that

$$\limsup_{T \to +\infty} \frac{2T C_\ell(0)}{T^{2-2\beta_\ell}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left(\frac{G_\ell(\tau)}{C_\ell(0)} - 1\right)^2 (1 + \tau)^{-2\beta_\ell} \, d\tau$$

$$\leq \varepsilon^2 \limsup_{T \to +\infty} \frac{2C_\ell(0)}{T^{1-2\beta_\ell}} \int_M^T (1 + \tau)^{-2\beta_\ell} \, d\tau \leq \varepsilon^2 \frac{4C_\ell(0)}{1 - 2\beta_\ell},$$

and likewise

$$\limsup_{T \to +\infty} \frac{4T C_\ell(0)}{T^{2-2\beta_\ell}} \int_M^T \left(1 - \frac{\tau}{T}\right) \left|\frac{G_\ell(\tau)}{C_\ell(0)} - 1\right| (1 + \tau)^{-2\beta_\ell} \, d\tau \leq \varepsilon \frac{8C_\ell(0)}{1 - 2\beta_\ell},$$

and (77) follows, ε being arbitrary. When $2\beta_\ell = 1$, then one can prove using the same arguments that

$$\lim_{T \to \infty} \frac{1}{T \log T} \int_{[0,T]^2} C_\ell^2(t - s) \, dt \, ds = 2C_\ell(0)^2$$

and the proof of the lemma is concluded. □

PROOF OF LEMMA 4.7.   For $\beta_\ell = 1$ from (75) we have, for $T > \max(1, M)$,

$$\frac{1}{T} \int_{[0,T]^2} C_\ell(t - s)^2 \, dt \, ds$$

$$\leq 2C_\ell(0)^2 \left(M + (\varepsilon + 1)^2 \int_{[M,T]} g_1(\tau)^2 \, d\tau\right) \leq 2C_\ell(0)^2 \left(M + (\varepsilon + 1)^2 \int_{\mathbb{R}} g_1(\tau)^2 \, d\tau\right)$$

$$\leq 2C_\ell(0)^2 \left(M + (\varepsilon + 1)^2 \int_{\mathbb{R}} |g_1(\tau)| \, d\tau\right) = 2C_\ell(0)^2 \left(M + 2\frac{(\varepsilon + 1)^2}{\alpha - 1}\right),$$

where we used the fact that $|g_1(\tau)| \leq 1$ for every $\tau \in \mathbb{R}$, see (10). □

PROOF OF LEMMA 4.8. Let us start with (45). For $\ell \in \mathcal{I}^\star$, $\beta_\ell < 1$ we have

$$\frac{1}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds$$

$$\leq \frac{2C_\ell(0)^2}{T^{1-2\beta_{\ell^\star}}} \left(M + (\varepsilon+1)^2 \int_{[M,T]} (1+\tau)^{-2\beta_{\ell^\star}} \, d\tau \right)$$

$$= \frac{2C_\ell(0)^2}{T^{1-2\beta_{\ell^\star}}} \left(M + \frac{(\varepsilon+1)^2}{-2\beta_{\ell^\star}+1} ((1+T)^{-2\beta_{\ell^\star}+1} - (1+M)^{-2\beta_{\ell^\star}+1}) \right)$$

$$\leq \frac{2C_\ell(0)^2}{T^{1-2\beta_{\ell^\star}}} \left(M + \frac{(\varepsilon+1)^2}{-2\beta_{\ell^\star}+1} (1+T)^{-2\beta_{\ell^\star}+1} \right)$$

$$\leq 2C_\ell(0)^2 \left(M + \frac{(\varepsilon+1)^2}{-2\beta_{\ell^\star}+1} \left(1+\frac{1}{M}\right)^{-2\beta_{\ell^\star}+1} \right),$$

where for the last inequality we recall that $T > \max(1, M)$. Let us now prove (46). From (75), for $\ell \notin \mathcal{I}^\star$, $\ell \geq 1$ and $\beta_\ell < 1$,

$$\frac{1}{T^{2-2\beta_{\ell^\star}}} \int_{[0,T]^2} C_\ell(t-s)^2 \, dt \, ds \leq \frac{2C_\ell(0)^2}{T^{1-2\beta_{\ell^\star}}} \left(M + (\varepsilon+1)^2 \int_{[M,T]} (1+\tau)^{-2\beta_{\ell^{\star\star}}} \, d\tau \right).$$

Now for $2\beta_{\ell^{\star\star}} = 1$ we have

$$\frac{1}{T^{1-2\beta_{\ell^\star}}} \int_{[M,T]} (1+\tau)^{-2\beta_{\ell^{\star\star}}} \, d\tau \leq 2m(\beta_{\ell^\star}),$$

while for $2\beta_{\ell^{\star\star}} < 1$ we have

$$\frac{1}{T^{1-2\beta_{\ell^\star}}} \int_{[M,T]} (1+\tau)^{-2\beta_{\ell^{\star\star}}} \, d\tau \leq \frac{1}{-2\beta_{\ell^{\star\star}}+1} \left(1+\frac{1}{M}\right)^{-2\beta_{\ell^{\star\star}}+1},$$

otherwise

$$\frac{1}{T^{1-2\beta_{\ell^\star}}} \int_{[M,T]} (1+\tau)^{-2\beta_{\ell^{\star\star}}} \, d\tau \leq \frac{1}{2\beta_{\ell^{\star\star}}-1} \left(\frac{1}{1+M}\right)^{2\beta_{\ell^{\star\star}}-1},$$

which concludes the proof. $\square$

Let us write

$$k_{\ell_1 \dots \ell_q}(T) = 2T \int_{[0,T]} \left(1 - \frac{\tau}{T}\right) C_{\ell_1}(\tau) \cdots C_{\ell_q}(\tau) \, d\tau$$

(79)

$$= 2T \left(\int_{[0,M]} \left(1 - \frac{\tau}{T}\right) C_{\ell_1}(\tau) \cdots C_{\ell_q}(\tau) \, d\tau \right.$$

$$\left. + \int_{[M,T]} \left(1 - \frac{\tau}{T}\right) C_{\ell_1}(\tau) \cdots C_{\ell_q}(\tau) \, d\tau \right).$$

PROOF OF LEMMA 4.13. The proof is similar to the proof of Lemma 4.7. Assume that there is at least one index $j \in \{1, \dots, q\}$ such that $\beta_{\ell_j} = 1$. Let $U := U(\ell_1, \dots, \ell_q) = \{j \in \{1, \dots, q\} : \beta_{\ell_j} = 1\}$. We have, from (79), for $T > \max(1, M)$,

$$\frac{k_{\ell_1 \dots \ell_q}(T)}{T} \leq 2C_{\ell_1}(0) \dots C_{\ell_q}(0) \left(M + (\varepsilon+1)^q \int_{[M,T]} (1+\tau)^{-\#U\alpha-(\beta_{\ell_1}+\dots\beta_{\ell_q}-\#U)} \, d\tau \right)$$

$$\leq 2C_{\ell_1}(0) \dots C_{\ell_q}(0) \left(M + (\varepsilon+1)^q \int_{[M,T]} (1+\tau)^{-\#U\alpha-(q-\#U)\min(\beta_0,\beta_{\ell^\star})} \, d\tau \right)$$

$$\leq 2C_{\ell_1}(0)\dots C_{\ell_q}(0)\Big(M + \frac{(\varepsilon+1)^q}{(q-\#U)\min(\beta_0,\beta_{\ell^\star})+\#U\alpha-1}$$

$$\times \Big(\frac{1}{1+M}\Big)^{(q-\#U)\min(\beta_0,\beta_{\ell^\star})+\#U\alpha-1}\Big)$$

$$\leq 2C_{\ell_1}(0)\dots C_{\ell_q}(0)\Big(M + \frac{1}{q\min(\beta_0,\beta_{\ell^\star})+\#U(\alpha-\min(\beta_0,\beta_{\ell^\star}))-1}$$

$$\times \Big(\frac{\varepsilon+1}{(1+M)^{\min(\beta_0,\beta_{\ell^\star})}}\Big)^q \Big(\frac{1}{1+M}\Big)^{\#U(\alpha-\min(\beta_0,\beta_{\ell^\star}))-1}\Big).$$

For $\alpha \geq 2$ we have $\#U(\alpha - \min(\beta_0, \beta_{\ell^\star})) - 1 \geq 0$ hence

$$\frac{k_{\ell_1\dots\ell_q}(T)}{T} \leq 2C_{\ell_1}(0)\dots C_{\ell_q}(0)\Big(M + \frac{1}{q\min(\beta_0,\beta_{\ell^\star})}\Big(\frac{\varepsilon+1}{(1+M)^{\min(\beta_0,\beta_{\ell^\star})}}\Big)^q\Big),$$

which concludes the proof. $\quad\square$

PROOF OF LEMMA 4.11.   This proof is similar to the one of Lemma 4.4. Consider ε, $M >$ 0 as in (36). Then, using Remark 4.3, we have

$$k_{\ell_1\ell_2\cdots\ell_q}(T) = \int_{[0,T]^2} C_{\ell_1}(t-s)C_{\ell_2}(t-s)\cdots C_{\ell_q}(t-s)\,dt\,ds$$

(80)
$$= 2T\int_0^M \Big(1 - \frac{\tau}{T}\Big)C_{\ell_1}(\tau)C_{\ell_2}(\tau)\cdots C_{\ell_q}(\tau)\,d\tau$$

$$+ 2T\int_M^T \Big(1 - \frac{\tau}{T}\Big)C_{\ell_1}(\tau)C_{\ell_2}(\tau)\cdots C_{\ell_q}(\tau)\,d\tau.$$

Now assume that $\beta_{\ell_1} + \cdots + \beta_{\ell_q} < 1$. For the first summand on the right hand side of (80) we have

$$\lim_{T\to\infty} \frac{2}{T^{1-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}}\int_0^M \Big(1 - \frac{\tau}{T}\Big)\big|C_{\ell_1}(\tau)C_{\ell_2}(\tau)\cdots C_{\ell_q}(\tau)\big|\,d\tau$$

$$\leq \lim_{T\to\infty} 2\frac{C_{\ell_1}(0)\cdots C_{\ell_q}(0)}{T^{1-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}}M = 0.$$

For the second summand on the right hand side of (80) we write

$$\int_M^T \Big(1 - \frac{\tau}{T}\Big)C_{\ell_1}(\tau)C_{\ell_2}(\tau)\cdots C_{\ell_q}(\tau)\,d\tau$$

$$= C_{\ell_1}(0)\cdots C_{\ell_q}(0)\int_M^T \Big(1 - \frac{\tau}{T}\Big)(1+\tau)^{-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}$$

(81)
$$\times \sum_{k=1}^{q} \sum_{\substack{k_1+\cdots+k_q=k \\ k_1,\dots,k_q\in\{0,1\}}} \Big(\frac{G_{\ell_1}(\tau)}{C_{\ell_1}(0)} - 1\Big)^{k_1}\cdots\Big(\frac{G_{\ell_q}(\tau)}{C_{\ell_q}(0)} - 1\Big)^{k_q}\,d\tau$$

$$+ C_{\ell_1}(0)\cdots C_{\ell_q}(0)\int_M^T \Big(1 - \frac{\tau}{T}\Big)(1+\tau)^{-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}\,d\tau.$$

For the first term on the right hand side of the previous equality it holds that

$$\lim_{T\to\infty} \frac{C_{\ell_1}(0)\cdots C_{\ell_q}(0)}{T^{1-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}}\int_M^T \Big(1 - \frac{\tau}{T}\Big)(1+\tau)^{-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}$$

(82)

$$\times \sum_{k=1}^{q} \sum_{\substack{k_1+\cdots+k_q=k \\ k_1,\ldots,k_q\in\{0,1\}}} \left(\frac{G_{\ell_1}(\tau)}{C_{\ell_1}(0)} - 1\right)^{k_1} \cdots \left(\frac{G_{\ell_q}(\tau)}{C_{\ell_q}(0)} - 1\right)^{k_q} d\tau = 0.$$

Let us prove (82). Actually, for $\tau > M$ we have

$$\left| \sum_{k=1}^{q} \sum_{\substack{k_1+\cdots+k_q=k \\ k_1,\ldots,k_q\in\{0,1\}}} \left(\frac{G_{\ell_1}(\tau)}{C_{\ell_1}(0)} - 1\right)^{k_1} \cdots \left(\frac{G_{\ell_q}(\tau)}{C_{\ell_q}(0)} - 1\right)^{k_q} \right| \le \sum_{k=1}^{q} \binom{q}{k} \varepsilon^k,$$

and (82) follows, ε being arbitrary. On the other hand, for the second summand on the right hand side of (81),

$$\lim_{T \to \infty} \frac{C_{\ell_1}(0) \cdots C_{\ell_q}(0)}{T^{1-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})}} \int_M^T \left(1 - \frac{\tau}{T}\right)(1+\tau)^{-(\beta_{\ell_1}+\cdots+\beta_{\ell_q})} d\tau$$

$$= \frac{C_{\ell_1}(0) \cdots C_{\ell_q}(0)}{(1 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q}))(2 - (\beta_{\ell_1} + \cdots + \beta_{\ell_q}))}.$$

Analogously, if $\beta_{\ell_1} + \cdots + \beta_{\ell_q} = 1$

$$\lim_{T \to \infty} \frac{k_{\ell_1\ldots\ell_q}(T)}{T \log T} = 2C_{\ell_1}(0) \cdots C_{\ell_q}(0).$$

Otherwise, if $\beta_{\ell_1} + \cdots + \beta_{\ell_q} > 1$, it immediately follows from equation (80) that, as $T \to +\infty$,

$$k_{\ell_1\ldots\ell_q}(T) = 2T \int_0^{+\infty} C_{\ell_1}(\tau)C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) d\tau + O(1).$$

Note that the limiting constant

$$\int_{\mathbb{R}} C_{\ell_1}(\tau)C_{\ell_2}(\tau) \cdots C_{\ell_q}(\tau) d\tau$$

in (57) is finite (see the proof of Proposition 4.12). $\quad\square$

PROOF OF LEMMA 4.14. The proof is similar to the proof of Lemma 4.8. For $\ell_1, \ldots, \ell_q \in \mathcal{I}^\star$, $\beta_{\ell_j} < 1$ for every j and $T > \max(1, M)$, from (79), we have

$$\frac{k_{\ell_1\ldots\ell_q}(T)}{T^{2-q\beta_{\ell^\star}}} \le \frac{2C_{\ell_1}(0)\ldots C_{\ell_q}(0)}{T^{1-q\beta_{\ell^\star}}} \left(M + (\varepsilon+1)^q \int_{[M,T]} (1+\tau)^{-q\beta_{\ell^\star}} d\tau\right)$$

$$\le 2C_{\ell_1}(0)\ldots C_{\ell_q}(0)\left(M + \frac{(\varepsilon+1)^q}{1-q\beta_{\ell^\star}}\left(1 + \frac{1}{M}\right)^{1-q\beta_{\ell^\star}}\right).$$

Finally for $(\ell_1, \ldots, \ell_q) \notin \mathcal{I}^\star$, $\ell_j \ge 1$, $\beta_{\ell_j} < 1$ for every j and $T > \max(1, M)$, from (79) (note that $\beta_{\ell_1} + \cdots \beta_{\ell_q} \ge \beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} > q\beta_{\ell^\star}$), we have

$$\frac{k_{\ell_1\ldots\ell_q}(T)}{T^{2-q\beta_{\ell^\star}}} \le \frac{2C_{\ell_1}(0)\ldots C_{\ell_q}(0)}{T^{1-q\beta_{\ell^\star}}} \left(M + (\varepsilon+1)^q \int_{[M,T]} (1+\tau)^{-(\beta_{\ell^{\star\star}}+(q-1)\beta_{\ell^\star})} d\tau\right)$$

$$\le 2C_{\ell_1}(0)\ldots C_{\ell_q}(0)\left(M + (\varepsilon+1)^q \frac{1}{T^{1-q\beta_{\ell^\star}}} \int_{[M,T]} (1+\tau)^{-(\beta_{\ell^{\star\star}}+(q-1)\beta_{\ell^\star})} d\tau\right).$$

Now if $\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} < 1$ then

$$\frac{1}{T^{1-q\beta_{\ell^\star}}} \int_{[M,T]} (1+\tau)^{-(\beta_{\ell^{\star\star}}+(q-1)\beta_{\ell^\star})} d\tau \le \frac{(1 + \frac{1}{M})^{-(\beta_{\ell^{\star\star}}+(q-1)\beta_{\ell^\star})+1}}{-(\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star}) + 1};$$

if $\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} = 1$, then

$$\frac{1}{T^{1-q\beta_{\ell^\star}}} \int_{[M,T]} (1+\tau)^{-1} d\tau \leq \frac{1}{T^{1-2\beta_{\ell^\star}}} \int_{[M,T]} (1+\tau)^{-1} d\tau \leq 2m(\beta_{\ell^\star}),$$

where $m(\beta_{\ell^\star})$ is a constant defined in Lemma 4.7. Finally if $\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} > 1$ then

$$\frac{1}{T^{1-q\beta_{\ell^\star}}} \int_{[M,T]} (1+\tau)^{-(\beta_{\ell^{\star\star}}+(q-1)\beta_{\ell^\star})} d\tau \leq \frac{(M+1)^{-(\beta_{\ell^{\star\star}}+(q-1)\beta_{\ell^\star}-1)}}{\beta_{\ell^{\star\star}} + (q-1)\beta_{\ell^\star} - 1}$$

and the proof is concluded. $\square$

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