ENTROPY DISSIPATION ESTIMATES FOR INHOMOGENEOUS ZERO-RANGE PROCESSES

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Introduced by Lu and Yau (*Comm. Math. Phys.* **156** (1993) 399–433), the martingale decomposition method is a powerful recursive strategy that has produced sharp log-Sobolev inequalities for homogeneous particle systems. However, the intractability of certain covariance terms has so far precluded applications to heterogeneous models. Here we demonstrate that the existence of an appropriate coupling can be exploited to bypass this limitation effortlessly. Our main result is a dimension-free modified log-Sobolev inequality for zero-range processes on the complete graph, under the only requirement that all rate increments lie in a compact subset of $(0, \infty)$. This settles an open problem raised by Caputo and Posta (*Probab. Theory Related Fields* **139** (2007) 65–87) and reiterated by Caputo, Dai Pra and Posta (*Ann. Inst. Henri Poincaré Probab. Stat.* **45** (2009) 734–753). We believe that our approach is simple enough to be applicable to many systems.

1. Introduction.

1.1. *Entropy dissipation estimates.* Consider a reversible Markov generator Q with respect to some probability distribution π on a finite state space Ω. In other words, Q is a $\Omega \times \Omega$ matrix with nonnegative off-diagonal entries, with each row summing up to 0, and satisfying the *local balance* equations

$$\pi(x)Q(x, y) = \pi(y)Q(y, x),$$

for all $x, y \in \Omega$. When Q is irreducible, the Markov semi-group $P_t = e^{tQ}$ generated by Q *mixes*: for any observable $f : \Omega \to (0, \infty)$, we have the pointwise convergence

$$P_t f \xrightarrow[t \to \infty]{} \mathbb{E}[f(X)],$$

where throughout the paper, X denotes a π-distributed random variable. A natural way to quantify this convergence consists in measuring the rate at which the *entropy*

$$\mathrm{Ent}(f) := \mathbb{E}[f(X) \log f(X)] - \mathbb{E}[f(X)] \log \mathbb{E}[f(X)],$$

decays along the semi-group. Specifically, one looks for a constant $\alpha > 0$, as large as possible, such that for all observables $f : \Omega \to (0, \infty)$ and all times $t \geq 0$,

$$\mathrm{Ent}(P_t f) \leq e^{-\alpha t} \mathrm{Ent}(f).$$

The optimal value of α is called the *entropy dissipation constant* and will be denoted by $\alpha(Q)$. Writing $\mathcal{E}(f, g) := -\mathbb{E}[f(X)(Qg)(X)]$ for the underlying Dirichlet form, one finds

$$\frac{d}{dt} \mathrm{Ent}(P_t f) = -\mathcal{E}(P_t f, \log P_t f).$$

Thus, $\alpha(Q)$ is more effectively characterized as the largest constant $\alpha > 0$ such that the following *modified log-Sobolev inequality* (MLSI) holds: for all observables $f : \Omega \to (0, \infty)$,

$$\mathcal{E}(f, \log f) \geq \alpha \mathrm{Ent}(f).$$

We refer to the tutorial paper [2] or the textbook [22] for more details on this fundamental functional inequality and its relation to hypercontractivity, concentration and mixing times. For intrinsic reasons exposed in [21], establishing sharp MLSI's for Markov chains on finite spaces remains a notoriously challenging task. A natural and important context where such entropic estimates have received a particular attention is that of interacting particle systems and, in particular, *zero-range* processes.

1.2. *Zero-range dynamics.* Introduced by Spitzer [24], the *zero-range process* (ZRP) is a generic conservative particle system in which individual jumps occur at a rate which only depends on the source, the destination, and the number of particles present at the source. We shall here focus on the mean-field version of the model, which is parameterized by the following ingredients:

- two integers $n, m \geq 1$ representing the numbers of sites and particles, respectively;
- a probability vector $p = (p_1, \ldots, p_n)$ specifying the law of the destination of a jump;
- a function $r_i : \{1, 2, \ldots\} \to (0, \infty)$ encoding the kinetics at site $i \in [n] := \{1, \ldots, n\}$.

The ZRP with these parameters is a continuous-time Markov chain on the state space

$$\Omega := \left\{ x = (x_1, \ldots, x_n) \in \mathbb{Z}_+^n : \sum_{i=1}^n x_i = m \right\},$$

where x_i represents the number of particles at site i. The action of the generator is given by

$$(Qf)(x) := \sum_{i=1}^n \sum_{j=1}^n r_i(x_i) p_j \big(f(x + \delta_j - \delta_i) - f(x) \big),$$

where $(\delta_1, \ldots, \delta_n)$ denotes the canonical n-dimensional basis, and with the convention that $r_i(0) = 0$ for all $i \in [n]$ (no jump from empty sites). This dynamic is easily seen to be reversible with respect to the explicit distribution

$$(1) \qquad\qquad \pi(x) \propto \prod_{i=1}^n \frac{p_i^{x_i}}{r_i(1) \cdots r_i(x_i)},$$

with the normalization being chosen so that π is a probability distribution on Ω. A somewhat degenerate but instructive situation is obtained with the linear choice

$$(2) \qquad\qquad r_i(\ell) = c \times \ell,$$

for all $i \in [n]$ and $\ell \geq 0$, where $c > 0$ is an arbitrary constant. In this case, the model trivializes in the sense that the m particles perform independent random walks, each jumping at rate c according to p. Thanks to the tensorization property of variance and entropy, the Poincaré and modified log-Sobolev constants of the whole system then reduce to those of a single particle, which are easily seen to be bounded away from 0 independently of p. In light of this, it is reasonable to expect that dimension-free functional inequalities will persist in the perturbative regime where the rate functions are *nearly linear*. Turning this intuition into rigorous estimates has been and continues to be a subject of active research, see, for example, [3, 4, 13, 15, 18, 23] on the Poincaré side, and [5, 6, 8–10, 12] on the log-Sobolev side.

1.3. *Result and related works.* Perhaps the most natural way to formalize the idea that the rate functions should be *nearly linear* consists in requiring that their increments all lie in a fixed compact subset of $(0, \infty)$.

ASSUMPTION 1. There are constants $\Delta, \delta > 0$ such that for every $i \in [n]$ and $\ell \in \mathbb{Z}_+$,

$$\delta \leq r_i(\ell + 1) - r_i(\ell) \leq \Delta.$$

In a remarkable work [5], Caputo, Dai Pra and Posta developed a novel method, based on the so-called Bochner–Bakry–Emery approach introduced in [1], to establish a dimension-free MLSI in the perturbative regime where the ratio Δ/δ is sufficiently small.

THEOREM 1 (Caputo, Dai Pra and Posta [5]). *Suppose that Assumption 1 holds with*

$$\frac{\Delta}{\delta} < 2, \tag{3}$$

and take $p = (\frac{1}{n}, \ldots, \frac{1}{n})$. *Then, for any number m of particles, the* ZRP *satisfies*

$$\alpha(Q) \geq 2\delta - \Delta. \tag{4}$$

In the so-called *homogeneous setting* where the rate function r_i is not allowed to depend on i, Assumption 1 can be relaxed at the price of considerable efforts, see [6]. However, the martingale approach used therein does not extend to inhomogeneous models in a natural way and, to the best of our knowledge, Theorem 1 constitutes the only available criterium for inhomogeneous rates. Although the bound (4) trivializes as Δ approaches the threshold 2δ, the authors predicted the persistence of a dimension-free MLSI beyond the perturbative regime (3). Specifically, they formulated the following conjecture, reiterated in [6].

CONJECTURE (Caputo, Dai Pra and Posta [5]). *Under Assumption 1, there is a dimension-free constant* $c(\delta, \Delta) > 0$ *such that for* $p = (\frac{1}{n}, \ldots, \frac{1}{n})$ *and any number m of particles,*

$$\alpha(Q) \geq c(\delta, \Delta).$$

Proving this requires new ideas, since it was noted in [5] that the convexity of $t \mapsto \text{Ent}(P_t f)$ fails as $\frac{\Delta}{\delta}$ gets large, making the Bochner–Bakry–Emery approach unapplicable. In the present paper, we establish the following strengthening of the above conjecture.

THEOREM 2 (Dimension-free MLSI). *Under Assumption 1, for any choice of the parameters p and m, the modified log-Sobolev constant of the* ZRP *satisfies*

$$\alpha(Q) \geq \frac{\delta^2}{2\Delta}.$$

This estimate is sharp up to a factor 2, as can already been seen in the linear case (2) (see [2], Example 3.10). More importantly, our main contribution lies in the simplicity of the method used to prove Theorem 2. Our starting point is the following elementary observation, which is a straightforward consequence of the product form (1).

FACT 1 (Recursive structure). If $X = (X_1, \ldots, X_n)$ has law π, then for any site $i \in [n]$, the conditional law of $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ given X_i coincides with the stationary law of a new ZRP with $n - 1$ sites, $m - X_i$ particles, rates $(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)$ and probability vector $(\frac{p_1}{1-p_i}, \ldots, \frac{p_{i-1}}{1-p_i}, \frac{p_{i+1}}{1-p_i}, \ldots, \frac{p_n}{1-p_i})$.

Since the new ZRP inherits Assumption 1 from the original one, a natural approach to Theorem 2 consists in proceeding by induction over the dimension n. This is in fact a classical strategy for establishing functional inequalities, known as the *martingale decomposition method*. Introduced by Lu and Yau [20] in the context of Kawasaki and Glauber dynamics, it has been successfully applied to various interacting particle systems [6, 11, 19, 25], as well as other Markov chains enjoying an appropriate recursive structure [7, 14, 16, 17]. To convert the above observation into an effective functional inequality however, one needs to estimate the Dirichlet form of the n-dimensional system in terms of that of the $(n-1)$-dimensional system. Because of the interaction between particles, this decomposition inevitably produces certain *cross terms*, whose intractibility has so far precluded applications to heterogeneous models. Our main contribution consists in showing that the existence of an appropriate coupling—see Proposition 1 below—allows one to bypass this limitation effortlessly.

REMARK 1 (Extensions).   We firmly believe that our idea is simple enough to be applicable to many other settings where the underlying dynamics are also repulsive. For example, our argument can easily be adapted to yield a MLSI with constant $\delta^2/2\Delta$ for the *inhomogeneous Bernoulli–Laplace model* (see, e.g., [5], Section 5) with rates in $[\delta, \Delta]$. We leave the details to the reader. See also [14] for another setting where our coupling idea has been successful.

2. Proofs.   From now onward, we suppose that Assumption 1 is satisfied.

2.1. *Monotone coupling.*   Using the fact that each rate function $r_i: \{1, 2, \ldots\} \to (0, \infty)$ is nondecreasing, we may enrich our ZRP by adding a *tagged particle* on top of it, whose position $(J(t): t \geq 0)$ evolves as follows: conditionally on the background process $(X(t): t \geq 0)$ being currently in some state $x \in \Omega$, we let the tagged particle jump across $[n]$ according to the Markov generator

$$L_x(i, j) := (r_i(x_i + 1) - r_i(x_i))(p_j - \mathbf{1}_{(i=j)}).$$

In more formal terms, we consider a Markov process $((X(t), J(t)): t \geq 0)$ taking values in the product space $\widehat{\Omega} := \Omega \times [n]$, and evolving according to the generator

$$\widehat{Q}((x, i), (y, j)) := Q(x, y)\mathbf{1}_{(i=j)} + L_x(i, j)\mathbf{1}_{(x=y)}.$$

We refer to this process as the *augmented* ZRP. An elementary but crucial observation is that:

(i) the background process $(X(t): t \geq 0)$ is a ZRP with parameters (n, m, p, r);
(ii) the aggregated process $(X(t) + \delta_{J(t)}: t \geq 0)$ is a ZRP with parameters $(n, m+1, p, r)$.

The existence of such a monotone coupling between zero-range processes with different numbers of particles is of course well known, and has been extensively used in the past. However, its consequences on the martingale approach do not seem to have been explored, and this is where our conceptual contribution lies. Specifically, our interest will here reside in the invariant law $\widehat{\pi}$ of $\widehat{Q}$, which is uniquely determined by the *global balance* equations

$$(5) \qquad \sum_{x \in \Omega} \widehat{\pi}(x, j)Q(x, y) + \sum_{i=1}^{n} \widehat{\pi}(y, i)L_y(i, j) = 0,$$

for all $(y, j) \in \widehat{\Omega}$. Although we do not have any explicit expression for $\widehat{\pi}$, we note that its first marginal has to be invariant under Q by construction, and is therefore simply the law π

defined at (1). We also note that, in the linear case (2), the generator L_x of the tagged particle becomes independent of the background state $x \in \Omega$, resulting in the product form

$$\widehat{\pi}(x, j) = \pi(x)p_j.$$

The next lemma states that under Assumption 1, $\widehat{\pi}$ is not far from this product measure.

LEMMA 1 (Product-measure approximation).   *For all* $(x, j) \in \widehat{\Omega}$, *we have*

$$\widehat{\pi}(x, j) \leq \frac{\Delta}{\delta}\pi(x)p_j.$$

PROOF.   Fix $j \in [n]$ and choose $y \in \Omega$ such that

$$\frac{\widehat{\pi}(y, j)}{\pi(y)} = \max_{x \in \Omega}\left\{\frac{\widehat{\pi}(x, j)}{\pi(x)}\right\}.$$

At the point (y, j), the first sum in the global balance equation (5) is nonpositive. Indeed,

$$\sum_{x \in \Omega}\widehat{\pi}(x, j)Q(x, y) \leq \frac{\widehat{\pi}(y, j)}{\pi(y)}\sum_{x \in \Omega}\pi(x)Q(x, y) = 0,$$

where the equality is simply the balance equation $\pi Q = 0$ at state y. Consequently, the second sum in (5) must be nonnegative. In other words,

$$0 \leq \sum_{i=1}^{n}\widehat{\pi}(y, i)L_y(i, j)$$

$$\leq \sum_{i=1}^{n}\widehat{\pi}(y, i)(\Delta p_j - \delta\mathbf{1}_{(i=j)}) = \Delta p_j \pi(y) - \delta\widehat{\pi}(y, j),$$

where we have used Assumption (1), and then the fact that the first marginal of $\widehat{\pi}$ is π.   □

These considerations lead us to the following result, which will constitute our main tool.

PROPOSITION 1 (Main coupling).   *Given a site* $i \in [n]$, *we may jointly construct a* π-*distributed random variable* X *and a* $[n] \setminus \{i\}$-*valued random variable* J *such that*

(6) $$\text{Law}(X - \delta_i + \delta_J | X_i = \ell) = \text{Law}(X | X_i = \ell - 1),$$

for each level $\ell \in \{1, \ldots, m\}$ *and moreover, for every* $(x, j) \in \Omega \times ([n] \setminus \{i\})$,

(7) $$\mathbb{P}(J = j | X = x) \leq \frac{\Delta p_j}{\delta(1 - p_i)}.$$

PROOF.   We simply let X_i be distributed according to the ith marginal of π and, given that $X_i = \ell$, we let $((X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n), J)$ be distributed according to the stationary distribution of the $(n - 1)$-dimensional augmented ZRP with $m - \ell$ background particles, rates $(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)$ and probability vector $(\frac{p_1}{1-p_i}, \ldots, \frac{p_{i-1}}{1-p_i}, \frac{p_{i+1}}{1-p_i}, \ldots, \frac{p_n}{1-p_i})$. Observation 1 ensures that the random variable $X := (X_1, \ldots, X_n)$ is distributed according to π. Moreover, property (7) is guaranteed by Lemma 1. Finally, the identity (6) is clear since under both distributions, the ith coordinate is almost-surely equal to $\ell - 1$ while the joint law of the remaining coordinates is the same, thanks to observation (ii) above.   □

2.2. *Single-site estimate.* In this section, we show how the above coupling implies—without effort—the crucial local MLSI needed for our inductive proof of Theorem 2. We start by introducing some notation. Let us decompose the Dirichlet form of the process as

$$(8) \qquad \mathcal{E}(f, g) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{E}_{ij}(f, g),$$

where $\mathcal{E}_{ij}(f, g)$ captures the contribution from all jumps with source i and destination j, that is,

$$\mathcal{E}_{ij}(f, g) := p_j \mathbb{E}[r_i(X_i)(f(X - \delta_i + \delta_j) - f(X))(g(X - \delta_i + \delta_j) - g(X))].$$

Given $f : \Omega \to (0, \infty)$ and a site $i \in [n]$, we define $f_i : \{0, \ldots, m\} \to (0, \infty)$ by

$$f_i(\ell) := \mathbb{E}[f(X)|X_i = \ell].$$

With this notation in hand, our aim is to establish the following local estimate, which relates the Dirichlet contribution from a single site i to the conditional entropy given X_i.

PROPOSITION 2 (Local MLSI). *For any observable $f : \Omega \to (0, \infty)$ and any site $i \in [n]$,*

$$\mathbb{E}[f_i(X_i) \log f_i(X_i)] - \mathbb{E}[f(X)] \log \mathbb{E}[f(X)] \le \frac{\Delta}{\delta^2} \sum_{j \in [n] \setminus \{i\}} \mathcal{E}_{ij}(f, \log f).$$

PROOF. Consider the bivariate function $\mathcal{H} : (0, \infty)^2 \to [0, \infty)$ defined by

$$\mathcal{H}(u, v) := (u - v)(\log u - \log v).$$

This function is convex, because its hessian matrix

$$\begin{bmatrix} \partial_{uu}\mathcal{H} & \partial_{uv}\mathcal{H} \\ \partial_{vu}\mathcal{H} & \partial_{vv}\mathcal{H} \end{bmatrix} = \frac{u + v}{(uv)^2} \begin{bmatrix} v^2 & -uv \\ -uv & u^2 \end{bmatrix}$$

is positive semi-definite at each point $(u, v) \in (0, \infty)^2$. Now, fix a site $i \in [n]$, and let (X, J) be as in Proposition 1. On the event $\{X_i \ge 1\}$, the definition of f_i and Property (6) imply

$$f_i(X_i) = \mathbb{E}[f(X)|X_i],$$
$$f_i(X_i - 1) = \mathbb{E}[f(X + \delta_J - \delta_i)|X_i].$$

Thus, the pair $(f_i(X_i), f_i(X_i - 1))$ is the conditional expectation of $(f(X), f(X - \delta_i + \delta_J))$ given X_i. By the conditional Jensen inequality, we deduce that on the event $\{X_i \ge 1\}$,

$$\mathcal{H}(f_i(X_i), f_i(X_i - 1)) \le \mathbb{E}[\mathcal{H}(f(X), f(X - \delta_i + \delta_J))|X_i].$$

Multiplying through by $(1 - p_i)r_i(X_i)$ (which is zero outside the event $\{X_i \ge 1\}$) and taking expectations, we obtain

$$(1 - p_i)\mathbb{E}[r_i(X_i)\mathcal{H}(f_i(X_i), f_i(X_i - 1))] \le (1 - p_i)\mathbb{E}[r_i(X_i)\mathcal{H}(f(X), f(X - \delta_i + \delta_J))]$$

$$\le \frac{\Delta}{\delta} \sum_{j \in [n] \setminus \{i\}} \mathcal{E}_{ij}(f, \log f),$$

where the second line uses Property (7). Thus, our task boils down to proving

$$\mathbb{E}[f_i(X_i) \log f_i(X_i)] - \mathbb{E}[f(X)] \log \mathbb{E}[f(X)] \le \frac{(1 - p_i)}{\delta} \mathbb{E}[r_i(X_i)\mathcal{H}(f_i(X_i), f_i(X_i - 1))].$$

This inequality is in fact a one-dimensional MLSI in disguise. Indeed, since $\mathbb{E}[f_i(X_i)] = \mathbb{E}[f(X)]$, the left-hand side is exactly the entropy of f_i under the law of X_i. On the other hand, the right-hand side is exactly $\frac{1}{\delta}\mathcal{E}_K(f_i, \log f_i)$, where $\mathcal{E}_K$ denotes the Dirichlet form of the birth-and-death generator K on $\{0, \ldots, m\}$ defined by

$$\forall \ell \in \{1, \ldots, m\}, \quad K(\ell, \ell - 1) := (1 - p_i) r_i(\ell),$$

$$\forall \ell \in \{0, \ldots, m - 1\}, \quad K(\ell, \ell + 1) := p_i \sum_{j \in [n] \setminus \{i\}} \mathbb{E}[r_j(X_j) | X_i = \ell].$$

Note that these rates are reversible with respect to the law of X_i, because they mimic the stationary zero-range dynamics at site i. Thus, the above inequality is simply

$$(9) \qquad\qquad\qquad \alpha(K) \geq \delta.$$

It remains to establish this one-dimensional MLSI. By Assumption 1, the death rates satisfy

$$K(\ell + 1, \ell) - K(\ell, \ell - 1) = (1 - p_i)(r_i(\ell + 1) - r_i(\ell)) \geq (1 - p_i)\delta.$$

Regarding the birth rates, we may invoke Property (6) to write

$$K(\ell - 1, \ell) - K(\ell, \ell + 1) = p_i \sum_{j \in [n] \setminus \{i\}} \mathbb{E}[r_j(X_j + \mathbf{1}_{(J=j)}) - r_j(X_j) | X_i = \ell] \geq p_i \delta.$$

For birth-and-death chains, these uniform bounds on the rate increments are known to imply the MLSI (9), see, for example, [5], Theorem 3.1, or [6], Lemma 2.2. $\square$

2.3. *The induction argument.* We are finally in position to prove Theorem 2 by induction over n. The claim is trivial for $n = 1$. We now assume that $n \geq 2$ and that the claim is already proven for $(n - 1)$-dimensional systems. Fix an observable $f : \Omega \to (0, \infty)$ and a site $i \in [n]$, and consider the $(n - 1)$-dimensional ZRP obtained by conditioning on X_i, as in Observation 1. Viewing f as a function of the $n - 1$ remaining variables, the induction hypothesis ensures that

$$\text{Ent}(f | X_i) \leq \frac{2\Delta}{\delta^2} \sum_{j,k \in [n] \setminus \{i\}} \frac{p_j}{1 - p_i} \mathbb{E}[r_k(X_k) \mathcal{H}(f(X), f(X + \delta_j - \delta_k)) | X_i],$$

where the entropy on the left-hand side is computed conditionally on X_i, that is,

$$\text{Ent}(f | X_i) := \mathbb{E}[f(X) \log f(X) | X_i] - f_i(X_i) \log f_i(X_i).$$

Taking expectations, we arrive at

$$\mathbb{E}[f(X) \log f(X)] \leq \mathbb{E}[f_i(X_i) \log f_i(X_i)] + \frac{\Delta}{\delta^2 (1 - p_i)} \sum_{j,k \in [n] \setminus \{i\}} \mathcal{E}_{jk}(f, \log f).$$

We now multiply by $1 - p_i$ and use Proposition 2 to obtain

$$(1 - p_i)\text{Ent}(f) \leq \frac{\Delta}{\delta^2} \sum_{j \in [n] \setminus \{i\}} \left((1 - p_i)\mathcal{E}_{ij}(f, \log f) + \sum_{k \in [n] \setminus \{i\}} \mathcal{E}_{kj}(f, \log f) \right)$$

$$\leq \frac{\Delta}{\delta^2} \sum_{j \in [n] \setminus \{i\}} \sum_{k=1}^{n} \mathcal{E}_{kj}(f, \log f).$$

Summing over all sites $i \in [n]$ and recalling (8), we conclude that

$$(n - 1)\text{Ent}(f) \leq \frac{2\Delta}{\delta^2}(n - 1)\mathcal{E}(f, \log f).$$

Since this is true for every $f : \Omega \to (0, \infty)$, we have just established a MLSI with constant $\frac{\delta^2}{2\Delta}$ for our n-dimensional ZRP. This completes our induction step.

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