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CRANK–NICOLSON SCHEME FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

BY YAOZHONG HU[1], YANGHUI LIU[2] AND DAVID NUALART[3]

[1]*Department of Mathematical and Statistical Sciences, University of Alberta, yaozhong@ualberta.ca*
[2]*Department of Mathematics, Baruch College CUNY, yanghui.liu@baruch.cuny.edu*
[3]*Department of Mathematics, University of Kansas, nualart@ku.edu*

We study the Crank–Nicolson scheme for stochastic differential equations (SDEs) driven by a multidimensional fractional Brownian motion with Hurst parameter $H > 1/2$. It is well known that for ordinary differential equations with proper conditions on the regularity of the coefficients, the Crank–Nicolson scheme achieves a convergence rate of n^{-2}, regardless of the dimension. In this paper we show that, due to the interactions between the driving processes, the corresponding Crank–Nicolson scheme for m-dimensional SDEs has a slower rate than for one-dimensional SDEs. Precisely, we shall prove that when the fBm is one-dimensional and when the drift term is zero, the Crank–Nicolson scheme achieves the convergence rate n^{-2H}, and when the drift term is nonzero, the exact rate turns out to be $n^{-\frac{1}{2}-H}$. In the general multidimensional case the exact rate equals $n^{\frac{1}{2}-2H}$. In all these cases the asymptotic error is proved to satisfy some linear SDE. We also consider the degenerated cases when the asymptotic error equals zero.

1. Introduction. This paper is concerned with the following stochastic differential equation (SDE for short) on $\mathbb{R}^d$ driven by a fractional Brownian motion (fBm for short)

$$X_t = x + \int_0^t V(X_s)\, dB_s, \quad t \in [0, T], \tag{1.1}$$

where $B = (B^0, B^1, \dots, B^m)$, and $(B^1, \dots, B^m)$ is an m-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. For notational convenience we denote $B_t^0 = t$ for $t \in [0, T]$ in order to include the drift term in (1.1). The integral on the right-hand side of (1.1) is of Riemann–Stieltjes type. It is well known that if the vector field $V = (V_0, V_1, \dots, V_m) : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^{m+1}, \mathbb{R}^d)$ has bounded partial derivatives which are Hölder continuous of order $\alpha > \frac{1}{H} - 1$, then there exists a unique solution for equation (1.1), which has bounded $\frac{1}{\gamma}$-variation on $[0, T]$ for any $\gamma < H$; see for example, [14, 23].

As in the Brownian motion case, the explicit solution of SDEs driven by fractional Brownian motions are rarely known. Thus one has to rely on numerical methods for simulations of these equations. Various time-discrete numerical approximation schemes for (1.1) have been considered in recent years. Recall that the classical Euler scheme is defined as follows:

$$\begin{aligned} X^n_{t_{k+1}} &= X^n_{t_k} + V\big(X^n_{t_k}\big)(B_{t_{k+1}} - B_{t_k}), \\ X^n_0 &= x, \end{aligned} \tag{1.2}$$

where $k = 0, 1, \dots, n-1$ and $t_k = kT/n$. This scheme is considered in [17, 18] for scalar SDEs, and generalized in [10, 15] to the multidimensional case. The solution of (1.2) has the

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exact strong convergence rate of n^{1-2H} when $H > \frac{1}{2}$, while in the case $H = \frac{1}{2}$ it converges to the corresponding Itô SDE

$$X_t = x + \int_0^t V(X_s)\delta B_s, \quad t \in [0, T],$$

where δ denotes the Itô stochastic integral. Note also that the Euler scheme is not convergent when $H < \frac{1}{2}$; see for example, [5]. A modified Euler scheme introduced in [10] generalizes the classical Euler scheme to the fBm case

$$\begin{aligned} X^n_{t_{k+1}} &= X^n_{t_k} + V(X^n_{t_k})(B_{t_{k+1}} - B_{t_k}) + \frac{1}{2}\sum_{j=1}^m \partial V_j V_j(X^n_{t_k})\left(\frac{T}{n}\right)^{2H}, \\ X^n_0 &= x. \end{aligned} \tag{1.3}$$

The modified Euler scheme has been shown to have a better convergence rate than (1.2). More precisely, the rate is $n^{\frac{1}{2}-2H}$ when $\frac{1}{2} < H < \frac{3}{4}$ and $n^{-1}\sqrt{\log n}$ when $H = \frac{3}{4}$, and in the case $\frac{3}{4} < H < 1$ the rate becomes n^{-1}. Weak convergence rates and asymptotic error distributions were also obtained for the modified Euler scheme. In [9], the authors considered Taylor schemes derived from the Taylor expansion in the one-dimensional case. In [11], the Taylor schemes and their modifications were introduced for SDEs driven by fBm's $B^1, \dots, B^m$, with Hurst parameters $H_1, \dots, H_m$, where $H_1, \dots, H_m \in (\frac{1}{2}, 1]$ are not necessarily equal. In [4], the Milstein scheme (or 2nd-order Taylor scheme) has been considered for the rough case $H < \frac{1}{2}$ and it is convergent as long as $H > \frac{1}{3}$. An extension of the result to mth order Taylor schemes is contained in [7]. In [5, 6], some 2nd and 3rd order implementable schemes are studied via the Wong–Zakai approximation.

The Crank–Nicolson (or Trapezoidal) scheme has been studied only recently. Recall that the Crank–Nicolson scheme for (1.1) is defined as follows:

$$\begin{aligned} X^n_{t_{k+1}} &= X^n_{t_k} + \frac{1}{2}[V(X^n_{t_{k+1}}) + V(X^n_{t_k})](B_{t_{k+1}} - B_{t_k}), \\ X^n_0 &= x, \end{aligned} \tag{1.4}$$

where again $t_k = kT/n$ for $k = 0, \dots, n-1$. In [16, 18], the Crank–Nicolson scheme is considered for SDEs with Hurst parameter $H \in (1/3, 1/2)$. It has been shown in [18] that if $V \in C^\infty_b$ the convergence rate of the Crank–Nicolson scheme is $n^{\frac{1}{2}-3H}$. This rate is exact in the sense that the renormalized error process $n^{3H-\frac{1}{2}}(X - X^n)$ converges weakly to a nonzero limit (see, e.g., [16]). Note however that due to the use of the Doss–Sussmann representation these results are applicable only to the scalar SDE setting, which corresponds to the case $m = d = 1$ and $V_0 \equiv 0$ in our notation. On the other hand, it has been conjectured in [19] that the Crank–Nicolson scheme has exact root mean square convergence rate $n^{\frac{1}{2}-2H}$.

In view of these results, our first goal is to answer the following question:

QUESTION 1. Is the Crank–Nicolson scheme still convergent in the multidimensional setting, and is the convergence rate the same as that of the scalar SDE?

Let us recall that in the case of deterministic ordinary differential equations (ODEs), either in the one-dimensional or multidimensional settings, and with proper regularity assumptions on the coefficients, the convergence rate of the Crank–Nicolson scheme is always n^{-2}. Surprisingly, as we will show in this paper, the Crank–Nicolson scheme (1.4) for SDEs has a very different feature comparing to the ODE cases. While the Crank–Nicolson scheme is still convergent, the convergence rate is largely "throttled" due to the interactions among the driving processes of the equation. More precisely, we will prove the following result. We consider the continuous time interpolation of the Crank–Nicolson scheme for $t \in [t_k, t_{k+1})$, $k = 0, \dots, n-1$:

$$X_t^n = X_{t_k}^n + \frac{1}{2}\big[V(X_{t_k}^n) + V(X_{t_{k+1}}^n)\big](B_t - B_{t_k}). \tag{1.5}$$

THEOREM 1.1. *Let X be the solution of equation (1.1) and let X^n be the continuous time interpolation of the Crank–Nicolson scheme $\{X_{t_0}^n, X_{t_1}^n, \dots, X_{t_n}^n\}$ defined by (1.5). Suppose that $V \in C_b^3$. Then for any $p \geq 1$ there exists a constant $K = K_p$ independent of n such that the following strong convergence result holds true for all $n \in \mathbb{N}$:*

$$\sup_{t\in[0,T]} \big(\mathbb{E}|X_t - X_t^n|^p\big)^{1/p} \leq K/\vartheta_n, \tag{1.6}$$

where ϑ_n is defined as

$$\vartheta_n = \begin{cases} n^{2H-\frac{1}{2}} & \textit{when } m > 1, \\ n^{H+\frac{1}{2}} & \textit{when } m = 1 \textit{ and } V_0 \not\equiv 0, \\ n^{2H} & \textit{when } m = 1 \textit{ and } V_0 \equiv 0. \end{cases}$$

Theorem 1.1 shows that if the driving process B is one-dimensional and there is no drift term, then the convergence rate of the Crank–Nicolson scheme (1.4) is n^{-2H}. This result coincides with the case of deterministic ODEs if we formally set $H = 1$, and also with the case of one-dimensional Brownian motion which corresponds to $H = \frac{1}{2}$ (see, e.g., [16, 18]). If a drift term is included in the equation, then the rate turns out to be $n^{-H-\frac{1}{2}}$. In the general case when B is multidimensional the convergence rate becomes $n^{\frac{1}{2}-2H}$, the same as that of the modified Euler scheme (1.3) with $\frac{1}{2} < H < \frac{3}{4}$. Note also that Theorem 1.1 gives a positive answer to the conjecture raised in [19] under this general assumption. The slowing down of convergence rate from one-dimensional case to multidimensional cases is due to the nonvanishing Lévy area term (see (3.1)). Indeed, in the one-dimensional case these Lévy area type processes disappear and the convergence of $X - X^n$ is dictated by some higher order terms.

The second part of the paper is motivated by the following question:

QUESTION 2. Are the convergence rates obtained in Theorem 1.1 exact? If yes, what is the limiting distributions of the scheme for both the one-dimensional and multidimensional cases?

To answer this question, we will consider the piecewise constant interpolations. Namely, we consider the processes $\tilde{X}^n$ and $\tilde{X}$:

$$\tilde{X}_t = X_{t_k} \quad \text{and} \quad \tilde{X}_t^n = X_{t_k}^n, \qquad t \in [t_k, t_{k+1}), k = 0, 1, \dots, n. \tag{1.7}$$

Recall that X and X^n are respectively the solutions of equations (1.1) and (1.4).

REMARK 1.2. The piecewise constant interpolation $\tilde{X}$ of the true solution X allows us to focus on the asymptotic error on the partition points. In fact, we will see that the interpolation (1.5) of X^n satisfies

$$\big(X_t - X_t^n\big) - \big(X_{t_k} - X_{t_k}^n\big) \sim n^{-2H}$$

for $t \in [t_k, t_{k+1})$ (see (4.34)). According to the rates stated in Theorem 1.1, this difference between the error at the partition points and at the nonpartition ones does not affect the rates of convergence of the scheme in all three cases. However, in the case of $m = 1$ and $V_0 \equiv 0$, it has a nontrivial contribution to the asymptotic error.

THEOREM 1.3. *Let $\tilde{X}$ and $\tilde{X}^n$ be the processes defined in* (1.7), *and suppose that $V \in C_b^3$. Denote by $\phi_{jj'}$ the Lie bracket between the vector fields V_j and $V_{j'}$:*

$$\phi_{jj'} = \partial V_j V_{j'} - \partial V_{j'} V_j, \quad j, j' = 0, 1, \dots, m, \tag{1.8}$$

where $\partial V_j V_{j'}$ denotes the inner product $\langle \partial V_j, V_{j'} \rangle = \sum_{i=1}^d V_{j'}^i \partial_i V_j$.

(i) *Suppose that $m > 1$. Then we have the convergence*

$$\big(\vartheta_n(\tilde{X} - \tilde{X}^n), B\big) \to (U, B) \tag{1.9}$$

in the Skorohod space $D([0,T]; \mathbb{R}^{d+m+1})$ as n tends to infinity. The above process U is the solution of the linear SDE on $[0, T]$

$$dU_t = \sum_{j=0}^m \partial V_j(X_t) U_t \, dB_t^j + T^{2H-\frac{1}{2}} \sqrt{\frac{\kappa}{2}} \sum_{1 \le j' < j \le m} \phi_{jj'}(X_t) \, dW_t^{j'j} \tag{1.10}$$

with $U_0 = 0$, where $W = (W^{j'j})_{1 \le j' < j \le m}$ is a standard $\frac{m(m-1)}{2}$-dimensional Brownian motion independent of B and κ is the constant defined by (3.4) *in Section* 3.

(ii) *Suppose that $m = 1$ and $V_0 \not\equiv 0$. Then the above convergence* (1.9) *still holds true. The process U is the solution of the linear SDE on $[0, T]$*

$$dU_t = \sum_{j=0,1} \partial V_j(X_t) U_t \, dB_t^j + T^{H+\frac{1}{2}} \sqrt{\frac{\varrho}{2}} \phi_{10}(X_t) \, dW_t, \quad U_0 = 0, \tag{1.11}$$

where W is a one-dimensional standard Brownian motion independent of B and ρ is the constant defined in (3.29) *in Section* 3.

(iii) *Suppose that $m = 1$ and $V_0 \equiv 0$. Then, we have the following convergence in $L^p(\Omega)$ for all $p \ge 1$ and $t \in [0, T]$:*

$$n^{2H}\big(\tilde{X}_t - \tilde{X}_t^n\big) \to U_t, \tag{1.12}$$

where the process U is the solution of the linear equation

$$dU_t = \partial V(X_t) U_t \, dB_t - \frac{T^{2H}}{4} \sum_{i,i'=1}^d \big(V^i V^{i'} \partial_i \partial_{i'} V\big)(X_t) \, dB_t, \quad U_0 = 0. \tag{1.13}$$

Theorem 1.3 shows that in the multidimensional cases, one obtains the central limit theorem for the renormalized error process. It is worth mentioning that the equation of the limiting process U does not depend on ϕ_{0j}, $j = 0, \dots, m$. This is due to the fact that ϕ_{0j} arises from the higher order terms of the expansion of the error process. In the scalar case the convergence of the error process holds in $L^p(\Omega)$. One could prove tightness in item (ii) of Theorem 1.3, but this requires an additional effort and will not be discussed in this paper, because L^p convergence is stronger than f.d.d. convergence and the tightness is not so relevant here. Theorem 1.3 implies in particular that, generally speaking, the convergence rates in Theorem 1.1 are exact. It is worth mentioning that the cutoff of the convergence rates observed in [10, 19] is not present in either of these cases. The Crank–Nicolson scheme provides us a first example in which the convergence is impacted by the dimension of the system.

We should point out that in the degenerated cases, for instance when the commutators are zero, Theorem 1.3 only says that the corresponding asymptotic error is equal to zero. In such situations, further investigations of the scheme are required. In the following two results, we consider two levels of degeneracy:

$$(D1) \quad \phi_{jj'} \equiv 0 \quad \text{for } j, j' = 1, \dots, m \quad \text{and} \quad (D2) \quad \phi_{jj'} \equiv 0 \quad \text{for } j, j' = 0, 1, \dots, m.$$

THEOREM 1.4. *Let X, X^n, and V be as in Theorem 1.1. Then for any $p \geq 1$ there exists a constant $K = K_p$ independent of n such that for all $n \in \mathbb{N}$:*

$$\sup_{t\in[0,T]} (\mathbb{E}|X_t - X_t^n|^p)^{1/p} \leq \begin{cases} Kn^{-H-\frac{1}{2}} & \text{when } (D1) \text{ holds}, \\ Kn^{-2H} & \text{when } (D2) \text{ holds}. \end{cases} \tag{1.14}$$

Note that these rates in (1.14) are exactly those obtained in Theorem 1.1. They tell us that the rates of convergence are dictated by Lie brackets of coefficients rather than by the dimension of the system.

The next theorem provides our main result on the asymptotic errors of the scheme in degenerated cases.

THEOREM 1.5. *Let the processes $\tilde{X}$ and $\tilde{X}^n$, the functions V and $\phi_{jj'}$, and the constant ρ be as in Theorem 1.3.*

(i) *Suppose that $(D1)$ holds, and let U be the solution of the equation on $[0, T]$:*

$$dU_t = \sum_{j=1}^m \partial V_j(X_t)U_t\, dB_t^j + T^{H+\frac{1}{2}}\sqrt{\frac{\varrho}{2}} \sum_{j=1}^m \phi_{j0}(X_t)\, dW_t^j, \quad U_0 = 0,$$

where $W = (W^1, \dots, W^m)$ is an m-dimensional Brownian motion independent of B. Then we have the convergence in $D([0, T]; \mathbb{R}^{d+m+1})$ as $n \to \infty$:

$$\big(n^{H+\frac{1}{2}}(\tilde{X} - \tilde{X}^n), B\big) \to (U, B).$$

(ii) *Suppose that $(D2)$ holds, and let U be the solution of the linear equation:*

$$dU_t = \sum_{j=0}^m \partial V_j(X_t)U_t\, dB_t^j - \frac{T^{2H}}{4} \sum_{j=1}^m \int_0^t \psi_{jjj}(s)\, dB_s^j + \sum_{j=0}^m \int_0^t \varphi_j(s)\, dB_s^j,$$

and $U_0 = 0$, where we denote $\psi_{jj'j''}(t) = \sum_{i,i'=1}^d (V_{j'}^i V_{j''}^{i'} \partial_{i'} \partial_i V_j)(X_t)$ and

$$\varphi_j(t) = \frac{1}{2} T^{2H} \sum_{j' \notin \{0,j\}} \left(\frac{1-2H}{4H+2} \psi_{jj'j'}(t) - \frac{1}{2H+1} \psi_{j'j'j}(t) \right), \quad t \in [0, T].$$

Then we have the following convergence in $L^p(\Omega)$ for all $p \geq 1$ and $t \in [0, T]$:

$$n^{2H}(\tilde{X}_t - \tilde{X}_t^n) \to U_t. \tag{1.15}$$

Our first step to prove Theorem 1.1–1.5 is based on an explicit expression of $X - X^n$ similar to that established in [10]. A significant difficulty is the integrability of the Malliavin derivatives of the approximation X^n. This is due to the fact that the Crank–Nicolson scheme (1.4) is determined by an implicit equation. This difficulty will be handled thanks to some fractional calculus techniques, see for example, [3, 11, 27]. A special attention has to be paid also to the Lévy area type processes mentioned above. Our approach to handle these processes relies on a combination of fractional calculus and Malliavin calculus tools. Let us mention that it is possible to extend our results to the rough case by the approaches introduced in [13].

The paper is structured as follows. In Section 2, we recall some basic results on the fBm's as well as some upper bound estimate results and limit theorem results on fractional integrals. In Section 3, we consider the moment estimates and the weak convergence of some Lévy area type processes. In Section 4, we consider the strong convergence, and then in Section 5 we prove Theorem 1.3 on the asymptotic error. Section 6 focuses on the degenerate cases. Some auxiliary results are stated and proved in the Appendix.

2. Preliminaries.

2.1. *Fractional Brownian motions*. We briefly review some basic facts about the stochastic calculus with respect to a fBm. The reader is referred to [20, 21] for further details. Let $B = \{B_t, t \in [0, T]\}$ be a one-dimensional fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, defined on some complete probability space $(\Omega, \mathscr{F}, P)$. Namely, B is a mean zero Gaussian process with covariance

$$\mathbb{E}(B_s B_t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

for $s, t \in [0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of the set of step functions on $[0, T]$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

It is easy to verify that

$$\langle \phi, \psi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \phi_u \psi_v |u-v|^{2H-2} \, du \, dv \tag{2.1}$$

for every pair of step functions $\phi, \psi \in \mathcal{H}$.

The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by B. We denote this isometry by $h \mapsto B(h)$. In this way, $\{B(h), h \in \mathcal{H}\}$ is an isonormal Gaussian process indexed by the Hilbert space $\mathcal{H}$.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variable of the form

$$F = f(B_{t_1}, \dots, B_{t_N}),$$

where $N \geq 1$, $t_1, \dots, t_N \in [0, T]$ and $f \in C_b^\infty(\mathbb{R}^N)$, namely, f and all its partial derivatives are bounded. The derivative operator D on F is defined as the $\mathcal{H}$-valued random variable

$$D_t F = \sum_{i=1}^N \frac{\partial f}{\partial x_i}(B_{t_1}, \dots, B_{t_N}) \mathbf{1}_{[0,t_i]}(t), \quad t \in [0, T].$$

For $p \geq 1$ we define the Sobolev space $\mathbb{D}_B^{1,p}$ (or simply $\mathbb{D}^{1,p}$) as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{\mathbb{D}^{1,p}} = \big(\mathbb{E}(|F|^p) + \mathbb{E}(\|DF\|_{\mathcal{H}}^p)\big)^{1/p}.$$

The above definition of the Sobolev space $\mathbb{D}^{1,p}$ can be extended to $\mathcal{H}$-valued random variables (see Section 1.2 in [21]). We denote by $\mathbb{D}_B^{1,p}(\mathcal{H})$ (or simply $\mathbb{D}^{1,p}(\mathcal{H})$) the corresponding Sobolev space.

We denote by δ the adjoint of the derivative operator D. We say $u \in \operatorname{Dom} \delta$ if there is a $\delta(u) \in L^2(\Omega)$ such that for any $F \in \mathbb{D}^{1,2}$ the following duality relationship holds

$$\mathbb{E}(\langle u, DF \rangle_{\mathcal{H}}) = \mathbb{E}(F \delta(u)). \tag{2.2}$$

The random variable $\delta(u)$ is also called the Skorohod integral of u with respect to the fBm B, and we use the notation $\delta(u) = \int_0^T u_t \delta B_t$. The following result is an example of application of the duality relationship that will be used later in the paper.

LEMMA 2.1. *Let B and $\tilde{B}$ be independent one-dimensional fBm's with Hurst parameter $H \in (\frac{1}{2}, 1)$. Take $h \in \mathcal{H} \otimes \mathcal{H}$, then the integral $\int_0^T \int_0^T h_{s,t} \delta B_s \delta \tilde{B}_t$ is well defined. Denote by D and $\tilde{D}$ the derivative operators associated with B and $\tilde{B}$, respectively. Take $F \in \mathbb{D}_{\tilde{B}}^{1,2}$ and assume that $\tilde{D} F \in \mathbb{D}_B^{1,2}(\mathcal{H})$. Then, applying the integration by parts twice, we obtain*

$$\mathbb{E}(\langle h, D\tilde{D}F \rangle_{\mathcal{H} \otimes \mathcal{H}}) = \mathbb{E}\left(F \int_0^T \int_0^T h_{s,t} \delta B_s \delta \tilde{B}_t\right). \tag{2.3}$$

2.2. *Weighted random sums.* In this subsection, we recall some estimates and limit results for Riemann–Stieltjes integrals of stochastic processes. Our main references are [3, 10, 11, 27]. Let us start with the definition of Hölder continuous functions in $L^p := L^p(\Omega)$. In the following $\|\cdot\|_p$ denotes the L^p-norm in the space L^p, where $p \geq 1$.

DEFINITION 2.2. Let $\beta \in (0,1)$ and $p \geq 1$. Let $f = \{f(t), t \in [a,b]\}$ be a continuous process such that $f(t) \in L^p$ for all $t \in [a,b]$. Then f is called a Hölder continuous function of order β in L^p if the following relation holds true for all $s, t \in [a,b]$:

$$\|f(t) - f(s)\|_p \leq K|t-s|^\beta.$$

We denote by $\|f\|_{\beta,p}$ the Hölder semi-norm

$$\|f\|_{\beta,p} = \sup\left\{\frac{\|f(t)-f(s)\|_p}{|t-s|^\beta} : t, s \in [a,b], t \neq s\right\}.$$

Our first result provides an upper-bound estimate for the L^p-norm of a Riemann–Stieltjes integral.

LEMMA 2.3. *Take $p \geq 1$, $p', q' > 1 : \frac{1}{p'} + \frac{1}{q'} = 1$ and $\beta, \beta' \in (0,1) : \beta + \beta' > 1$. Let $f(t)$, $g(t)$, $t \in [a,b]$ be Hölder continuous functions of order β and β' in $L^{pp'}$ and $L^{pq'}$, respectively. Then the Riemann–Stieltjes integral $\int_a^b f\,dg$ is well defined in L^p, and we have the estimate*

$$\left\|\int_a^b f\,dg\right\|_p \leq \left(K\|f\|_{\beta,pp'} + \|f(a)\|_{pp'}\right)\|g\|_{\beta',pq'}(b-a)^{\beta'}, \tag{2.4}$$

where K is a constant depending only on the parameters p, p', q', β, β'.

PROOF. The proof is based on the fractional integration by parts formula (see [27]), following the arguments used in the proof of Lemma A.1 in [10]. □

Given a double sequence of random variables $\zeta = \{\zeta_{k,n}, n \in \mathbb{N}, k = 0, 1, \ldots, n\}$, for each $t \in [0,T]$ we set

$$g_n(t) := \sum_{k=0}^{\lfloor nt/T \rfloor} \zeta_{k,n}, \tag{2.5}$$

where $\lfloor nt/T \rfloor$ denotes the integer part of nt/T. We recall the following result from [11], which provides an upper-bound estimate for weighted random sums (or the so-called discrete integrals) of the process g_n.

LEMMA 2.4. *Let p, p', q', β, β' be as in Lemma 2.3. Let f be a Hölder continuous function of order β in $L^{pp'}$. Let g_n be as in (2.5) such that for any $j, k = 0, 1, \ldots, n$ we have*

$$\mathbb{E}\left(|g_n(kT/n) - g_n(jT/n)|^{pq'}\right) \leq K\left(|k-j|/n\right)^{\beta' pq'}.$$

Then the following estimate holds true for $i, j = 0, 1, \ldots, n$, $i > j$:

$$\left\|\sum_{k=j+1}^{i} f(t_k)\zeta_{k,n}\right\|_p \leq K\left(\|f\|_{\beta,pp'} + \|f(t_j)\|_{pp'}\right)\left(\frac{i-j}{n}\right)^{\beta'}.$$

Let us now recall some limit theorems for weighted random sums. The first result says that if the "weight-free" random sum (2.5) converges weakly and if the weight process satisfies certain regularity assumption, then the weighted random sum also converges weakly. The reader is referred to [3] for further details.

PROPOSITION 2.5. *Let g_n be defined in* (2.5). *Assume that g_n satisfies the inequality*

$$\mathbb{E}\big(\big|g_n(kT/n) - g_n(jT/n)\big|^4\big) \le K\big(|k-j|/n\big)^2$$

for $j, k = 0, 1, \dots, n$. Suppose further that the finite-dimensional distributions of g_n converge stably to those of $W = \{W_t, t \in [0, T]\}$, where W is a standard Brownian motion independent of g_n.

Let $f = \{f(t), t \in [0, T]\}$ be a Hölder continuous process of order β for $\beta > 1/2$. Consider the Riemann–Stieltjes integral $\int_0^t f(s)\,dW_s$. Recall that $\lfloor nt/T \rfloor$ denotes the integer part of nt/T.

Then the finite-dimensional distributions of $\sum_{k=0}^{\lfloor nt/T \rfloor} f(t_k)\zeta_{k,n}$ converge stably to those of $\int_0^t f(s)\,dW_s$.

Recall that a sequence of random vectors F_n converges stably to a random vector F, where F is defined on an extension $(\Omega', \mathscr{F}', \mathbb{P}')$ of the original probability $(\Omega, \mathscr{F}, \mathbb{P})$, if $(F_n, Z) \to (F, Z)$ weakly for any $\mathscr{F}$-measurable random variable Z. The reader is referred to [1, 12, 25] for further details on stable convergence.

The following result can be viewed as the L^p-convergence version of Proposition 2.5 (see [10]).

PROPOSITION 2.6. *Take $\beta, \lambda \in (0, 1) : \beta + \lambda > 1$. Let $p \ge 1$ and $p', q' > 1$ such that $\frac{1}{p'} + \frac{1}{q'} = 1$ and $pp' > \frac{1}{\beta}$, $pq' > \frac{1}{\lambda}$. Let g_n be defined in* (2.5). *Suppose that the following two conditions hold true*:

(i) *For $t \in [0, T]$, we have the convergence $g_n(t) \to z(t)$ in $L^{pq'}$*;
(ii) *For $j, k = 0, 1, \dots, n$ we have the relation*

$$\mathbb{E}\big(\big|g_n(kT/n) - g_n(jT/n)\big|^{pq'}\big) \le K\big(|k-j|/n\big)^{\lambda pq'}.$$

Let $f = \{f(t), t \in [0, T]\}$ be a continuous process such that $\mathbb{E}(\|f\|_\beta^{pp'}) \le K$ and $\mathbb{E}(|f(0)|^{pp'}) \le K$. Then for each $t \in [0, T]$ we have the convergence:

$$\lim_{n\to\infty} \sum_{k=0}^{\lfloor nt/T \rfloor} f(t_k)\zeta_{k,n} = \int_0^t f(s)\,dz(s),$$

where the limit is understand as the limit in L^p.

3. Lévy area type processes. Let $B = \{B_t, t \ge 0\}$ be a one-dimensional fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, and let $\tilde{B} = \{\tilde{B}_t, t \ge 0\}$ be a Hölder continuous process of order $\beta > \frac{1}{2}$. Let $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be the uniform partition on $[0, T]$ and take $t_{n+1} = \frac{n+1}{n}T$. For $t \in [t_l, t_{l+1}) \cap [0, T]$, $l = 0, \dots, n$, define the Lévy area type process on $[0, T]$

$$Z_n(t) = \sum_{k=0}^{l}\bigg(\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} d\tilde{B}_u\,dB_s - \int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}} d\tilde{B}_u\,dB_s\bigg). \tag{3.1}$$

In this section, we study the convergence rate and the asymptotic distribution of the sequence $\{Z_n, n \in \mathbb{N}\}$. We focus on two cases: (i) $\tilde{B}$ is an independent copy of B; and (ii) $\tilde{B}$ is the identity function: $\tilde{B}_t = t$ for $t \ge 0$.

3.1. *Case* (*i*). For simplicity, we denote by μ the measure on the plane $\mathbb{R}^2$ given by

$$\mu(ds\,dt) = H(2H-1)|s-t|^{2H-2}\,ds\,dt. \tag{3.2}$$

For each $p \in \mathbb{Z}$ we set

$$Q(p) = \int_{\substack{p<s<p+1\\0<t<1}} \int_{\substack{p<v<s\\0<u<t}} \mu(dv\,du)\mu(ds\,dt),$$
$$R(p) = \int_{\substack{p<s<p+1\\0<t<1}} \int_{\substack{p<v<s\\t<u<1}} \mu(dv\,du)\mu(ds\,dt).$$

We have the following result on the process Z_n.

PROPOSITION 3.1. *Let Z_n be the process defined by* (3.1) *and let $\tilde{B}$ be an independent copy of B. Then, there exists a constant K depending on H and T such that for $t, s \in \Pi$ we have*

$$n^{4H-1}\mathbb{E}\big(|Z_n(t) - Z_n(s)|^2\big) \le K|t-s|. \tag{3.3}$$

Furthermore, the finite-dimensional distributions of $(n^{2H-\frac{1}{2}}Z_n(t), B_t, \tilde{B}_t, t \in [0,T])$ converge weakly to those of $(T^{2H-\frac{1}{2}}\sqrt{2\kappa}W_t, B_t, \tilde{B}_t, t \in [0,T])$ as n tends to infinity, where $W = \{W_t, t \in [0,T]\}$ is a standard Brownian motion independent of $(B, \tilde{B})$, and

$$\kappa = \sum_{p\in\mathbb{Z}} \big(Q(p) - R(p)\big). \tag{3.4}$$

REMARK 3.2. Figure 1 provides the graph of the parameter κ as a function of H on $(\frac{1}{2}, 1)$. We observe that κ converges to $\frac{1}{2}$ as H tends to $\frac{1}{2}$ which corresponds to the Brownian motion, and it approaches zero as H tends to one.

PROOF OF PROPOSITION 3.1. The proof is divided into several steps.

Step 1. In this step, we show the convergence of $n^{4H-1}\mathbb{E}(Z_n(t)^2)$ and derive its limit as $n \to \infty$. We first calculate the second moment of $Z_n(t)$. Note that when $\tilde{B}$ is an independent

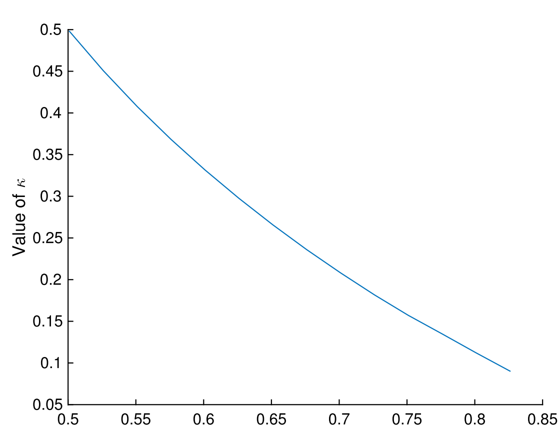

FIG. 1. *The value of κ.*

copy of B we have

$$Z_n(t) = \sum_{k=0}^{\lfloor nt/T \rfloor} \left(\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \delta \tilde{B}_u \delta B_s - \int_{t_k}^{t_{k+1}} \int_{s}^{t_{k+1}} \delta \tilde{B}_u \delta B_s \right) \tag{3.5}$$
$$= \sum_{k=0}^{\lfloor nt/T \rfloor} \int_0^T \int_0^T \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u) \delta \tilde{B}_u \delta B_s,$$

where δ denotes the Skorohod integral and

$$\beta_{\frac{k}{n}}(s) = \mathbf{1}_{[t_k,t_{k+1}]}(s), \qquad \gamma_{t_k,s}(u) = \mathbf{1}_{[t_k,s]}(u) - \mathbf{1}_{[s,t_{k+1}]}(u). \tag{3.6}$$

By the integration by parts formula (2.3) and taking into account the expression of $Z_n(t)$ in (3.5) we obtain

$$\mathbb{E}(Z_n(t)^2) = \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{[0,T]^4} \tilde{D}_{u'} D_{s'} Z_n(t) \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u) \mu(du\, du') \mu(ds\, ds'), \tag{3.7}$$

where D and $\tilde{D}$ are the derivative operators associated with B and $\tilde{B}$, respectively. It is clear that

$$\tilde{D}_{u'} D_{s'} Z_n(t) = \sum_{k=0}^{\lfloor nt/T \rfloor} \beta_{\frac{k}{n}}(s') \gamma_{t_k,s'}(u').$$

Therefore, we obtain the expression

$$\mathbb{E}(Z_n(t)^2) = \sum_{k,k'=0}^{\lfloor nt/T \rfloor} \int_{[0,T]^4} \beta_{\frac{k'}{n}}(s') \beta_{\frac{k}{n}}(s) \gamma_{t_{k'},s'}(u') \gamma_{t_k,s}(u) \mu(du\, du') \mu(ds\, ds'). \tag{3.8}$$

By changing the variables from (u, u', s, s') to $\frac{T}{n}(u, u', s, s')$, we obtain

$$\mathbb{E}(Z_n(t)^2) = \left(\frac{T}{n}\right)^{4H} \sum_{k,k'=0}^{\lfloor nt/T \rfloor} \int_{\substack{k'<s'<k'+1 \\ k<s<k+1}} \int_{0<u,u'<n} \varphi_{k',s'}(u') \varphi_{k,s}(u) \mu(du\, du') \mu(ds\, ds'),$$

where $\varphi_{k,s}(u) = \mathbf{1}_{[k,s]}(u) - \mathbf{1}_{[s,k+1]}(u)$. Denote $\varphi^0_{k,s}(u) = \mathbf{1}_{[k,s]}(u)$, $\varphi^1_{k,s}(u) = \mathbf{1}_{[s,k+1]}(u)$, and set

$$e_{ij} = \int_{\substack{k'<s'<k'+1 \\ k<s<k+1}} \int_{0<u,u'<n} \varphi^i_{k',s'}(u') \varphi^j_{k,s}(u) \mu(du\, du') \mu(ds\, ds').$$

Then we can write

$$\mathbb{E}(Z_n(t)^2) = \left(\frac{T}{n}\right)^{4H} \sum_{k,k'=0}^{\lfloor nt/T \rfloor} \sum_{i,j=0,1} (-1)^{i+j} e_{ij}.$$

It is easy to see that $e_{00} = e_{11} = Q(k-k')$ and $e_{01} = e_{10} = R(k-k')$. Therefore,

$$\mathbb{E}(Z_n(t)^2) = 2\left(\frac{T}{n}\right)^{4H} \sum_{k,k'=0}^{\lfloor nt/T \rfloor} [Q(k-k') - R(k-k')]. \tag{3.9}$$

Taking $p = k - k'$ on the right-hand side of (3.9), we obtain

$$\mathbb{E}(Z_n(t)^2)$$
$$= 2\left(\frac{T}{n}\right)^{4H} \left(\sum_{p=0}^{\lfloor nt/T \rfloor} \sum_{k=p}^{\lfloor nt/T \rfloor} [Q(p) - R(p)] + \sum_{p=-\lfloor nt/T \rfloor}^{-1} \sum_{k=0}^{\lfloor nt/T \rfloor + p} [Q(p) - R(p)] \right) \tag{3.10}$$
$$:= q_1 + q_2.$$

We decompose q_1 as follows

$$\begin{aligned} q_1 &= 2\Big(\frac{T}{n}\Big)^{4H} \sum_{p=0}^{\lfloor nt/T \rfloor} \Big(\Big\lfloor \frac{nt}{T} \Big\rfloor - p + 1\Big)\big(Q(p) - R(p)\big) \\ &= 2\Big(\frac{T}{n}\Big)^{4H} \Big(\Big\lfloor \frac{nt}{T} \Big\rfloor \sum_{p=0}^{\lfloor nt/T \rfloor} \big(Q(p) - R(p)\big) - \sum_{p=0}^{\lfloor nt/T \rfloor} (p-1)\big(Q(p) - R(p)\big)\Big) \\ &:= q_{11} + q_{12}. \end{aligned}$$

By the mean value theorem for the integrals appearing in the definitions of Q and P, it is easy to show that $|Q(p) - R(p)| \le K p^{4H-5}$ for $p > 0$. This implies that $\sum_{p=0}^{\infty}(Q(p) - R(p))$ is convergent and we also have

$$\Big| \sum_{p=0}^{\lfloor nt/T \rfloor} (p-1)\big(Q(p) - R(p)\big) \Big| \le K\big(n^{4H-3} \vee 1\big).$$

Here $a \vee b$ denotes the maximum of a and b and notice that the upper bound 1 is needed when $4H - 3 < 0$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} n^{4H-1} q_{11} &= \lim_{n\to\infty} 2n^{4H-1}\Big(\frac{T}{n}\Big)^{4H} \Big\lfloor \frac{nt}{T} \Big\rfloor \sum_{p=0}^{\lfloor nt/T \rfloor} \big(Q(p) - R(p)\big) \\ &= 2tT^{4H-1} \sum_{p=0}^{\infty} \big(Q(p) - R(p)\big) \end{aligned} \tag{3.11}$$

and

$$\lim_{n\to\infty} n^{4H-1} q_{12} = 0. \tag{3.12}$$

In summary, from (3.11) and (3.12), we obtain

$$\lim_{n\to\infty} n^{4H-1} q_1 = 2tT^{4H-1} \sum_{p=0}^{\infty} \big(Q(p) - R(p)\big). \tag{3.13}$$

In a similar way, we can prove the following convergence for q_2

$$\lim_{n\to\infty} n^{4H-1} q_2 = 2tT^{4H-1} \sum_{p=-\infty}^{-1} \big(Q(p) - R(p)\big). \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.10) yields

$$\lim_{n\to\infty} n^{4H-1} \mathbb{E}\big(Z_n(t)^2\big) = 2T^{4H-1}\kappa t, \tag{3.15}$$

where recall that κ is a constant defined in (3.4).

Step 2. In this step, we show inequality (3.3). This inequality is obvious when $s = t$. In the following we consider the case when $t > s$.

Take $t \in \Pi$. By the definition of q_1 we have

$$\begin{aligned} q_1 &\le 2\Big(\frac{T}{n}\Big)^{4H} \sum_{p=0}^{\lfloor nt/T \rfloor} \Big(\frac{nt}{T} - p + 1\Big)\big|Q(p) - R(p)\big| \\ &\le 2\Big(\frac{T}{n}\Big)^{4H} \Big(\frac{nt}{T} + 1\Big) \sum_{p=0}^{\infty} \big|Q(p) - R(p)\big|. \end{aligned}$$

In the same way, we can show that

$$q_2 \le 2\left(\frac{T}{n}\right)^{4H}\left(\frac{nt}{T}+1\right)\sum_{p=-\infty}^{-1}\big|Q(p)-R(p)\big|.$$

Applying these two inequalities to (3.10) we obtain

$$n^{4H-1}\mathbb{E}\big(Z_n(t)^2\big) \le K\left(t+\frac{T}{n}\right) \tag{3.16}$$

for $t \in \Pi$, where K is a constant depending on H, T. Take $s, t \in \Pi$ such that $s < t$. Inequality (3.3) then follows by replacing t in (3.16) by $t - s - \frac{T}{n}$ and noticing that $Z_n(t) - Z_n(s)$ and $Z_n(t - s - \frac{T}{n})$ are equal in distribution and thus have the same second moments.

Step 3. Take $s, t \in [0, T]$ such that $s < t$. In this step, we derive the limit of the quantity $n^{4H-1}\mathbb{E}(Z_n(t)Z_n(s))$. Denote $\eta(t) = t_k$ for $t \in [t_k, t_{k+1})$, $k = 0, 1, \dots, n$. Then we have $Z_n(t) = Z_n(\eta(t))$. Since $Z_n(\eta(t)) - Z_n(\eta(s))$ and $Z_n(\eta(t) - \eta(s) - \frac{T}{n})$ have the same distribution, we have

$$\begin{aligned}\mathbb{E}\big(\big|Z_n(t)-Z_n(s)\big|^2\big) &= \mathbb{E}\big(\big|Z_n\big(\eta(t)\big)-Z_n\big(\eta(s)\big)\big|^2\big)\\ &= \mathbb{E}\left(\left|Z_n\left(\eta(t)-\eta(s)-\frac{T}{n}\right)\right|^2\right).\end{aligned} \tag{3.17}$$

Note that $0 < (t - s) - (\eta(t) - \eta(s) - \frac{T}{n}) < 2\frac{T}{n}$, so either $Z_n(\eta(t) - \eta(s) - \frac{T}{n}) = Z_n(t - s)$ or $Z_n(\eta(t) - \eta(s) - \frac{T}{n}) = Z_n(t - s - \frac{T}{n})$. In both cases we have

$$\lim_{n\to\infty} n^{4H-1}\mathbb{E}\left(\left|Z_n\left(\eta(t)-\eta(s)-\frac{T}{n}\right)\right|^2\right) = \lim_{n\to\infty} n^{4H-1}\mathbb{E}\big(\big|Z_n(t-s)\big|^2\big). \tag{3.18}$$

Indeed, the identity is clear in the first case. In the second case we write

$$\begin{aligned}&n^{4H-1}\left(\mathbb{E}\big(\big|Z_n(t-s)\big|^2\big)-\mathbb{E}\left(\left|Z_n\left(t-s-\frac{T}{n}\right)\right|^2\right)\right)\\ &\quad= n^{4H-1}\mathbb{E}\left(\left(Z_n(t-s)-Z_n\left(t-s-\frac{T}{n}\right)\right)\left(Z_n(t-s)+Z_n\left(t-s-\frac{T}{n}\right)\right)\right).\end{aligned}$$

Then, applying Hölder's inequality and the estimate (3.16) to the right-hand side, yields

$$n^{4H-1}\left|\mathbb{E}\big(\big|Z_n(t-s)\big|^2\big)-\mathbb{E}\left(\left|Z_n\left(t-s-\frac{T}{n}\right)\right|^2\right)\right| \le K\frac{T}{n}. \tag{3.19}$$

This implies, in particular, that the right-hand side of (3.19) converges to zero as $n \to \infty$ and thus relation (3.18) holds.

Substituting (3.18) into (3.17) and with the help of (3.15) we obtain

$$\lim_{n\to\infty} n^{4H-1}\mathbb{E}\big(\big|Z_n(t)-Z_n(s)\big|^2\big) = 2T^{4H-1}\kappa(t-s). \tag{3.20}$$

By expanding the left-hand side of (3.20) and using (3.15), we obtain

$$\lim_{n\to\infty} n^{4H-1}\mathbb{E}\big(Z_n(t)Z_n(s)\big) = 2T^{4H-1}\kappa(t\wedge s), \qquad s,t \in [0,T]. \tag{3.21}$$

Step 4. In this step we prove the weak convergence of the finite-dimensional distributions of $(n^{2H-\frac{1}{2}}Z_n, B, \tilde{B})$. Given $r_1, \dots, r_L \in [0, T]$, $L \in \mathbb{N}$, we need to show that the random vector

$$\Theta_L^n := \big(n^{2H-\frac{1}{2}}\big(Z_n(r_1),\dots,Z_n(r_L)\big), B_{r_1},\dots,B_{r_L},\tilde{B}_{r_1},\dots,\tilde{B}_{r_L}\big)$$

converges in law to

$$\Theta_L := \big(T^{2H-\frac{1}{2}}\sqrt{2\kappa}\big(W(r_1), \dots, W(r_L)\big), B_{r_1}, \dots, B_{r_L}, \tilde{B}_{r_1}, \dots, \tilde{B}_{r_L}\big)$$

as n tends to infinity, where recall that $W = \{W_t, t \in [0, T]\}$ is a standard Brownian motion independent of $(B, \tilde{B})$. According to [24] (see also Theorem 6.2.3 in [20]), this is true if we can show the weak convergence of each component of Θ_L^n to the corresponding component of Θ_L and the convergence of its covariance matrix to that of Θ_L.

The convergence of the covariance of $n^{2H-\frac{1}{2}} Z_n(r_i)$ and $n^{2H-\frac{1}{2}} Z_n(r_j)$ to that of $T^{2H-\frac{1}{2}}\sqrt{2\kappa} W(r_i)$ and $T^{2H-\frac{1}{2}}\sqrt{2\kappa} W(r_j)$ follows from (3.21). The covariance of $n^{2H-\frac{1}{2}} Z_n(r_i)$ and $(B_{r_j}, \tilde{B}_{r_j})$ is zero since they are in different chaoses, so the limit of the covariance is zero, which equals the covariance of $T^{2H-\frac{1}{2}}\sqrt{2\kappa} W(r_i)$ and $(B_{r_j}, \tilde{B}_{r_j})$ since W and B are independent.

By the fourth moment theorem (see [22] and also Theorem 5.2.7 in [20]) and taking into account (3.21), to show the weak convergence of the components of Θ_L^n it remains to show that the limits of their fourth moments exist, and

$$\lim_{n\to\infty} n^{8H-2}\mathbb{E}\big(Z_n(t)^4\big) = 3 \lim_{n\to\infty} n^{8H-2}\big(\mathbb{E}\big(Z_n(t)^2\big)\big)^2 \tag{3.22}$$

for $t \in [0, T]$.

Applying the integration by parts formula (2.3) to $\mathbb{E}(Z_n(t)^4)$ and taking into account the expression of $Z_n(t)$ in (3.5), we obtain

$$\begin{aligned} \mathbb{E}\big(Z_n(t)^4\big) &= \mathbb{E}\big(Z_n(t)^3 \cdot Z_n(t)\big) \\ &= \sum_{k=0}^{\lfloor nt/T \rfloor} \mathbb{E} \int_{[0,T]^4} \tilde{D}_{u'} D_{s'}\big[Z_n(t)^3\big] \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u) \mu\big(du\, du'\big) \mu\big(ds\, ds'\big), \end{aligned} \tag{3.23}$$

where D and $\tilde{D}$ are the differential operators associated with B and $\tilde{B}$, respectively. We expand the second derivative $\tilde{D}_{u'} D_{s'}[Z_n(t)^3]$ as follows:

$$\tilde{D}_{u'} D_{s'}\big[Z_n(t)^3\big] = 3Z_n(t)^2 \tilde{D}_{u'} D_{s'} Z_n(t) + 6Z_n(t) \tilde{D}_{u'} Z_n(t) D_{s'} Z_n(t).$$

Substituting the above identity into (3.23), we obtain

$$\mathbb{E}\big(Z_n(t)^4\big) = d_1 + d_2,$$

where

$$d_2 = 6 \sum_{k=0}^{\lfloor nt/T \rfloor} \mathbb{E} \int_{[0,T]^4} Z_n(t) \tilde{D}_{u'} Z_n(t) D_{s'} Z_n(t) \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u) \mu\big(du\, du'\big) \mu\big(ds\, ds'\big) \tag{3.24}$$

and

$$d_1 = 3\mathbb{E}\big(Z_n(t)^2\big) \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{[0,T]^4} \tilde{D}_{u'} D_{s'} Z_n(t) \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u) \mu\big(du\, du'\big) \mu\big(ds\, ds'\big). \tag{3.25}$$

Substituting (3.7) into d_1, we obtain

$$d_1 = 3\mathbb{E}\big(Z_n(t)^2\big)\mathbb{E}\big(Z_n(t)^2\big). \tag{3.26}$$

The term d_2 is more sophisticated. We shall prove in Section A.1 the following fact:

$$\lim_{n\to\infty} n^{8H-2} d_2 = 0. \tag{3.27}$$

The identity (3.26) and the convergence (3.27) together imply the identity (3.22). This completes the proof. □

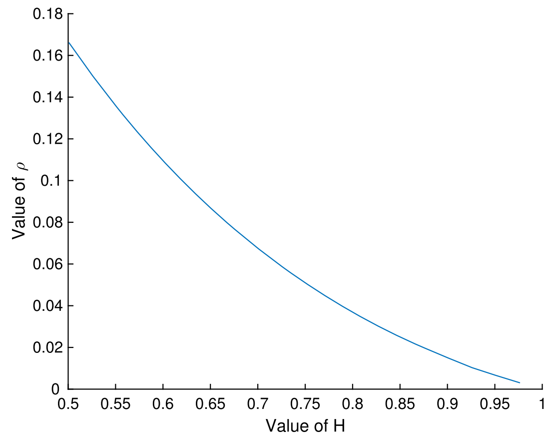

FIG. 2. *The value of ρ.*

3.2. *Case (ii).* In this subsection we consider the process Z_n in (3.1) with the assumption that $\tilde{B}_t = t$, $t \in [0,T]$. In order to distinguish this new Z_n from the one in the previous subsection, we will denote it by z_n. For each $p \in \mathbb{Z}$, we define two quantities

$$\tilde{Q}(p) = \int_{\substack{p<s<p+1\\0<t<1}} \int_{\substack{p<v<s\\0<u<t}} dv\,du\,\mu(ds\,dt), \qquad \tilde{R}(p) = \int_{\substack{p<s<p+1\\0<t<1}} \int_{\substack{p<v<s\\t<u<1}} dv\,du\,\mu(ds\,dt),$$

where recall that $\mu(ds\,dt) = H(2H-1)|s-t|^{2H-2}\,ds\,dt$ is a measure on $\mathbb{R}^2$.

PROPOSITION 3.3. *Let z_n be the process defined in (3.1) where we take $\tilde{B}_t = t$, $t \in [0,T]$. Then, there exists a constant K depending on H and T, such that for $t, s \in \Pi$*

$$n^{2H+1}\mathbb{E}\big((z_n(t) - z_n(s))^2\big) \le K|t-s|. \tag{3.28}$$

Moreover, the finite-dimensional distributions of the process $(n^{H+\frac{1}{2}} z_n, B)$ converge weakly to those of $(\sqrt{2\varrho}\,T^{H+\frac{1}{2}} W, B)$ as $n \to \infty$, where W is a Brownian motion independent of B and

$$\varrho := \sum_{p\in\mathbb{Z}} \big(\tilde{Q}(p) - \tilde{R}(p)\big). \tag{3.29}$$

REMARK 3.4. Figure 2 provides the graph of the parameter ρ versus H on $(\frac{1}{2}, 1)$. We see that ρ converges to $\frac{1}{6}$ as H tends to $\frac{1}{2}$, and ρ approaches zero as H tends to one.

PROOF OF PROPOSITION 3.3. The proof will be done in several steps.

Step 1. We first calculate the second moment of $z_n(t)$. We rewrite $z_n(t)$ as

$$z_n(t) = \sum_{k=0}^{\lfloor nt/T \rfloor} \int_0^T \int_0^T \beta_{\frac{k}{n}}(s)\gamma_{t_k,s}(u)\,du\,\delta B_s,$$

where $\beta_{\frac{k}{n}}(s)$ and $\gamma_{t_k,s}(u)$ are as in (3.6), and then applying the covariance formula (2.1) we obtain

$$\mathbb{E}\big(z_n(t)^2\big) = \sum_{k,k'=0}^{\lfloor nt/T \rfloor} \int_{[0,T]^4} \beta_{\frac{k'}{n}}(s')\beta_{\frac{k}{n}}(s)\gamma_{t_{k'},s'}(u')\gamma_{t_k,s}(u)\,du\,du'\mu(ds\,ds'). \tag{3.30}$$

Note that, in comparison with formula (3.8), in the right-hand side of (3.30) the measure μ appears only once and this expression does not have the same symmetry of variables as

(3.8). In the following we compute $\mathbb{E}(z_n(t)^2)$ in detail, which differs from the computations performed in Proposition 3.1.

We apply a change of variables from (u, u', s, s') to $\frac{T}{n}(u, u', s, s')$ to obtain

$$
\begin{aligned}
&\mathbb{E}\big(z_n(t)^2\big)\\
&\quad= \left(\frac{T}{n}\right)^{2H+2} \sum_{k,k'=0}^{\lfloor nt/T\rfloor} \int_{\substack{k'<s'<k'+1\\ k<s<k+1}} \int_{0<u,u'<n} \varphi_{k',s'}(u')\varphi_{k,s}(u)\,du\,du'\,\mu(ds\,ds')\\
&\quad= \left(\frac{T}{n}\right)^{2H+2} \sum_{k,k'=0}^{\lfloor nt/T\rfloor} \sum_{i,j=0,1} (-1)^{i+j}\tilde{e}_{ij},
\end{aligned}
$$

where

$$
\tilde{e}_{ij} = \int_{\substack{k'<s'<k'+1\\ k<s<k+1}} \int_{0<u,u'<n} \varphi^i_{k',s'}(u')\varphi^j_{k,s}(u)\,du\,du'\,\mu(ds\,ds'),
$$

and $\varphi^0_{k,s} = \mathbf{1}_{[k,s]}$, $\varphi^1_{k,s} = \mathbf{1}_{[s,k+1]}$ and $\varphi_{k,s} = \varphi^0_{k,s} - \varphi^1_{k,s}$ are defined as in the previous subsection. It is easy to see that

$$
\tilde{e}_{00} = \tilde{Q}(k-k') \quad \text{and} \quad \tilde{e}_{10} = \tilde{R}(k-k'). \tag{3.31}
$$

By the change of variables from (s, s') to $(k+1-s, k+1-s')$, we obtain

$$
\tilde{e}_{11} = \int_{k-k'}^{k-k'+1} \int_0^1 \big(s' - (k-k')\big)s\,\mu(ds\,ds') = \tilde{Q}(k-k'), \tag{3.32}
$$

where the second equation follows by exchanging the orders of the two integrals. By changing the variables from (s, s') to $(k'+1-s, k'+1-s')$ for $\tilde{e}_{11}$, we obtain

$$
\tilde{e}_{01} = \int_0^1 \int_{k'-k}^{k'-k+1} (1-s')\big(s - (k'-k)\big)\mu(ds\,ds') = \tilde{R}(k'-k),
$$

and, therefore,

$$
\sum_{k,k'=0}^{\lfloor nt/T\rfloor} \tilde{e}_{01} = \sum_{k,k'=0}^{\lfloor nt/T\rfloor} \tilde{R}(k'-k) = \sum_{k,k'=0}^{\lfloor nt/T\rfloor} \tilde{R}(k-k'). \tag{3.33}
$$

In summary, from (3.31), (3.32) and (3.33), we obtain

$$
\begin{aligned}
\mathbb{E}\big(z_n(t)^2\big) &= 2\left(\frac{T}{n}\right)^{2H+2} \sum_{k,k'=0}^{\lfloor nt/T\rfloor} \big(\tilde{Q}(k-k') - \tilde{R}(k-k')\big)\\
&= 2\left(\frac{T}{n}\right)^{2H+2} \left(\sum_{p=0}^{\lfloor nt/T\rfloor} \sum_{k'=0}^{\lfloor nt/T\rfloor - p} \big(\tilde{Q}(p) - \tilde{R}(p)\big) \right.\\
&\qquad \left. + \sum_{p=-\lfloor nt/T\rfloor}^{-1} \sum_{k'=-p}^{\lfloor nt/T\rfloor} \big(\tilde{Q}(p) - \tilde{R}(p)\big) \right)\\
&:= \tilde{q}_1 + \tilde{q}_2.
\end{aligned} \tag{3.34}
$$

Step 2. In this step we show inequality (3.28). Since $|\tilde{Q}(p) - \tilde{R}(p)| \sim p^{2H-3}$ for sufficiently large p, it is easy to see that the series $\sum_{p\in\mathbb{Z}} |\tilde{Q}(p) - \tilde{R}(p)|$ is convergent. So we have the estimates

$$
\tilde{q}_1 \le 2\left(\frac{T}{n}\right)^{2H+2} \left(\frac{nt}{T} + 1\right) \sum_{p=0}^{\infty} |\tilde{Q}(p) - \tilde{R}(p)| \tag{3.35}
$$

and

$$\tilde{q}_2 \le 2\left(\frac{T}{n}\right)^{2H+2}\left(\frac{nt}{T}+1\right)\sum_{p=-\infty}^{-1}|\tilde{Q}(p)-\tilde{R}(p)|. \tag{3.36}$$

Applying (3.35) and (3.36) to (3.34), yields

$$n^{2H+1}\mathbb{E}(z_n(t)^2) \le K\left(t+\frac{T}{n}\right). \tag{3.37}$$

Take $s,t \in \Pi$. By replacing t in (3.37) by $t-s-\frac{T}{n}$ and noticing that $z_n(t)-z_n(s)$ and $z_n(t-s-\frac{T}{n})$ have the same distribution, we obtain

$$\begin{aligned} n^{2H+1}\mathbb{E}(|z_n(t)-z_n(s)|^2) &= n^{2H+1}\mathbb{E}\left(\left|z_n\left(t-s-\frac{T}{n}\right)\right|^2\right) \\ &\le K(t-s). \end{aligned} \tag{3.38}$$

This completes the proof of (3.28).

Step 3. In this step, we show the convergence of the process $(n^{H+\frac{1}{2}}z_n, B)$. Note that the finite-dimensional distributions of $(n^{H+\frac{1}{2}}z_n, B)$ are Gaussian, so to show their convergences it suffices to show the convergences of their covariances. We first consider the convergence of $n^{2H+1}\mathbb{E}(|z_n(t)|^2)$. To this aim, we write

$$\begin{aligned} \tilde{q}_1 &= 2\left(\frac{T}{n}\right)^{2H+2}\sum_{p=0}^{\lfloor nt/T\rfloor}\left(\left\lfloor\frac{nt}{T}\right\rfloor-p+1\right)(\tilde{Q}(p)-\tilde{R}(p)) \\ &= 2\left(\frac{T}{n}\right)^{2H+2}\left(\left\lfloor\frac{nt}{T}\right\rfloor\sum_{p=0}^{\lfloor nt/T\rfloor}(\tilde{Q}(p)-\tilde{R}(p))-\sum_{p=0}^{\lfloor nt/T\rfloor}(p-1)(\tilde{Q}(p)-\tilde{R}(p))\right) \\ &:= \tilde{q}_{11}+\tilde{q}_{12}. \end{aligned} \tag{3.39}$$

First, it is easy to verify the following convergence

$$\begin{aligned} \lim_{n\to\infty} n^{2H+1}\tilde{q}_{11} &= \lim_{n\to\infty} 2n^{2H+1}\left(\frac{T}{n}\right)^{2H+2}\left\lfloor\frac{nt}{T}\right\rfloor\sum_{p=0}^{\lfloor nt/T\rfloor}(\tilde{Q}(p)-\tilde{R}(p)) \\ &= 2T^{2H+1}t\sum_{p=0}^{\infty}(\tilde{Q}(p)-\tilde{R}(p)). \end{aligned} \tag{3.40}$$

On the other hand, since $|\sum_{p=0}^{\lfloor nt/T\rfloor}(p-1)(\tilde{Q}(p)-\tilde{R}(p))| \le Kn^{2H-1}$, we have the convergence

$$\lim_{n\to\infty} n^{2H+1}\tilde{q}_{12} = 0. \tag{3.41}$$

Putting together (3.40) and (3.41), and taking into account (3.39), we obtain

$$\lim_{n\to\infty} n^{2H+1}\tilde{q}_1 = 2T^{2H+1}t\sum_{p=0}^{\infty}(\tilde{Q}(p)-\tilde{R}(p)). \tag{3.42}$$

The quantity $\tilde{q}_2$ can be considered in a similar way. We can show that

$$\lim_{n\to\infty} n^{2H+1}\tilde{q}_2 = 2T^{2H+1}t\sum_{p=-\infty}^{-1}(\tilde{Q}(p)-\tilde{R}(p)). \tag{3.43}$$

Applying (3.42) and (3.43) to (3.34) we obtain

$$\lim_{n\to\infty} n^{2H+1}\mathbb{E}(z_n(t)^2) = 2T^{2H+1}\varrho t. \tag{3.44}$$

Take $s, t \in [0, T]$. By the same argument as in (3.21) and with the help of (3.37) and (3.44), we can show that

$$\lim_{n\to\infty} n^{2H+1}\mathbb{E}(z_n(t)z_n(s)) = 2T^{2H+1}\varrho(t\wedge s). \tag{3.45}$$

On the other hand, by some elementary computation (see Section A.2), one can show that

$$\lim_{n\to\infty} \mathbb{E}(z_n(t)B_r) = 0. \tag{3.46}$$

Therefore, combining (3.45) and (3.46), we conclude that the covariances of the finite-dimensional distributions of $(n^{H+\frac{1}{2}}z_n, B)$ converge to those of $(\sqrt{2\varrho}T^{H+\frac{1}{2}}W, B)$. The proof is now complete. □

4. The strong convergence. We recall that X is the solution of equation (1.1) and X^n is the continuous time interpolation of the Crank–Nicolson scheme defined in (1.5). In this section we prove Theorem 1.1 and some auxiliary results.

PROOF OF THEOREM 1.1. The proof is divided into several steps.

Step 1: Decomposition of the error process. Denote $Y_t := X_t - X_t^n$, $t \in [0, T]$, and for convenience we write $\eta(t) = t_k$ for $t \in [t_k, t_{k+1})$ and $\epsilon(t) = t_{k+1}$ for $t \in (t_k, t_{k+1}]$. Equations (1.1) and (1.5) allows us to write

$$\begin{aligned} Y_t &= \int_0^t [V(X_s) - V(X_s^n)]\,dB_s + \frac{1}{2}\int_0^t [V(X_s^n) - V(X_{\eta(s)}^n)]\,dB_s \\ &\quad + \frac{1}{2}\int_0^t [V(X_s^n) - V(X_{\epsilon(s)}^n)]\,dB_s \\ &= \sum_{j=0}^m \sum_{i=1}^d \int_0^t V_{ji}(s) Y_s^i\,dB_s^j + \frac{1}{2}J_1(t) + \frac{1}{2}J_2(t), \end{aligned} \tag{4.1}$$

where we have set for $t \in [0, T]$:

$$V_{ji}(s) = \int_0^1 \partial_i V_j(\theta X_s + (1-\theta)X_s^n)\,d\theta, \tag{4.2}$$

$$\begin{aligned} J_1(t) &= \int_0^t [V(X_s^n) - V(X_{\eta(s)}^n)]\,dB_s, \\ J_2(t) &= \int_0^t [V(X_s^n) - V(X_{\epsilon(s)}^n)]\,dB_s, \end{aligned} \tag{4.3}$$

and we denote by ∂_i the partial differential operator with respect to the ith variable, that is, $\partial_i f(x) = \frac{\partial f}{\partial x_i}(x)$ for $f \in C^1$. Notice that by the chain rule for the Young integral, we obtain

$$\begin{aligned} V(X_s^n) - V(X_{\eta(s)}^n) &= \sum_{i=1}^d \partial_i V(X_{\eta(s)}^n)(X_s^{n,i} - X_{\eta(s)}^{n,i}) \\ &\quad + \sum_{i,i'=1}^d \int_{\eta(s)}^s \int_{\eta(s)}^u \partial_{i'}\partial_i V(X_v^n)\,dX_v^{n,i'}\,dX_u^{n,i}. \end{aligned}$$

Substituting the above expression into $J_1(t)$, we obtain the following decomposition for $J_1(t)$

$$J_1(t) = R_0(t) + R_1(t), \quad t \in [0, T], \tag{4.4}$$

where we denote

$$R_1(t) = \int_0^t \left[\sum_{i,i'=1}^{d} \int_{\eta(s)}^{s} \int_{\eta(s)}^{u} \partial_{i'} \partial_i V(X_v^n)\, dX_v^{n,i'}\, dX_u^{n,i} \right] dB_s \tag{4.5}$$

and

$$\begin{aligned} R_0(t) &= \int_0^t \left[\sum_{i=1}^{d} \partial_i V(X_{\eta(s)}^n)(X_s^{n,i} - X_{\eta(s)}^{n,i}) \right] dB_s \\ &= \frac{1}{2} \sum_{i=1}^{d} \sum_{j,j'=0}^{m} \int_0^t \partial_i V_j(X_{\eta(s)}^n) \left([V_{j'}^i(X_{\epsilon(s)}^n) + V_{j'}^i(X_{\eta(s)}^n)] \int_{\eta(s)}^{s} dB_u^{j'} \right) dB_s^j \end{aligned} \tag{4.6}$$

and in the second equation of (4.6) we have used relation (1.5).

In a similar way as for (4.4), we have

$$J_2(t) = -\tilde{R}_0(t) + \tilde{R}_1(t), \quad t \in [0, T],$$

where

$$\tilde{R}_1(t) = \int_0^t \left[\sum_{i,i'=1}^{d} \int_{s}^{\epsilon(s)} \int_{u}^{\epsilon(s)} \partial_{i'} \partial_i V(X_v^n)\, dX_v^{n,i'}\, dX_u^{n,i} \right] dB_s \tag{4.7}$$

and

$$\tilde{R}_0(t) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j,j'=0}^{m} \int_0^t \partial_i V_j(X_{\epsilon(s)}^n) \left([V_{j'}^i(X_{\epsilon(s)}^n) + V_{j'}^i(X_{\eta(s)}^n)] \int_{s}^{\epsilon(s)} dB_u^{j'} \right) dB_s^j.$$

We will need a further decomposition of the processes J_1 and J_2. To this aim, let us introduce the processes I_1 and I_2 defined on Π, namely, for $t \in \Pi \setminus \{0\}$

$$I_1(t) = \sum_{j,j'=0}^{m} \int_0^t (\partial V_j V_{j'})(X_{\eta(s)}^n) \int_{\eta(s)}^{s} dB_u^{j'}\, dB_s^j, \tag{4.8}$$

$$I_2(t) = \sum_{j,j'=0}^{m} \int_0^t (\partial V_j V_{j'})(X_{\eta(s)}^n) \int_{s}^{\epsilon(s)} dB_u^{j'}\, dB_s^j, \tag{4.9}$$

and for $t = 0$ we set $I_1(0) = I_2(0) = 0$. Here $\partial = (\partial_1, \dots, \partial_d)$ and $\partial V_j V_{j'}$ means $\sum_{i=1}^{d} \partial_i V_j V_{j'}^i$.

To make the computations more clear we will replace integrals on $[0, t]$ by summations of integrals over the intervals $[t_k, t_{k+1}]$, with $0 \le k \le nt/T - 1$. Subtracting (4.9) from (4.8) we obtain the following "Lévy area term"

$$I_1(t) - I_2(t) = E_1(t) := \sum_{j,j'=0}^{m} \sum_{k=0}^{nt/T-1} (\partial V_j V_{j'})(X_{t_k}^n) \zeta_{t_k,t_{k+1}}^{j'j}, \tag{4.10}$$

where we denote

$$\zeta_{st}^{ij} = \int_s^t \int_s^u dB_v^i\, dB_u^j - \int_s^t \int_u^t dB_v^i\, dB_u^j, \quad 0 \le s \le t \le T.$$

Notice that Fubini's theorem implies that $\zeta_{s,t}^{ij} = -\zeta_{s,t}^{ji}$. So expression (4.10) can be rewritten as

$$E_1(t) = \sum_{j'<j} \sum_{k=0}^{nt/T-1} \phi_{jj'}(X_{t_k}^n) \zeta_{t_k,t_{k+1}}^{j'j}, \quad t \in \Pi, \tag{4.11}$$

where recall that $\phi_{jj'} = \partial V_j V_{j'} - \partial V_{j'} V_j$. It is worth mentioning that when B is one-dimensional we have $E_1 \equiv 0$.

With the above preparations we decompose $J_1(t) + J_2(t)$ for $t \in \Pi$ as follows:

$$\begin{aligned} &J_1(t) + J_2(t) \\ &\quad = \big(I_1(t) - I_2(t)\big) + \big(R_0(t) - I_1(t)\big) + \big(I_2(t) - \tilde{R}_0(t)\big) + R_1(t) + \tilde{R}_1(t) \\ &\quad := E_1(t) + E_2(t) + E_3(t) + E_4(t) + E_5(t). \end{aligned} \tag{4.12}$$

Step 2: Upper-bound for the Crank–Nicolson scheme. For any function f over $[0, T]$ we denote $\|f\|_{s,t,\beta} = \sup_{u,v\in[s,t]} |f_s - f_t|/|s - t|^\beta$ and write $\|f\|_\beta := \|f\|_{0,T,\beta}$. It follows from Lemma 8.4 in [11] that there exists a constant K such that

$$\|X^n\|_\infty \vee \|X^n\|_\beta \le K + K\|B\|_\beta^{1/\beta}. \tag{4.13}$$

Furthermore, there exist constants K_0 and K_0' independent of n such that for $0 \le s < t \le T$ and $(t - s)^\beta \|B\|_\beta \le K_0$, we have

$$\|X^n\|_{s,t,\beta} \le K_0' \|B\|_\beta. \tag{4.14}$$

Step 3: Estimates of E_e, $1 \le e \le 5$. In this step we show that

$$\sum_{e=1}^{5} \big\|E_e(t) - E_e(s)\big\|_p \le K(t - s)^{\frac{1}{2}}/\vartheta_n, \quad s, t \in \Pi. \tag{4.15}$$

We divide this step into two parts:

Step 3.1. Take $s, t \in \Pi$ such that $s \le t$. When $e = 2, 3, 4, 5$, we are going to show that

$$\big\|E_e(t) - E_e(s)\big\|_p \le Kn^{-2H}(t - s)^{\frac{1}{2}}, \quad s, t \in \Pi, \tag{4.16}$$

where recall that $\|\cdot\|_p$ denotes the L^p-norm.

First, subtracting (4.8) from (4.6) we obtain the following expression for E_2

$$\begin{aligned} &E_2(t) \\ &\quad = \sum_{i=1}^{d} \sum_{j,j'=0}^{m} \sum_{k=0}^{nt/T-1} \frac{1}{2} \partial_i V_j(X_{t_k}^n) \cdot [V_{j'}^i(X_{t_{k+1}}^n) - V_{j'}^i(X_{t_k}^n)] \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB_u^{j'} \, dB_s^j \\ &\quad = \frac{1}{4} \sum_{i=1}^{d} \sum_{j,j',j''=0}^{m} \sum_{k=0}^{nt/T-1} \partial_i V_j(X_{t_k}^n) \cdot \int_{t_k}^{t_{k+1}} \langle \partial V_{j'}^i(X_v^n), V_{j''}(X_{t_{k+1}}^n) + V_{j''}(X_{t_k}^n) \rangle \, dB_v^{j''} \\ &\qquad \cdot \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB_u^{j'} \, dB_s^j, \end{aligned} \tag{4.17}$$

where in the second equation we have applied the chain rule to $V_{j'}^i(X_{t_{k+1}}^n) - V_{j'}^i(X_{t_k}^n)$ and also equation (1.5) for X^n. For convenience, let us put $h_{t_k}^n = \partial_i V_j(X_{t_k}^n)[V_{j''}^{i'}(X_{t_{k+1}}^n) + V_{j''}^{i'}(X_{t_k}^n)]$ and $f_v = \partial_{i'} V_{j'}^i(X_v^n)$, where we have omitted the dependence of h and f on the indices i, i', j, j', j'' for simplicity. Then the above expression becomes

$$E_2(t) = \frac{1}{4} \sum_{i,i'=1}^{d} \sum_{j,j',j''=0}^{m} \sum_{k=0}^{nt/T-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} f_v h_{t_k}^n \, dB_u^{j'} \, dB_s^j \, dB_v^{j''}. \tag{4.18}$$

Observe that some elementary computations show that the triple integral in (4.18) is equal to

$$\int_{t_k}^{t_{k+1}}\int_{t_k}^{v}\int_{t_k}^{s} f_v h^n_{t_k}\,dB^{j'}_u\,dB^j_s\,dB^{j''}_v+\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\int_{t_k}^{v} f_v h^n_{t_k}\,dB^{j'}_u\,dB^{j''}_v\,dB^j_s$$
$$+\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\int_{t_k}^{u} f_v h^n_{t_k}\,dB^{j''}_v\,dB^{j'}_u\,dB^j_s.$$

Substituting the above expression into (4.18), we then obtain an expression of $E_2(t)$ of the form (A.24). Now with the help of the estimate of X^n in (4.13), it is clear that f and h^n satisfy the conditions in Lemma A.2. Therefore, applying Lemma A.2 to $E_2(t)$, we obtain the desired estimate (4.16) for $e=2$. Regarding E_e, $e=3,4,5$ we note that, like in the case of E_2 they are sums of weighted triple integrals of B. Precisely, recall that $dX^{n,i}_t=\sum_{j=1}^m\frac{V^i_j(X^n_{t_k})+V^i_j(X^n_{t_{k+1}})}{2}\,dB^j_t$ for $t\in[t_k,t_{k+1})$. Therefore, we have

$$E_4=\sum_{i,i'=1}^{d}\sum_{j,j'=1}^{m}\sum_{k=0}^{nt/T-1}\frac{V^{i'}_{j'}(X^n_{t_k})+V^{i'}_{j'}(X^n_{t_{k+1}})}{2}\cdot\frac{V^i_j(X^n_{t_k})+V^i_j(X^n_{t_{k+1}})}{2}$$
$$\times\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\int_{t_k}^{u}\partial_{i'}\partial_i V_{j''}(X^n_v)\,dB^{j'}_v\,dB^j_u\,dB^{j''}_s,$$
$$E_5=\sum_{i,i'=1}^{d}\sum_{j,j'=1}^{m}\sum_{k=0}^{nt/T-1}\frac{V^{i'}_{j'}(X^n_{t_k})+V^{i'}_{j'}(X^n_{t_{k+1}})}{2}\cdot\frac{V^i_j(X^n_{t_k})+V^i_j(X^n_{t_{k+1}})}{2}$$
$$\times\int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}\int_{u}^{t_{k+1}}\partial_{i'}\partial_i V_{j''}(X^n_v)\,dB^{j'}_v\,dB^j_u\,dB^{j''}_s,$$

and

$$E_3=\sum_{i=1}^{d}\sum_{j,j'=0}^{m}\sum_{k=0}^{nt/T-1}\left((\partial_i V_j V^i_{j'})(X^n_{t_k})-\partial_i V_j(X^n_{t_{k+1}})\frac{V^i_{j'}(X^n_{t_{k+1}})+V^i_{j'}(X^n_{t_k})}{2}\right)$$
$$\times\int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}dB^{j'}_u\,dB^j_s$$
$$=-\sum_{i=1}^{d}\sum_{j,j'=0}^{m}\sum_{k=0}^{nt/T-1}\frac{V^{i'}_{j''}(X^n_{t_k})+V^{i'}_{j''}(X^n_{t_{k+1}})}{2}\cdot\int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}dB^{j'}_u\,dB^j_s$$
$$\times\int_{t_k}^{t_{k+1}}\frac{\partial_{i'}(\partial_i V_j V^i_{j'})(X^n_v)+\partial_{i'}(\partial_i V_j)(X^n_v)V^i_{j'}(X^n_{t_k})}{2}\,dB^{j''}_v.$$

Using a similar argument to that in the proof of Lemma A.2 we obtain relation (4.16) for $e=3,4,5$. This completes the proof of (4.16).

Step 3.2. It remains to consider the process $E_1(t)$, $t\in\Pi$. We decompose E_1 in the following way:

$$E_1(t)=\sum_{0\neq j'<j}\sum_{k=0}^{nt/T-1}\phi_{jj'}(X^n_{t_k})\zeta^{j'j}_{t_k,t_{k+1}}+\sum_{0=j'<j}\sum_{k=0}^{nt/T-1}\phi_{jj'}(X^n_{t_k})\zeta^{j'j}_{t_k,t_{k+1}}\tag{4.19}$$
$$:=E_{11}(t)+E_{12}(t).$$

Expression (4.19) and Lemma 2.4 together suggest to consider the following "weight-free" random sum corresponding to E_{11}

$$g_n(t) = n^{2H-\frac{1}{2}} \sum_{0\neq j'<j} \sum_{k=0}^{\lfloor nt/T \rfloor} \zeta_{t_k,t_{k+1}}^{j'j}.$$

It follows from relation (3.3) in Proposition 3.1 that g_n satisfies the assumptions in Lemma 2.4. Indeed, by Proposition 3.1 the following estimate holds true for all $s,t \in \Pi$

$$\mathbb{E}\big(|g_n(t) - g_n(s)|^2\big)^{\frac{1}{2}} \leq K|t-s|^{\frac{1}{2}}. \tag{4.20}$$

Furthermore, notice that $g_n(t) - g_n(s)$ belongs to the second chaos generated by B. Therefore, a hypercontractivity argument (see, e.g., [21]) implies that relation (4.20) also holds when the L^2-norm is replaced by L^p-norm, $p \geq 1$. Take $f = \phi_{jj'}(X^n_\cdot)$, $\beta' = \frac{1}{2}$, $\frac{1}{2} < \beta < H$, $p = p' = q' = 2$. Then applying Lemma 2.4 to E_{11} and g_n we obtain the estimate

$$\|E_{11}(t) - E_{11}(s)\|_p \leq K n^{-2H+\frac{1}{2}}(t-s)^{\frac{1}{2}}, \quad s,t \in \Pi. \tag{4.21}$$

We can proceed in a similar way to show the estimate for E_{12}. First, define the "weight-free" random sum corresponding to $E_{12}(t)$

$$\tilde{g}_n(t) = n^{1/2+H} \sum_{0=j'<j} \sum_{k=0}^{\lfloor nt/T \rfloor} \zeta_{t_k,t_{k+1}}^{j'j}.$$

Then as in (4.20), estimate (3.28) in Proposition 3.3 together with an hypercontractivity argument, yields that $\tilde{g}_n$ satisfies the conditions in Lemma 2.4 for $\beta' = \frac{1}{2}$ and $p = q' = 2$. Taking $\frac{1}{2} < \beta < H$, $q' = 2$ and $f = \phi_{jj'}(X^n_\cdot)$ as before and applying Lemma 2.4 to E_{12}, we obtain the estimate

$$\|E_{12}(t) - E_{12}(s)\|_p \leq K n^{-H-\frac{1}{2}}(t-s)^{\frac{1}{2}}, \quad s,t \in \Pi. \tag{4.22}$$

In summary of relations (4.16), (4.21) and (4.22), and taking into account the fact that $E_{11} = 0$ when $m = 1$ and $E_{11} = E_{12} = 0$ when $m = 1$ and $V_0 \equiv 0$, we obtain the desired estimate (4.15).

Step 4: Upper-bounds for the Jacobian. Our proof of Theorem 1.1 is based on a linearization argument. This step aims at studying linear equations involved in the linearization argument.

Let $\Lambda^n = (\Lambda^{n,i}_{i'})_{1\leq i,i'\leq d}$ be the solution of the linear equation for $t \in [0,T]$:

$$\Lambda^{n,i}_{i'}(t) = \delta^i_{i'} + \sum_{j=0}^m \sum_{i''=1}^d \int_0^t V^i_{ji''}(s)\Lambda^{n,i''}_{i'}(s)\,dB^j_s, \quad i,i' = 1,\dots,d. \tag{4.23}$$

Here $V^i_{ji''}$ is defined by (4.2), and $\delta^i_{i'}$ is the Kronecker function whose value is one for $i = i'$ and zero otherwise. The $d \times d$ matrix $\Lambda^n(t)$ is invertible, and we denote its inverse by $\Gamma^n(t)$. With an elementary application of the product rule to $\Gamma^n\Lambda^n$, we can verify that Γ^n solves the equation

$$\Gamma^{n,i}_{i'}(t) = \delta^i_{i'} - \sum_{j=0}^m \sum_{i''=1}^d \int_0^t \Gamma^{n,i}_{i''}(s)V^{i''}_{ji'}(s)\,dB^j_s, \quad i,i' = 1,\dots,d, t \in [0,T].$$

With the help of Lemma 3.2 (ii) in [10] together with the estimate (4.14), we have

$$\|\Lambda^n\|_\infty \vee \|\Lambda^n\|_\beta \vee \|\Gamma^n\|_\infty \vee \|\Gamma^n\|_\beta \leq K e^{K\|B\|_\beta^{1/\beta}}.$$

We can now apply Fernique's lemma to get for $p \geq 1$

$$\big\| \|\Lambda^n\|_\infty \big\|_p \vee \big\| \|\Lambda^n\|_\beta \big\|_p \vee \big\| \|\Gamma^n\|_\infty \big\|_p \vee \big\| \|\Gamma^n\|_\beta \big\|_p \leq K. \tag{4.24}$$

The above preparations provide an explicit expression of the error process

$$Y_t = \frac{1}{2} \Lambda_t^n \sum_{i=1}^{2} \int_0^t \Gamma_s^n \, dJ_i(s), \qquad t \in [0, T]. \tag{4.25}$$

Indeed, applying the product rule to the right-hand side of (4.25), we see that Y in (4.25) satisfies (4.1).

Let $\Lambda = (\Lambda_{i'}^i)_{1 \leq i, i' \leq d}$ be the solution of the equation

$$\Lambda_{i'}^i(t) = \delta_{i'}^i + \sum_{j=0}^{m} \sum_{i''=1}^{d} \int_0^t \partial_{i''} V_j^i(X_s) \Lambda_{i'}^{i''}(s) \, dB_s^j, \tag{4.26}$$

for $t \in [0, T]$ and denote by $\Gamma(t)$ the inverse of $\Lambda(t)$. As before, we have

$$\Gamma_{i'}^i(t) = \delta_{i'}^i - \sum_{j=0}^{m} \sum_{i''=1}^{d} \int_0^t \Gamma_{i''}^{n,i}(s) \partial_{i'} V_j^{i''}(X_s) \, dB_s^j$$

for $t \in [0, T]$, and it follows from Lemma 3.1 in [10] that relation (4.24) still holds when Λ^n and Γ^n in (4.24) are replaced by Λ and Γ, respectively.

Step 5: Estimates of $\Gamma^n Y$. Multiplying both sides of (4.25) by Γ_t^n we have

$$\Gamma_t^n Y_t = \frac{1}{2} \sum_{i=1}^{2} \int_0^t \Gamma_u^n \, dJ_i(u).$$

Writing $\Gamma_u^n = \Gamma_{\eta(u)}^n + (\Gamma_u^n - \Gamma_{\eta(u)}^n)$ we then get the following decomposition for $s, t \in \Pi$, $s \leq t$

$$\sum_{i=1}^{2} \int_s^t \Gamma_u^n \, dJ_i(u) = \sum_{i=1}^{2} \int_s^t \Gamma_{\eta(u)}^n \, dJ_i(u) + \sum_{i=1}^{2} \int_s^t \int_{\eta(u)}^u d\Gamma_v^n \, dJ_i(u). \tag{4.27}$$

Regarding the first term in (4.27), we apply the relation (4.12) to get

$$\sum_{i=1}^{2} \int_s^t \Gamma_{\eta(u)}^n \, dJ_i(u) = \sum_{e=1}^{5} \sum_{k=ns/T}^{nt/T-1} \Gamma_{t_k}^n \big(E_e(t_{k+1}) - E_e(t_k) \big). \tag{4.28}$$

For convenience, let us write the right-hand side of (4.28) in terms of an integral

$$\sum_{t_k=s}^{t-\frac{T}{n}} \Gamma_{t_k}^n \big(E_e(t_{k+1}) - E_e(t_k) \big) =: \int_s^t \Gamma_{\eta(u)}^n \, dE_e(u). \tag{4.29}$$

Note that equation (4.29) is only valid for $s, t \in \Pi$ since E_e, $e = 1, \dots, 5$ are only defined on Π. Now substituting (4.28) into (4.27) and taking into account (4.29), we get

$$\sum_{i=1}^{2} \int_s^t \Gamma_u^n \, dJ_i(u) = \sum_{e=1}^{5} \int_s^t \Gamma_{\eta(u)}^n \, dE_e(u) + \sum_{i=1}^{2} \int_s^t \int_{\eta(u)}^u d\Gamma_v^n \, dJ_i(u). \tag{4.30}$$

In order to bound $\Gamma^n Y$, we first estimate the quantity $\int_s^t \Gamma_{\eta(u)}^n \, dE_e(u)$ in (4.30). This can be done with the help of Lemma 2.4 as in the proof of (4.21). Indeed, take $\hat{g}_n(t) = \vartheta_n E_e(t)$, $t \in \Pi$ and $f = \Gamma^n$, and let β, β', p, p', q' be as before. Then estimate (4.15) shows that $\hat{g}_n$

satisfies the conditions in Lemma 2.4. Applying Lemma 2.4 to $\int_s^t \Gamma_{\eta(u)}^n \, dE_e(u)$ and invoking expression (4.29), we obtain

$$\left\| \sum_{e=1}^{5} \int_s^t \Gamma_{\eta(u)}^n \, dE_e(u) \right\|_p \le K(t-s)^{\frac{1}{2}}/\vartheta_n, \quad s,t \in \Pi. \tag{4.31}$$

Let us turn to the second term in (4.27). By the definition of Γ^n and J_1, we have

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} \int_{t_k}^{u} d\Gamma_v^n \, dJ_1(u) \\ &\quad = \Bigg(\frac{1}{2} \sum_{j,j'=0}^{m} \sum_{i,i',i''=1}^{d} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \big(-\Gamma_{i'}^{n,i'''}(v) V_{j,i}^{i'}(v) \big) \, dB_v^j \\ &\qquad \times \int_{t_k}^{u} \partial_{i''} V_{j'}^{i}(X_r^n) \, dX_r^{n,i''} \, dB_u^{j'} \Bigg)_{1 \le i''' \le d}. \end{aligned} \tag{4.32}$$

Therefore, writing

$$\int_s^t \int_{\eta(u)}^{u} d\Gamma_v^n \, dJ_1(u) = \sum_{k=ns/T}^{nt/T-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} d\Gamma_v^n \, dJ_1(u) \tag{4.33}$$

and then applying (4.32) and some elementary decompositions of multiple integrals, we can show that the right-hand side of (4.33) is of the form (A.24). Applying Lemma A.2, we obtain

$$\left\| \int_s^t \int_{\eta(u)}^{u} d\Gamma_v^n \, dJ_i(u) \right\|_p \le K n^{-2H} (t-s)^{1/2} \tag{4.34}$$

for $i = 1$. This estimate still holds true in the case $i = 2$, and the proof is similar. Substituting (4.31) and (4.34) into (4.27), we obtain the estimate

$$\left\| \sum_{i=1}^{2} \int_s^t \Gamma_u^n \, dJ_i(u) \right\|_p \le K(t-s)^{\frac{1}{2}}/\vartheta_n, \quad s,t \in \Pi. \tag{4.35}$$

Applying Lemma 2.3 and taking into account the expression of J_i in (4.3), we can show that

$$\left\| \int_{t_k}^{t} \Gamma_u^n \, dJ_i(u) \right\|_p \le K n^{-2H}, \quad t \in [t_k, t_{k+1}], i = 1, 2. \tag{4.36}$$

Combining this estimate with (4.35), we obtain the inequality

$$\sup_{t \in [0,T]} \left\| \sum_{i=1}^{2} \int_0^t \Gamma_u^n \, dJ_i(u) \right\|_p \le K/\vartheta_n. \tag{4.37}$$

Step 6: Conclusion. Inequality (1.6) follows by applying Hölder's inequality to (4.25) and using estimate (4.37) and estimate (4.24) for Λ^n. □

The following result provides an estimate on the increments of the error process.

LEMMA 4.1. *Under the assumptions and notation of Theorem 1.1, the error process $Y = X - X^n$ satisfies the following relation for all $s,t \in \Pi$*

$$\mathbb{E}\big(|Y_t - Y_s|^p\big)^{1/p} \le K|t-s|^{\frac{1}{2}}/\vartheta_n. \tag{4.38}$$

PROOF. Invoking expression (4.25) of Y, we can write

$$Y_t - Y_s = \frac{1}{2}(\Lambda_t^n - \Lambda_s^n) \sum_{i=1}^{2} \int_0^t \Gamma_u^n \, dJ_i(u) + \frac{1}{2}\Lambda_s^n \sum_{i=1}^{2} \int_s^t \Gamma_u^n \, dJ_i(u). \tag{4.39}$$

Inequality (4.38) then follows by applying Hölder's inequality to (4.39) and by taking into account estimates (4.24) and (4.35) and the fact that $\|\Lambda_t^n - \Lambda_s^n\|_p \le \|\|\Lambda^n\|_\beta\|_p \cdot (t-s)^\beta$. This completes the proof. □

The following lemma is a convergence result for the processes Λ^n and Γ^n.

LEMMA 4.2. *Take β such that $\frac{1}{2} < \beta < H$. Let Λ^n and Λ be the solutions of equations (4.23) and (4.26), respectively, and let Γ^n and Γ be their inverses. Then we have*

$$\|\Lambda^n - \Lambda\|_{\beta,p} + \|\Gamma^n - \Gamma\|_{\beta,p} \le K n^{1-2\beta}. \tag{4.40}$$

PROOF. See Section A.4. □

We end this section with the following technical results. For convenience let us write, as in (4.29),

$$\sum_{t_k=s}^{t-T/n} \Gamma_{t_k}^n \big(E_{11}(t_{k+1}) - E_1(t_k)\big) =: \int_s^t \Gamma_{\eta(u)}^n \, dE_{11}(u) \quad \text{for } s, t \in \Pi.$$

LEMMA 4.3. *We continue to use the notation of Theorem 1.1. Let $s, t \in \Pi$, $s \le t$. If $m > 1$, then we have the estimate*

$$\sup_{s,t\in\Pi} \left\| \sum_{i=1}^{2} \int_s^t \Gamma_u^n \, dJ_i(u) - \int_s^t \Gamma_{\eta(u)}^n \, dE_{11}(u) \right\|_p \le K n^{-\frac{1}{2}-H}. \tag{4.41}$$

In the case $m = 1$, we have

$$\sup_{s,t\in\Pi} \left\| \sum_{i=1}^{2} \int_s^t \Gamma_u^n \, dJ_i(u) - \int_s^t \Gamma_{\eta(u)}^n \, dE_{12}(u) \right\|_p \le K n^{-2H}. \tag{4.42}$$

Suppose that $m = 1$ and $V_0 \equiv 0$. Then, for $\beta < H$, we can find a constant $K = K_\beta$ such that

$$\sup_{t\in[0,T]} \left\| \sum_{i=1}^{2} \int_0^t \Gamma_u^n \, dJ_i(u) - \sum_{e=2}^{5} \int_0^{\eta(t)} \Gamma_{\eta(u)}^n \, dE_e(u) \right\|_p \le K_\beta n^{1-4\beta}. \tag{4.43}$$

PROOF. By subtracting $\int_s^t \Gamma_{\eta(u)}^n \, dE_{11}(u)$ from both sides of (4.27) and taking into account the expression (4.28) we obtain

$$\begin{aligned} &\sum_{i=1}^{2} \int_s^t \Gamma_u^n \, dJ_i(u) - \int_s^t \Gamma_{\eta(u)}^n \, dE_{11}(u) \\ &\quad = \int_s^t \Gamma_{\eta(u)}^n \, dE_{12}(u) + \sum_{e=2}^{5} \int_s^t \Gamma_{\eta(u)}^n \, dE_e(u) + \sum_{i=1}^{2} \int_s^t \int_{\eta(u)}^u d\Gamma_v^n \, dJ_i(u). \end{aligned} \tag{4.44}$$

As in the proof of (4.31), we can show that the first two terms on the right-hand side of (4.44) are bounded by $Kn^{-\frac{1}{2}-H}$ and Kn^{-2H}, respectively. Furthermore, thanks to (4.34), we have that the third term is bounded by Kn^{-2H}. Putting these bounds together, we obtain the desired estimate (4.41). Relation (4.42) follows from a similar argument and is left to the reader. Please refer to Section A.5 for a proof of estimate (4.43). □

5. Asymptotic error distributions. In this section we prove Theorem 1.3.

PROOF OF THEOREM 1.3. The proof is done in several steps.

Step 1. Suppose that $m > 1$ or $V_0 \not\equiv 0$. We first observe that, by Theorem 13.5 in [2] together with (4.38), the proof of the weak convergence of $(\vartheta_n(\tilde{X} - \tilde{X}^n), B)$ can be reduced to showing the convergence of its finite-dimensional distributions (f.d.d.). We also note that, by (4.25) we have

$$\tilde{X}_t - \tilde{X}_t^n = X_{t_k} - X_{t_k}^n = \frac{1}{2}\Lambda_{t_k}^n \sum_{i=1}^{2} \int_0^{t_k} \Gamma_u^n \, dJ_i(u), \quad t \in [t_k, t_{k+1}).$$

Step 2. In this step we assume that $m > 1$. Set

$$S^n(t) = \frac{1}{2}\Lambda_{t_k}^n \int_0^{t_k} \Gamma_{\eta(s)}^n \, dE_{11}(s), \quad t \in [t_k, t_{k+1}).$$

We first observe that, applying relation (4.41) in Lemma 4.3 and relation (4.36), we obtain that $\vartheta_n \| S^n(t) - (\tilde{X}_t - \tilde{X}_t^n)\|_p$ is uniformly bounded by $\vartheta_n n^{-\frac{1}{2}-H}$ and thus it converges to zero as $n \to \infty$. This implies that the limit of the finite-dimensional distributions of $(\vartheta_n(\tilde{X} - \tilde{X}^n), B)$ is equal to that of $(\vartheta_n S^n, B)$.

To further reduce the problem, we set

$$S(t) = \frac{1}{2}\Lambda_{t_k} \int_0^{t_{k+1}} \Gamma_{\eta(s)} \, dE_{11}(s), \quad t \in [t_k, t_{k+1})$$

and calculate

$$\begin{aligned} S^n(t_k) - S(t_k) = {} & \frac{1}{2} \sum_{0 \neq j' < j} \int_0^{t_k} \left[\Lambda_{t_k}^n \Gamma_{\eta(s)}^n \phi_{jj'}(X_{\eta(s)}^n) - \Lambda_{t_k} \Gamma_{\eta(s)} \phi_{jj'}(X_{\eta(s)})\right] d\zeta_{\eta(s),s}^{j'j} \\ & - \frac{1}{2}\Lambda_{t_k} \int_{t_k}^{t_{k+1}} \Gamma_{\eta(s)} \, dE_{11}(s). \end{aligned} \tag{5.1}$$

By Lemma 2.3 it follows that the L^p-norm of the second term in the right-hand side of (5.1) is bounded by Kn^{-2H}. On the other hand, applying Lemmas 4.1 and 4.2 to $f_s = \Lambda_t^n \Gamma_s^n \phi_{jj'}(X_s^n) - \Lambda_t \Gamma_s \phi_{jj'}(X_s)$, one can show that

$$\|f\|_{\beta,p} \le Kn^{1-2\beta}. \tag{5.2}$$

Taking $\zeta_{k,n} = \zeta_{t_k,t_{k+1}}^{j'j}$, applying Lemma 2.4 and taking into account (5.2) we then obtain that the first term in the right-hand side of (5.1) is bounded by $Kn^{1-2\beta+1/2-2H}$. Plugging these two estimates into (5.1), we obtain

$$\left\| S^n(t) - S(t) \right\|_p \le Kn^{1-2\beta+1/2-2H} \vee n^{-2H}$$

for $t \in \Pi$, and thus for $t \in [0, T]$. This implies in particular that to show the f.d.d. convergence of $(\vartheta_n S^n, B)$ is the same as to show that of $(\vartheta_n S, B)$.

Applying Proposition 2.5 to the process $(\vartheta_n S, B)$ and taking into account the weak convergence result in Proposition 3.1, we conclude that the f.d.d. of $(\vartheta_n S, B)$ converge to that of (U, B), where

$$U_t = T^{2H-\frac{1}{2}} \sqrt{\frac{\kappa}{2}} \Lambda_t \sum_{1 \le j' < j \le m} \int_0^t \Gamma_s \phi_{jj'}(X_s) \, dW_s^{j'j}.$$

The convergence (1.9) follows from the fact that $\{U_t, t \in [0, T]\}$ solves the SDE (1.10).

Step 3. We turn to the case $m = 1$ and $V_0 \not\equiv 0$. Using similar arguments as in Step 2, relation (4.42) implies that in order to show the weak limit (for the convergence in f.d.d.) of $(\vartheta_n(\tilde{X} - \tilde{X}^n), B)$ it suffices to consider that of $(\vartheta_n \tilde{S}^n, B)$, where

$$\tilde{S}_t^n = \frac{1}{2}\Lambda_{\eta(t)}^n \int_0^{\eta(t)} \Gamma_{\eta(s)}^n \, dE_{12}(s).$$

With the help of Lemma 4.2 we obtain that the convergence of the f.d.d. of $(\vartheta_n \tilde{S}^n, B)$ is the same as that of $(\vartheta_n \tilde{S}, B)$, where

$$\tilde{S}_t = \frac{1}{2}\Lambda_{\eta(t)} \sum_{k=0}^{\lfloor nt/T \rfloor} \Gamma_{t_k}\phi_{10}(X_{t_k})\zeta_{t_k,t_{k+1}}^{01}.$$

Applying Proposition 2.5 to $\tilde{S}$ and taking into account the weak convergence result in Proposition 3.3, we obtain that the f.d.d. of $(\vartheta_n \tilde{S}, B)$ converges to that of $(\tilde{U}, B)$, where

$$\tilde{U}_t = T^{H+\frac{1}{2}}\sqrt{\frac{\varrho}{2}}\Lambda_t \int_0^t \Gamma_s \phi_{10}(X_s)\, dW_s.$$

Convergence (1.9) then follows from the fact that $\tilde{U}$ is the solution of equation (1.11).

Step 4. We now consider the scalar case when $m = 1$ and $V_0 \equiv 0$. Convergence (1.12) is clear for $t = 0$, and for convenience we take $t > 0$ from now on. In a similar way as in Step 2, with the help of estimate (4.43) and Lemma 4.2 we are able to reduce the proof of the L^p-convergence of $n^{2H}(\tilde{X}_t - \tilde{X}_t^n)$ into that of the quantity

$$\frac{1}{2}n^{2H}\Lambda_{\eta(t)} \sum_{e=2}^{5} \sum_{k=0}^{\lfloor nt/T \rfloor - 1} \Gamma_{t_k}\big(E_e(t_{k+1}) - E_e(t_k)\big). \tag{5.3}$$

It now remains to show that the quantity in (5.3) converges to the solution of equation (1.13).

Observe that, by (4.17), we have, for $t \in \Pi$,

$$\begin{aligned}
&\sum_{k=0}^{nt/T-1} \Gamma_{t_k}\big(E_2(t_{k+1}) - E_2(t_k)\big) \\
&\quad = \frac{1}{4}\sum_{i=1}^{d} \sum_{k=0}^{nt/T-1} \Gamma_{t_k}\partial_i V(X_{t_k}^n) \\
&\qquad \times \left(\int_{t_k}^{t_{k+1}} \langle \partial V^i(X_v^n), V(X_{t_{k+1}}^n) + V(X_{t_k}^n)\rangle \, dB_v\right)\left(\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} dB_u \, dB_s\right).
\end{aligned}$$

Consider the following modification of this summation:

$$\begin{aligned}
\tilde{E}_2(t) &= \frac{1}{2}\sum_{i=1}^{d} \sum_{k=0}^{nt/T-1} \Gamma_{t_k}\big(\partial_i V\langle \partial V^i, V\rangle\big)(X_{t_k}) \int_{t_k}^{t_{k+1}} dB_v \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} dB_u \, dB_s \\
&= \frac{1}{4}\sum_{i=1}^{d} \sum_{k=0}^{nt/T-1} \Gamma_{t_k}\big(\partial_i V\langle \partial V^i, V\rangle\big)(X_{t_k})(B_{t_k,t_{k+1}})^3.
\end{aligned}$$

It is then easy to show that, for $t \in \Pi$,

$$n^{2H}\left(\sum_{k=0}^{nt/T-1} \Gamma_{t_k}\big(E_2(t_{k+1}) - E_2(t_k)\big) - \tilde{E}_2(t)\right) \to 0 \quad \text{in } L^p \text{ as } n \to \infty. \tag{5.4}$$

In a similar way we introduce

$$\tilde{E}_3(t) = -\frac{1}{4}\sum_{i=1}^{d}\sum_{k=0}^{nt/T-1} \Gamma_{t_k}\big([\partial(\partial_i V V^i)]V + V^i[\partial(\partial_i V)]V\big)(X_{t_k}) \cdot (B_{t_k,t_{k+1}})^3$$

and

$$\tilde{E}_4(t) = \tilde{E}_5(t) = \frac{1}{6}\sum_{i',i=1}^{d}\sum_{k=0}^{nt/T-1} \Gamma_{t_k}\big(V^{i'} V^i \partial_i \partial_{i'} V\big)(X_{t_k})(B_{t_k,t_{k+1}})^3.$$

Then we have the convergence in L^p for $e = 3, 4, 5$:

$$n^{2H}\left(\sum_{k=0}^{nt/T-1} \Gamma_{t_k}\big(E_e(t_{k+1}) - E_e(t_k)\big) - \tilde{E}_e(t)\right) \to 0. \tag{5.5}$$

The convergences in (5.4) and (5.5) together imply that to show the convergence (1.12) it suffices to consider the quantity $n^{2H}\sum_{e=2}^{5}\tilde{E}_e(t)$.

With an elementary computation we get

$$\sum_{e=2}^{5}\tilde{E}_e(t) = -\frac{1}{6}\sum_{i',i=1}^{d}\sum_{k=0}^{nt/T-1} \Gamma_{t_k}\big(V^{i'} V^i \partial_i \partial_{i'} V\big)(X_{t_k})(B_{t_k,t_{k+1}})^3, \quad t \in \Pi. \tag{5.6}$$

Take $f_t = \Gamma_t(V^{i'} V^i \partial_i \partial_{i'} V)(X_t)$ and $\zeta_{k,n} = (B_{t_{k+1}} - B_{t_k})^3$. Applying Proposition 2.6 to (5.6) and taking into account Lemma A.1(ii), we obtain

$$\frac{1}{2}\Lambda_{\eta(t)}\left(n^{2H}\sum_{e=2}^{5}\tilde{E}_e\big(\eta(t)\big)\right) \to \bar{U}_t$$

in L^p for $t \in [0, T]$, where

$$\bar{U}_t = -\frac{T^{2H}}{4}\sum_{i',i=1}^{d}\Lambda_t \int_0^t \Gamma_s\big(V^{i'} V^i \partial_i \partial_{i'} V\big)(X_t)\, dB_s.$$

The convergence (1.12) follows from the fact that the process $\bar{U}$ verifies the equation (1.13). □

6. The degenerated cases. In this section we consider the degenerated cases in which the limits in Theorem 1.3 are equal to zero. We consider the strong convergence of the Crank–Nicolson scheme in the first subsection and then focus on the asymptotic error in the second subsection.

6.1. *The strong convergence.* In this subsection, we prove Theorem 1.4.

PROOF OF THEOREM 1.4. Recall that $\phi_{jj'}$, E_{11}, E_{12} and E_e, $e = 1, \dots, 5$ are defined in (1.8), (4.12) and (4.19). Suppose that $\phi_{jj'} \equiv 0$ for all $j, j' = 1, \dots, m$. In this case we have $E_{11} \equiv 0$ and thus $E_1 = E_{12}$. Applying (4.16) to E_e, $e = 2, 3, 4, 5$, and (4.22) to E_{12} and taking into account the identity $E_1 = E_{12}$, we obtain the estimate

$$\sum_{e=1}^{5}\big\|E_e(t) - E_e(s)\big\|_p \le K(t-s)^{\frac{1}{2}} n^{-\frac{1}{2}-H}, \quad s, t \in \Pi.$$

In a similar way as in the proof of Theorem 1.1, Step 5 and 6, we obtain estimate (1.14) in this case.

We turn to the case $\phi_{ij} \equiv 0$ for $i, j = 0, 1, \dots, m$. Note that in this case $E_1 \equiv 0$, and so applying (4.16), we get

$$\sum_{e=1}^{5} \| E_e(t) - E_e(s) \|_p \le K(t-s)^{\frac{1}{2}} n^{-2H}, \quad s, t \in \Pi.$$

Following the lines of the proof of Theorem 1.1, Step 5 and 6 again, we obtain (1.14). □

6.2. *The asymptotic error.* In this subsection, we prove Theorem 1.5.

PROOF OF THEOREM 1.5. The proof for Item (i) in Theorem 1.5 follows the lines in Step 3 of the proof of Theorem 1.3, and will be left to the reader.

In the following we prove Item (ii). With the preparations in the first two subsections, we can consider the asymptotic error of the scheme in a similar way as in the proof of Theorem 1.3, Step 4. Indeed, with the help of Lemma 4.2, in view of the expression of the error process Y in (4.25) and the expression of $J_1 + J_2$ in (4.12) in terms of E_e, $e = 1, \dots, 5$, we can show that the L^2-limit of $n^{2H} Y_t$ for $t \in \Pi$ is equal to that of

$$\frac{1}{2} n^{2H} \sum_{e=2}^{5} \Lambda_t \sum_{k=0}^{\lfloor nt/T \rfloor} \Gamma_{t_k} \big(E_e(t_{k+1}) - E_e(t_k) \big),$$

where we have used the fact that $E_1 \equiv 0$ due to the assumption that $\phi_{jj'} \equiv 0$ for $j, j' = 0, \dots, m$. With the help of estimate (4.38) in Lemma 4.1, one can further reduce the the L^2-convergence of $n^{2H} Y_t$ to that of

$$\frac{1}{2} n^{2H} \sum_{i=2}^{5} \Lambda_t \bar{E}_i(t), \tag{6.1}$$

where

$$\bar{E}_2(t) = \frac{1}{2} \sum_{i,i'=1}^{d} \sum_{j,j',j''=0}^{m} \sum_{k=0}^{\lfloor nt/T \rfloor} \Gamma_{t_k} \big(\partial_i V_j \cdot V_{j''}^{i'} \partial_{i'} V_{j'}^{i} \big)(X_{t_k}) \times \int_{t_k}^{t_{k+1}} dB_v^{j''} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dB_u^{j'} \, dB_s^{j},$$

$$\bar{E}_3(t) = -\frac{1}{2} \sum_{i,i'=1}^{d} \sum_{j,j',j''=0}^{m} \sum_{k=0}^{\lfloor nt/T \rfloor} \Gamma_{t_k} \big(V_{j''}^{i'} \partial_{i'} (\partial_i V_j V_{j'}^{i}) + V_{j'}^{i} V_{j''}^{i'} \partial_{i'} \partial_i V_j \big)(X_{t_k}) \times \int_{t_k}^{t_{k+1}} dB_v^{j''} \int_{t_k}^{t_{k+1}} \int_{s}^{t_{k+1}} dB_u^{j'} \, dB_s^{j},$$

$$\bar{E}_4(t) = \sum_{i,i'=1}^{d} \sum_{j,j',j''=0}^{m} \sum_{k=0}^{\lfloor nt/T \rfloor} \Gamma_{t_k} \big(V_{j''}^{i'} V_{j'}^{i} \partial_{i'} \partial_i V_j \big)(X_{t_k}) \times \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \int_{t_k}^{u} dB_v^{j''} \, dB_u^{j'} \, dB_s^{j},$$

$$\bar{E}_5(t) = \sum_{i,i'=1}^{d} \sum_{j,j',j''=0}^{m} \sum_{k=0}^{\lfloor nt/T \rfloor} \Gamma_{t_k} \big(V_{j''}^{i'} V_{j'}^{i} \partial_{i'} \partial_i V_j \big)(X_{t_k}) \times \int_{t_k}^{t_{k+1}} \int_{s}^{t_{k+1}} \int_{u}^{t_{k+1}} dB_v^{j''} \, dB_u^{j'} \, dB_s^{j}.$$

Now as in the proof of the convergence of (5.6), by applying Proposition 2.6 to (6.1) and taking into account Proposition A.3 and A.5, Corollary A.4 and A.6, and Lemma A.7, we obtain that

$$\lim_{n\to\infty}\frac{1}{2}n^{2H}\sum_{i=2}^{5}\Lambda_t\bar{E}_i(t) \tag{6.2}$$
$$=\Lambda_t\Big(\sum_{j\in\Xi}\int_0^t\Big(-\frac{T^{2H}}{4}\Gamma_s\psi_{jjj}(s)\Big)\,dB_s^j+\sum_{j\in\bar{\Xi}}\int_0^t\Gamma_s\varphi_j(s)\,dB_s^j\Big),$$

where ψ and φ are given in Theorem 1.5. The convergence (1.15) then follows by observing that the right-hand side of (6.2) satisfies the equation of U in Theorem 1.5(ii). $\square$

APPENDIX

A.1. Proof of (3.27). The proof will be done in seven steps.

Step 1. In this step, we derive a decomposition for d_2. First, applying the integration by parts formula (2.3), we obtain

$$\begin{aligned}&\mathbb{E}\big(Z_n(t)\tilde{D}_{u'}Z_n(t)D_{s'}Z_n(t)\big)\\ &=\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{[0,T]^4}\big[D_{r'}\tilde{D}_{u'}Z_n(t)\big]\big[\tilde{D}_{v'}D_{s'}Z_n(t)\big]\\ &\quad\times\beta_{\frac{k}{n}}(r)\gamma_{t_k,r}(v)\mu(dv\,dv')\mu(dr\,dr')\\ &=\sum_{k,k_3,k_4=0}^{\lfloor nt/T\rfloor}\int_{[0,T]^4}\beta_{\frac{k_3}{n}}(r')\gamma_{t_{k_3},r'}(u')\beta_{\frac{k_4}{n}}(s')\gamma_{t_{k_4},s'}(v')\\ &\quad\times\beta_{\frac{k}{n}}(r)\gamma_{t_k,r}(v)\mu(dv\,dv')\mu(dr\,dr'),\end{aligned} \tag{A.1}$$

where the second equation follows from the fact that

$$\tilde{D}_vD_rZ_n(t)=\sum_{k=0}^{\lfloor nt/T\rfloor}\beta_{\frac{k}{n}}(r)\gamma_{t_k,r}(v),\qquad t\in[0,T].$$

Substituting expression (A.1) into (3.24), we obtain

$$\begin{aligned}d_2=6\sum_{k_1,k_2,k_3,k_4=0}^{\lfloor nt/T\rfloor}&\int_{\substack{t_{k_4}<s'<t_{k_4}\\ t_{k_1}<s<t_{k_1}}}\int_{0<u,u'<T}\int_{\substack{t_{k_3}<r'<t_{k_3}\\ t_{k_2}<r<t_{k_2}}}\int_{0<v,v'<T}\gamma_{t_{k_3},r'}(u')\\ &\times\gamma_{t_{k_4},s'}(v')\gamma_{t_{k_2},r}(v)\gamma_{t_{k_1},s}(u)\mu(dv\,dv')\mu(dr\,dr')\mu(du\,du')\mu(ds\,ds').\end{aligned}$$

By changing the variables from (v,v',r,r',u,u',s,s') to $\frac{T}{n}(v,v',r,r',u,u',s,s')$ and exchanging the orders of integrals associated with $\mu(du\,du')$ and $\mu(dr\,dr')$ we obtain

$$d_2=6\Big(\frac{T}{n}\Big)^{8H}\sum_{k_1,k_2,k_3,k_4=0}^{\lfloor nt/T\rfloor}c(k_1,k_2,k_3,k_4),$$

where

$$\begin{aligned}c(k_1,k_2,k_3,k_4)=&\int_{\substack{k_4<s'<k_4+1\\ k_1<s<k_1+1}}\int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}}\int_{0<v,v',u,u'<n}\varphi_{k_3,r'}(u')\varphi_{k_4,s'}(v')\\ &\times\varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\mu(dv\,dv')\mu(du\,du')\mu(dr\,dr')\mu(ds\,ds'),\end{aligned} \tag{A.2}$$

and recall that

$$\varphi_{k,s}(u)=\varphi^0_{k,s}(u)-\varphi^1_{k,s}(u),\qquad \varphi^0_{k,s}(u)=\mathbf{1}_{[k,s]}(u),\qquad \varphi^1_{k,s}(u)=\mathbf{1}_{[s,k+1]}(u), \tag{A.3}$$

where $\mathbf{1}_{[a,b]}$ denotes the indicator function of the interval $[a,b]$.

Now we denote

$$I:=\left\{(k_1,k_2,k_3,k_4):k_1,k_2,k_3,k_4=0,1,\dots,\left\lfloor\frac{nt}{T}\right\rfloor\right\}.$$

Take $i,j=1,2,3,4$, and denote by I_{ij} the set of (k_1,k_2,k_3,k_4) in I such that $|k_i-k_j|>2$, that is, $I_{ij}=\{(k_1,k_2,k_3,k_4)\in I:|k_i-k_j|>2\}$. Denote by I^c_{ij} the complement of I_{ij}. We decompose I as follows:

$$I=\bigcup_{l=1}^{8}M_l,$$

where

$$\begin{aligned}
M_1 &= I_{42}\cap I_{41}\cap I_{31}\cap I_{32};\\
M_2 &= \left(I^c_{42}\cap I_{41}\cap I_{31}\cap I_{32}\right)\cup\left(I_{42}\cap I_{41}\cap I^c_{31}\cap I_{32}\right)\\
&:= M_{21}+M_{22};\\
M_3 &= \left(I_{42}\cap I^c_{41}\cap I_{31}\cap I_{32}\right)\cup\left(I_{42}\cap I_{41}\cap I_{31}\cap I^c_{32}\right);\\
M_4 &= \left(I^c_{42}\cap I^c_{41}\cap I_{31}\cap I_{32}\right)\cup\left(I_{42}\cap I^c_{41}\cap I^c_{31}\cap I_{32}\right)\\
&\qquad\cup\left(I^c_{42}\cap I_{41}\cap I_{31}\cap I^c_{32}\right)\cup\left(I_{42}\cap I_{41}\cap I^c_{31}\cap I^c_{32}\right)\\
&:= M_{41}\cup M_{42}\cup M_{43}\cup M_{44};\\
M_5 &= I_{42}\cap I^c_{41}\cap I_{31}\cap I^c_{32};\\
M_6 &= I^c_{42}\cap I_{41}\cap I^c_{31}\cap I_{32};\\
M_7 &= \left(I^c_{42}\cap I^c_{41}\cap I^c_{31}\cap I_{32}\right)\cup\left(I^c_{42}\cap I^c_{41}\cap I_{31}\cap I^c_{32}\right)\\
&\qquad\cup\left(I^c_{42}\cap I_{41}\cap I^c_{31}\cap I^c_{32}\right)\cup\left(I_{42}\cap I^c_{41}\cap I^c_{31}\cap I^c_{32}\right);\\
M_8 &= I^c_{42}\cap I^c_{41}\cap I^c_{31}\cap I^c_{32}.
\end{aligned}$$

This decomposition of I puts similar cases together into one group and allows us to treat different cases in each group M_i simultaneously.

For any subset M of I, we denote

$$d_2(M):=6\left(\frac{T}{n}\right)^{8H}\sum_{(k_1,k_2,k_3,k_4)\in M}c(k_1,k_2,k_3,k_4).$$

It is clear that

$$d_2=\sum_{l=1}^{8}d_2(M_l).$$

Thus to show (3.27) it suffices to show that $n^{8H-2}d_2(M_l)\to 0$ as $n\to\infty$ for each $l=1,\dots,8$.

Step 2. In this step, we show the convergence of $n^{8H-2}d_2(M_7)$ and $n^{8H-2}d_2(M_8)$. Since

$$\left|\varphi_{k,s}(u)\right|\le\mathbf{1}_{[k,k+1]}(u), \tag{A.4}$$

we have

$$\left|c(k_1,k_2,k_3,k_4)\right| \le 1.$$

Applying this inequality to $d_2(M_7)$, we obtain

$$\left|d_2(M_7)\right| \le 6\left(\frac{T}{n}\right)^{8H} \sum_{(k_1,k_2,k_3,k_4)\in M_7} 1.$$

Note that

$$M_7 \subset \{|k_i - k_j| \le 6 \text{ for } i, j = 1,2,3,4\},$$

so the number of elements in M_7 is less than $2 \cdot 6^3 n$. This implies that

$$\left|d_2(M_7)\right| \le 2\cdot 6^4 n\left(\frac{T}{n}\right)^{8H}.$$

It follows from this estimate that $n^{8H-2} d_2(M_7) \to 0$ as $n \to 0$. Note that $M_8 \subset \{|k_i - k_j| \le 4 \text{ for } i, j = 1,2,3,4\}$. So in the same way, we can show that $n^{8H-2} d_2(M_8) \to 0$.

Step 3. In this step, we consider $d_2(M_5)$ and $d_2(M_6)$. For $(k_1,k_2,k_3,k_4) \in M_5$, we have $|k_2 - k_4| > 2$ and $|k_1 - k_3| > 2$. By the mean value theorem and with the help of (A.4), it is easy to see that

$$\left|c(k_1,k_2,k_3,k_4)\right| \le K|k_2 - k_4|^{2H-2}|k_1 - k_3|^{2H-2}. \tag{A.5}$$

Applying (A.5) to $d_2(M_5)$, we obtain

$$\left|d_2(M_5)\right| \le K\left(\frac{T}{n}\right)^{8H} \sum_{k_1,k_2,k_3,k_4\in M_5} |k_2 - k_4|^{2H-2}|k_1 - k_3|^{2H-2}. \tag{A.6}$$

Note that for $(k_1,k_2,k_3,k_4) \in M_5$ we have $|k_1 - k_4| \le 2$ and $|k_2 - k_3| \le 2$, so

$$\begin{aligned} |k_2 - k_4| &\le |k_2 - k_3| + |k_3 - k_1| + |k_1 - k_4| \\ &\le 3|k_3 - k_1|. \end{aligned}$$

Applying this inequality to the right-hand side of (A.6), yields

$$\begin{aligned} \left|d_2(M_5)\right| &\le K\left(\frac{T}{n}\right)^{8H} \sum_{(k_1,k_2,k_3,k_4)\in M_5} |k_2 - k_4|^{2H-2}|k_4 - k_2|^{2H-2} \\ &\le K\left(\frac{T}{n}\right)^{8H} \sum_{k_2,k_4:|k_2-k_4|>2} |k_2 - k_4|^{4H-4}. \end{aligned}$$

By taking $p = k_2 - k_4$, we obtain

$$\begin{aligned} \left|d_2(M_5)\right| &\le K\left(\frac{T}{n}\right)^{8H} \sum_{k_2=0}^{n} \sum_{n\ge|p|>2} |p|^{4H-4} \\ &\le Kn\left(\frac{T}{n}\right)^{8H} \left(n^{4H-3} \vee 1\right). \end{aligned}$$

It follows from the above estimate that $n^{8H-2} d_2(M_5)$ converges to zero as n tends to infinity. The proof of the convergence $n^{8H-2} d_2(M_6) \to 0$ is similar. Instead of (A.5), we have the estimate

$$\left|c(k_1,k_2,k_3,k_4)\right| \le K|k_1 - k_4|^{2H-2}|k_2 - k_3|^{2H-2}$$

for $(k_1,k_2,k_3,k_4) \in M_6$.

Step 4. In this step, we derive a new expression for $c(k_1, k_2, k_3, k_4)$. Recall that $\varphi_{k_4,s'}(v') = \varphi^0_{k_4,s'}(v') - \varphi^1_{k_4,s'}(v')$ (see (A.3)). Substituting this identity into (A.2), we obtain

$$c(k_1, k_2, k_3, k_4) = c_0(k_1, k_2, k_3, k_4) - c_1(k_1, k_2, k_3, k_4), \tag{A.7}$$

where

$$c_i(k_1, k_2, k_3, k_4) = \int_{\substack{k_4<s'<k_4+1\\ k_1<s<k_1+1}} \int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}} \int_{0<v,v',u,u'<n} \varphi_{k_3,r'}(u')\varphi^i_{k_4,s'}(v') \times \varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\mu(dv\,dv')\mu(du\,du')\mu(dr\,dr')\mu(ds\,ds').$$

By exchanging the orders of the integrals associated with v' and s' in c_1, we obtain

$$\begin{aligned} &c_1(k_1, k_2, k_3, k_4) \\ &\quad= \int_{\substack{k_4<v'<k_4+1\\ k_1<s<k_1+1}} \int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}} \int_{0<v,u,u'<n} \int_{k_4<s'<v'} |v - v'|^{2H-2}|s - s'|^{2H-2} \\ &\quad\quad\times \varphi_{k_3,r'}(u')\varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\,dv\,ds'\mu(du\,du')\mu(dr\,dr')\,ds\,dv', \end{aligned}$$

which, by switching the notations s' and v', is equal to

$$\begin{aligned} &\int_{\substack{k_4<s'<k_4+1\\ k_1<s<k_1+1}} \int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}} \int_{0<v,u,u'<n} \int_{k_4<v'<s'} |v - s'|^{2H-2}|s - v'|^{2H-2}\varphi_{k_3,r'}(u') \\ &\quad\times \varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\,dv\,dv'\mu(du\,du')\mu(dr\,dr')\,ds\,ds'. \end{aligned}$$

Substituting the above expression of c_1 into (A.7), we obtain

$$\begin{aligned} &c(k_1, k_2, k_3, k_4) \\ &\quad= \int_{\substack{k_4<s'<k_4+1\\ k_1<s<k_1+1}} \int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}} \int_{0<v,u,u'<n} \int_{k_4<v'<s'} \phi(s, s', v, v')\varphi_{k_3,r'}(u') \\ &\quad\quad\times \varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\,dv\,dv'\mu(du\,du')\mu(dr\,dr')\,ds\,ds', \end{aligned} \tag{A.8}$$

where we denote

$$\phi(s, s', v, v') = |v - v'|^{2H-2}|s - s'|^{2H-2} - |v - s'|^{2H-2}|s - v'|^{2H-2}. \tag{A.9}$$

Step 5. We turn to $d_2(M_4)$. It is easy to show that

$$d_2(M_{4i}) = d_2(M_{4j}), \quad i, j = 1, 2, 3, 4. \tag{A.10}$$

As an example, we show that $d_2(M_{41}) = d_2(M_{44})$. The other identities in (A.10) can be shown in a similar way. First, by exchanging the orders of integrals associated with $\mu(dr\,dr')$ and $\mu(ds\,ds')$ and integrals associated with $\mu(dv\,dv')$ and $\mu(du\,du')$, we obtain

$$c(k_1, k_2, k_3, k_4) = \int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}} \int_{\substack{k_4<s'<k_4+1\\ k_1<s<k_1+1}} \int_{0<v,v',u,u'<n} \varphi_{k_3,r'}(u')\varphi_{k_4,s'}(v') \times \varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\mu(du\,du')\mu(dv\,dv')\mu(ds\,ds')\mu(dr\,dr').$$

Replacing $(v, v', u, u', r, r', s, s')$ by $(u, u', v, v', s, s', r, r')$ in the above expression, we obtain

$$\begin{aligned} c(k_1, k_2, k_3, k_4) &= \int_{\substack{k_3<s'<k_3+1\\ k_2<s<k_2+1}} \int_{\substack{k_4<r'<k_4+1\\ k_1<r<k_1+1}} \int_{0<v,v',u,u'<n} \varphi_{k_3,s'}(v')\varphi_{k_4,r'}(u') \\ &\quad\times \varphi_{k_2,s}(u)\varphi_{k_1,r}(v)\mu(dv\,dv')\mu(du\,du')\mu(dr\,dr')\mu(ds\,ds') \\ &= c(k_2, k_1, k_4, k_3). \end{aligned}$$

So we have

$$d_2(M_{41}) = 6\left(\frac{T}{n}\right)^{8H} \sum_{(k_1,k_2,k_3,k_4)\in M_{41}} c(k_2,k_1,k_4,k_3) = d_2(M_{44}),$$

where the second identity follows by replacing (k_1,k_2,k_3,k_4) by (k_2,k_1,k_4,k_3).

The identities in (A.10) imply that to show the convergence $n^{8H-2}d_2(M_4)\to 0$, it suffices to show that $n^{8H-2}d_2(M_{44})\to 0$ as $n\to\infty$.

Take $(k_1,k_2,k_3,k_4)\in M_{44}$. Then we have $|k_1-k_4|>2$ and $|k_2-k_4|>2$, and thus the quantities $|v-v'|$, $|s-s'|$, $|v-s'|$, $|s-v'|$ in ϕ are larger than one. This allows us to apply the mean value theorem to ϕ to obtain the estimate

$$\text{(A.11)}\quad \left|\phi(s,s',v,v')\right| \le K\left(|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-2} + |k_4-k_1|^{2H-2}|k_4-k_2|^{2H-3}\right).$$

Applying (A.11) to (A.8) and taking into account (A.4), we obtain

$$\text{(A.12)}\quad \left|c(k_1,k_2,k_3,k_4)\right| \le K\left(|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-2} + |k_4-k_1|^{2H-2}|k_4-k_2|^{2H-3}\right).$$

Since $|k_1-k_2|\le|k_1-k_3|+|k_3-k_2|\le 4$, we have $|k_4-k_1|\le 3|k_4-k_2|$. This applied to (A.12) yields

$$\left|c(k_1,k_2,k_3,k_4)\right| \le K|k_4-k_1|^{4H-5},$$

and thus

$$\begin{aligned}\left|d_2(M_{44})\right| &\le 6\left(\frac{T}{n}\right)^{8H} \sum_{(k_1,k_2,k_3,k_4)\in M_{44}} \left|c(k_1,k_2,k_3,k_4)\right| \\ &\le 6\left(\frac{T}{n}\right)^{8H} \sum_{k_1,k_4:|k_1-k_4|>2} K|k_4-k_1|^{4H-5}.\end{aligned}$$

By taking $p=k_1-k_4$, we obtain

$$\left|d_2(M_{44})\right| \le 6\left(\frac{T}{n}\right)^{8H} \sum_{k_1=0}^{n}\sum_{n\ge|p|>2} p^{4H-5} \le 12n\left(\frac{T}{n}\right)^{8H}\sum_{p=3}^{\infty} p^{4H-5},$$

which implies that $n^{8H-2}d_2(M_{44})\to 0$ as $n\to\infty$.

Step 6. In this step, we consider $d_2(M_2)$ and $d_2(M_3)$. As in Step 4, it is easy to show that $d_2(M_{21}) = d_2(M_{22})$. So to show that $n^{8H-2}d_2(M_2)\to 0$, it suffices to show that $n^{8H-2}d_2(M_{22})\to 0$ as $n\to\infty$.

Take $(k_1,k_2,k_3,k_4)\in M_{22}$, we have $|k_1-k_4|>2$, $|k_2-k_4|>2$, and so inequality (A.11) holds. Applying (A.11) to $d_2(M_{22})$ and taking $p_1=k_1-k_4$ and $p_2=k_4-k_2$, we obtain

$$\begin{aligned}&\left|d_2(M_{22})\right| \\ &\quad\le K\left(\frac{T}{n}\right)^{8H} \sum_{(k_1,k_2,k_3,k_4)\in M_{22}} \left(|k_4-k_1|^{2H-3}|k_4-k_2|^{2H-2} + |k_4-k_1|^{2H-2}|k_4-k_2|^{2H-3}\right) \\ &\quad\le K\left(\frac{T}{n}\right)^{8H} \sum_{k_4=1}^{n}\sum_{p_1,p_2:n\ge|p_1|,|p_2|\ge 2} \left(|p_1|^{2H-3}|p_2|^{2H-2} + |p_1|^{2H-2}|p_2|^{2H-3}\right).\end{aligned}$$

It is easy to see from the above estimate that $n^{8H-2}|d_2(M_{22})|\le Kn^{2H-2}$, which converges to zero as n tends to infinity. The proof for the convergence $n^{8H-2}d_2(M_3)\to 0$ follows the same lines.

Step 7. It remains to show that $n^{8H-2}d_2(M_1) \to 0$ as $n \to \infty$. To do this, we first derive a new expression for $c(k_1,k_2,k_3,k_4)$. Recall $\varphi_{k_3,r'}(u') = \varphi^0_{k_3,r'}(u') - \varphi^1_{k_3,r'}(u')$ in (A.3). Substituting this identity into (A.8), we obtain

$$c(k_1,k_2,k_3,k_4) = \tilde{c}_0(k_1,k_2,k_3,k_4) - \tilde{c}_1(k_1,k_2,k_3,k_4), \tag{A.13}$$

where

$$\begin{aligned}
&\tilde{c}_i(k_1,k_2,k_3,k_4) \\
&\quad = \int_{\substack{k_4<s'<k_4+1\\ k_1<s<k_1+1}} \int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}} \int_{0<v,u,u'<n} \int_{k_4<v'<s'} \phi(s,s',v,v')\varphi^i_{k_3,r'}(u') \\
&\qquad \times \varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\,dv\,dv'\mu(du\,du')\mu(dr\,dr')\,ds\,ds',
\end{aligned}$$

where recall that ϕ is defined by (A.9). As in Step 3, by exchanging the order of the integrals associated with the variables r' and u', and then switching the notations r' and u', we obtain

$$\begin{aligned}
\tilde{c}_1 = &\int_{\substack{k_4<s'<k_4+1\\ k_1<s<k_1+1}} \int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}} \int_{0<v,u<n} \int_{k_3<u'<r'} \int_{k_4<v'<s'} \phi(s,s',v,v') \\
&\times |u-r'|^{2H-2}|r-u'|^{2H-2}\varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\,dv\,dv'\,du\,du'\,dr\,dr'\,ds\,ds'.
\end{aligned}$$

Substituting the above expression of $\tilde{c}_1$ into (A.13), we obtain

$$\begin{aligned}
&c(k_1,k_2,k_3,k_4) \\
&\quad = \int_{\substack{k_4<s'<k_4+1\\ k_1<s<k_1+1}} \int_{\substack{k_3<r'<k_3+1\\ k_2<r<k_2+1}} \int_{0<v,u<n} \int_{k_3<u'<r'} \int_{k_4<v'<s'} \phi(s,s',v,v') \\
&\qquad \times \phi(r,r',u,u')\varphi_{k_2,r}(v)\varphi_{k_1,s}(u)\,dv\,dv'\,du\,du'\,dr\,dr'\,ds\,ds'.
\end{aligned} \tag{A.14}$$

Take $(k_1,k_2,k_3,k_4) \in M_1$, then it is clear that the inequality (A.11) holds true, and in the same way, we can show that

$$\left|\phi(r,r',u,u')\right| \le K\left(|k_1-k_3|^{2H-3}|k_2-k_3|^{2H-2} + |k_1-k_3|^{2H-2}|k_1-k_3|^{2H-3}\right). \tag{A.15}$$

Applying inequalities (A.11) and (A.15) to (A.14) and taking $p_1 = k_3 - k_1$, $p_2 = k_2 - k_3$, $p_3 = k_4 - k_2$, we obtain

$$\begin{aligned}
\left|d_2(M_1)\right| &\le 6\left(\frac{T}{n}\right)^{8H} \sum_{(k_1,k_2,k_3,k_4)\in M_1} \left|c(k_1,k_2,k_3,k_4)\right| \\
&\le Kn^{-8H} \sum_{k_1=0}^{n} \sum_{(p_1,p_2,p_3)\in J} \left(|p_1|^{2H-3}|p_2|^{2H-2} + |p_1|^{2H-2}|p_2|^{2H-3}\right) \\
&\qquad \times \tilde{c}(p_1,p_2,p_3) \\
&= Kn^{-8H} \sum_{k_1=0}^{n} \sum_{p_1,p_2: n\ge|p_1|,|p_2|>2} \left(|p_1|^{2H-3}|p_2|^{2H-2} + |p_1|^{2H-2}|p_2|^{2H-3}\right) \\
&\qquad \times \sum_{p_3\in J(p_1,p_2)} \tilde{c}(p_1,p_2,p_3),
\end{aligned} \tag{A.16}$$

where

$$\begin{aligned}
\tilde{c}(p_1,p_2,p_3) &= |p_1+p_2+p_3|^{2H-3}|p_3|^{2H-2} + |p_1+p_2+p_3|^{2H-2}|p_3|^{2H-3}, \\
J &= \left\{(p_1,p_2,p_3) : n \ge |p_1|,|p_2|,|p_3|,|p_1+p_2+p_3| > 2\right\},
\end{aligned}$$

and

$$J(p_1, p_2) = \{p_3 : n \ge |p_3|, |p_1 + p_2 + p_3| > 2\}.$$

We claim that $\sum_{p_3 \in J(p_1,p_2)} \tilde{c}(p_1, p_2, p_3)$ is uniformly bounded in (p_1, p_2) by a constant. Take p_1, p_2 such that $n \ge |p_1|, |p_2| > 2$. Since when $|p_1 + p_2 + p_3| < |p_3|$ we have $|p_1 + p_2 + p_3|^\alpha > |p_3|^\alpha$ for $\alpha = 2H - 2$ and $\alpha = 2H - 3$, we can write

$$\begin{aligned} \sum_{p_3 \in J(p_1,p_2):|p_1+p_2+p_3|<|p_3|} \tilde{c}(p_1, p_2, p_3) &\le \sum_{p_3 \in J(p_1,p_2)} |p_1 + p_2 + p_3|^{4H-5} \\ &\le 2 \sum_{p=3}^{\infty} p^{4H-5}. \end{aligned} \tag{A.17}$$

Similarly, we have

$$\sum_{p_3 \in J(p_1,p_2):|p_1+p_2+p_3|\ge|p_3|} \tilde{c}(p_1, p_2, p_3) \le \sum_{p_3 : n \ge |p_3| > 2} |p_3|^{4H-5} \le 2 \sum_{p=3}^{\infty} p^{4H-5}. \tag{A.18}$$

In summary of (A.17) and (A.18), we have shown that

$$\sum_{p_3 \in J(p_1,p_2)} \tilde{c}(p_1, p_2, p_3) \le 4 \sum_{p=2}^{\infty} p^{4H-5}. \tag{A.19}$$

Applying inequality (A.19) to (A.16), we obtain the estimate

$$\left|d_2(M_1)\right| \le K n^{-6H},$$

which implies that $n^{8H-2} d_2(M_1) \to 0$ as $n \to \infty$.

A.2. Proof of (3.46). By the integration by parts formula (2.2), we obtain

$$\mathbb{E}\big(z_n(t) B_r\big) = \sum_{k=0}^{\lfloor nt/T \rfloor} \int_0^r \int_0^T \int_0^T \beta_{\frac{k}{n}}(s) \gamma_{t_k,s}(u)\, du\, \mu\big(ds\, ds'\big).$$

By changing the variables from (u, s, s') to $\frac{T}{n}(u, s, s')$ in the above expression, we obtain

$$\mathbb{E}\big(z_n(t) B_r\big) = \left(\frac{T}{n}\right)^{2H+1} \sum_{k=0}^{\lfloor nt/T \rfloor} \int_0^{\frac{nr}{T}} \int_k^{k+1} \int_0^n \varphi_{k,s}(u) \left|s - s'\right|^{2H-2} du\, ds\, ds', \tag{A.20}$$

where $\varphi_{k,s}(u)$ is defined in (A.3). Let us denote $I_1(k) = [k-2, k+2] \cap [0, \frac{nr}{T}]$ and $I_2(k) = [0, \frac{nr}{T}] \setminus I_1(k)$, and set

$$A_i = \left(\frac{T}{n}\right)^{2H+1} \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{I_i(k)} \int_k^{k+1} \int_0^n \varphi_{k,s}(u) \left|s - s'\right|^{2H-2} du\, ds\, ds'.$$

Then it is easy to show that

$$\mathbb{E}\big(z_n(t) B_r\big) = A_1 + A_2.$$

So to prove that $n^{H+\frac{1}{2}} \mathbb{E}(z_n(t) B_r) \to 0$ it suffices to show that $n^{H+\frac{1}{2}} A_i \to 0$ as $n \to \infty$ for $i = 1, 2$.

We can write

$$
\begin{aligned}
A_1 &\le \left(\frac{T}{n}\right)^{2H+1} \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{k-2}^{k+2} \int_k^{k+1} \int_k^{k+1} |s-s'|^{2H-2}\, du\, ds\, ds' \\
&\le Kn\left(\frac{T}{n}\right)^{2H+1},
\end{aligned}
$$

so we have $n^{H+\frac{1}{2}} A_1 \to 0$ as $n \to \infty$.

Now we turn to A_2. By exchanging the orders of the integrals with respect to u and s, we have

$$
\int_k^{k+1} \int_0^n \mathbf{1}_{[k,s]}(u) |s-s'|^{2H-2}\, du\, ds = \int_k^{k+1} \int_k^s |u-s'|^{2H-2}\, du\, ds. \tag{A.21}
$$

Substituting (A.21) into A_2, we obtain

$$
A_2 = \left(\frac{T}{n}\right)^{2H+1} \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{I_2(k)} \int_k^{k+1} \int_k^s \left(|s-s'|^{2H-2} - |u-s'|^{2H-2}\right) du\, ds\, ds'. \tag{A.22}
$$

Note that for $s' \in I_2(k)$ we have

$$
\left| \int_k^{k+1} \int_k^s \left(|s-s'|^{2H-2} - |u-s'|^{2H-2}\right) du\, ds \right| \le K|k-s'|^{2H-3},
$$

so

$$
\begin{aligned}
|A_2| &\le \left(\frac{T}{n}\right)^{2H+1} \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{I_2(k)} K|k-s'|^{2H-3}\, ds' \\
&= K\left(\frac{T}{n}\right)^{2H+1} \sum_{k=0}^{\lfloor nt/T \rfloor} \left(2^{2H-2} - (n-k)^{2H-2} + 2^{2H-2} - k^{2H-2}\right) \\
&\le K\left(\frac{T}{n}\right)^{2H+1} \sum_{k=0}^{n} 2^{2H-2} = Kn\left(\frac{T}{n}\right)^{2H+1},
\end{aligned}
$$

which implies that $n^{H+\frac{1}{2}} A_2 \to 0$ as $n \to \infty$. This completes the proof.

A.3. Estimates of some triple integrals. In this subsection, we provide estimates for some triple integrals which have been used in the main body of the paper.

LEMMA A.1.

(i) *For $t \in \Pi$, we define*

$$
G_t = \sum_{k=0}^{nt/T-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \int_{t_k}^{u} dB_v^1\, dB_u^2\, dB_s^3, \tag{A.23}
$$

where B^1, B^2, B^3 are independent processes, and are either fBm's with Hurst parameter $H > \frac{1}{2}$ or are equal to the identity function. Let $p \ge 1$. Then we have

$$
\|G_t - G_s\|_p \le Kn^{-2H}|t-s|^{\frac{1}{2}}, \quad s, t \in \Pi.
$$

(ii) *Let B a one-dimensional fBm with Hurst parameter $H > \frac{1}{2}$. Take $p \geq 1$ and $t \in [0, T]$. We have the following convergence in L^p:*

$$n^{2H} \sum_{k=0}^{\lfloor nt/T \rfloor -1} (B_{t_{k+1}} - B_{t_k})^3 \to 3T^{2H} B_t.$$

PROOF. The result in (i) follows from Proposition 5.10 in [11] with $r' = 3$ and $H_\alpha = H + H + H$. The convergence in (ii) follows immediately from results in [8] or [26]. □

We need the following technical lemma.

LEMMA A.2. *Let f and g be β-Hölder continuous stochastic processes on $[0, T]$ and $\mathbb{E}(\|f\|_\beta^p) + \mathbb{E}(\|g\|_\beta^p) \leq K$ for all $\frac{1}{2} < \beta < H$ and $p \geq 1$, and let h^n, $n \in \mathbb{N}$ be processes on $[0, T]$ such that*

$$\|h_t^n - h_s^n\|_p \leq K(t - s)^\beta, \quad s, t \in \Pi : s \leq t.$$

Let B^1, B^2 and B^3 be as in Lemma A.1. For each $i, j = 1, 2, 3$ we denote

$$\tilde{G}_t^{ij} = \sum_{k=0}^{nt/T-1} h_{t_k}^n \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_3} \int_{t_k}^{s_2} f_{s_i} g_{s_j} \, dB_{s_1}^1 \, dB_{s_2}^2 \, dB_{s_3}^3, \quad t \in \Pi. \tag{A.24}$$

Then the following estimate holds true for all $s, t \in \Pi$:

$$\|\tilde{G}_t^{ij} - \tilde{G}_s^{ij}\|_p \leq K n^{-2H} |t - s|^{\frac{1}{2}}. \tag{A.25}$$

PROOF. We decompose $\tilde{G}^{ij}$ as follows:

$$\begin{aligned}
\tilde{G}_t^{ij} = {} & \sum_{k=0}^{nt/T-1} f_{t_k} g_{t_k} h_{t_k}^n \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_3} \int_{t_k}^{s_2} dB_{s_1}^1 \, dB_{s_2}^2 \, dB_{s_3}^3 \\
& + \sum_{k=0}^{nt/T-1} f_{t_k} h_{t_k}^n \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_3} \int_{t_k}^{s_2} \int_{t_k}^{s_j} dg_{s_4} \, dB_{s_1}^1 \, dB_{s_2}^2 \, dB_{s_3}^3 \\
& + \sum_{k=0}^{nt/T-1} h_{t_k}^n \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_3} \int_{t_k}^{s_2} \int_{t_k}^{s_i} g_{s_j} \, df_{s_4} \, dB_{s_1}^1 \, dB_{s_2}^2 \, dB_{s_3}^3.
\end{aligned} \tag{A.26}$$

Applying Proposition 5.10 in [11] with $r' = 4$ and $H_\alpha = \beta + H + H + H$ to the second and third terms on the right-hand side of (A.26), and applying Lemma 2.4 to the first term and taking into account the estimate in Lemma A.1(i), we obtain inequality (A.25). □

A.4. Proof of Lemma 4.2. By the definition of J_1 (see (4.3)), we have

$$\begin{aligned}
\int_s^t \Gamma_u^n \, dJ_1(u) &= \sum_{i=1}^d \int_s^t \Gamma_u^n \int_{\eta(u)}^u \partial_i V(X_v^n) \, dX_v^{n,i} \, dB_u \\
&= \sum_{i=1}^d \sum_{k=\lfloor ns/T \rfloor}^{\lfloor nt/T \rfloor} \int_{t_k \vee s}^{t_{k+1} \wedge t} \Gamma_u^n \int_{\eta(u)}^u \partial_i V(X_v^n) \, dX_v^{n,i} \, dB_u
\end{aligned}$$

for $s, t \in [0, T]$. Applying the Minkovski inequality to the right-hand side of the above equation, and then taking into account Lemma 8.2 in [11] as well as the integrability of Γ^n, X^n and B in the sense of Definition 2.2, we obtain the estimate

$$\text{(A.27)} \qquad \begin{aligned} \left\| \int_s^t \Gamma_u^n \, dJ_1(u) \right\|_p &\le \sum_{i=1}^d \sum_{k=\lfloor ns/T \rfloor}^{\lfloor nt/T \rfloor} n^{-\beta}(t_{k+1} \wedge t - t_k \vee s)^\beta \\ &\le K|t-s|^\beta n^{1-2\beta}. \end{aligned}$$

In the same way we can show that estimate (A.27) holds while J_1 is replaced by J_2. Applying these two estimates to

$$Y_t = \frac{1}{2}\Lambda_t^n \sum_{i=1}^2 \int_0^t \Gamma_s^n \, dJ_i(s),$$

we obtain

$$\text{(A.28)} \qquad \|Y\|_{\beta,p} \le K n^{1-2\beta}.$$

We denote $\Phi := \Lambda - \Lambda^n$. Subtracting (4.23) from (4.26), we can write

$$\begin{aligned} \Phi_{i'}^i(t) &= \sum_{j=0}^m \sum_{i''=1}^d \int_0^t \left[\partial_{i''} V_j^i(X_s)\Lambda_{i'}^{i''}(s) - V_{ji''}^i(s)\Lambda_{i'}^{n,i''}(s)\right] dB_s^j \\ &= \sum_{j=0}^m \sum_{i''=1}^d \int_0^t \partial_{i''} V_j^i(X_s)\Phi_{i'}^{i''}(s)\, dB_s^j \\ &\quad + \sum_{j=0}^m \sum_{i''=1}^d \int_0^t \left[\partial_{i''} V_j^i(X_s) - V_{ji''}^i(s)\right]\Lambda_{i'}^{n,i''}(s)\, dB_s^j. \end{aligned}$$

The following identity is an easy consequence of the product rule

$$\text{(A.29)} \qquad \Lambda(t) - \Lambda^n(t) = \sum_{i,i'=1}^d \sum_{j=0}^m \Lambda(t) \int_0^t \Gamma_{i'}(s)\left[\partial_i V_j^{i'}(X_s) - V_{ji}^{i'}(s)\right]\Lambda^{n,i}(s)\, dB_s^j.$$

Denote

$$\tilde V(X_s, X_s^n) = \int_0^1 \int_0^1 \partial\partial_{i'} V_j^{i''}\big(\lambda X_s + (1-\lambda)\big(\theta X_s + (1-\theta)X_s^n\big)\big)(1-\theta)\, d\lambda\, d\theta.$$

One can show that

$$\partial_{i'} V_j^{i''}(X_s) - V_{ji'}^{i''}(s) = \tilde V(X_s, X_s^n) Y_s.$$

Then (A.29) becomes

$$\Lambda_i(t) - \Lambda_i^n(t) = \Lambda_t \int_0^t dg_s \cdot Y_s,$$

where

$$g_t = \sum_{i',i''=1}^d \sum_{j=0}^m \int_0^t \Gamma_{i''}(s)\tilde V(X_s, X_s^n)\Lambda_i^{n,i'}(s)\, dB_s^j.$$

Applying Lemma 2.3 and taking into account the estimate (A.28), we obtain

$$\left\| \Lambda_t \int_s^t dg_s \cdot Y_s \right\|_p \le K n^{1-2\beta}(t-s)^\beta,$$

which implies the estimate for $\|\Lambda - \Lambda\|_{\beta,p}$. The estimate for the quantity $\Gamma - \Gamma^n$ can be shown similarly.

A.5. Proof of (4.43). Since $E_1 = 0$, it is clear that

$$\sum_{i=1}^{2}\int_0^t \Gamma_u^n \, dJ_i(u) - \sum_{e=2}^{5}\int_0^{\eta(t)} \Gamma_{\eta(u)}^n \, dE_e(u)$$

$$(A.30)\qquad = \sum_{i=1}^{2}\int_0^t \Gamma_u^n \, dJ_i(u) - \sum_{i=1}^{2}\int_0^{\eta(t)} \Gamma_{\eta(u)}^n \, dJ_i(u)$$

$$= \sum_{i=1}^{2}\int_0^t \int_{\eta(s)}^{s} d\Gamma_u^n \, dJ_i(s) + \sum_{i=1}^{2}\int_{\eta(t)}^{t} \Gamma_{\eta(u)}^n \, dJ_i(u).$$

In the following, we estimate the L^p-norms of the two terms on the right-hand side of (A.30).

Recall that I_1 and I_2 are defined by (4.8)–(4.9). For $t \in [0, T]$, we set

$$I(t) = \sum_{i=1}^{d}\int_0^t (V^i \partial_i V)(X_{\eta(s)}^n)(B_s - B_{\eta(s)}) \, dB_s.$$

It is clear that $I(t_k) = I_1(t_k) = I_2(t_k)$ for $k = 0, 1, \dots, n$. As in (4.12), we introduce the decomposition

$$J_1(t) + J_2(t) = (R_0(t) - I(t)) + (I(t) - \tilde{R}_0(t)) + R_1(t) + \tilde{R}_1(t)$$

$$:= E_2(t) + E_3(t) + E_4(t) + E_5(t)$$

for $t \in [0, T]$, where R_0, R_1, $\tilde{R}_0$ and $\tilde{R}_1$ are defined as before. Note that the E_2 and E_3 defined here are extensions of similar terms in (4.12) from Π to $[0, T]$. Applying Lemma 8.2 in [11] to (4.5) and (4.7), we obtain

$$(A.31)\qquad \|E_e\|_{t_k, t_{k+1}, \beta} \le K e^{K\|B\|_\beta^{1/\beta}} n^{-2\beta}, \quad e = 4, 5.$$

Similarly, we can show that inequality (A.31) also holds for $e = 2, 3$. Therefore, we obtain

$$(A.32)\qquad \|J_1 + J_2\|_{[t_k, t_{k+1}], \beta} = \left\| \sum_{e=2}^{5} E_e \right\|_{[t_k, t_{k+1}], \beta} \le K e^{K\|B\|_\beta^{1/\beta}} n^{-2\beta}.$$

Applying Lemma 8.2 in [11] and with the help of the estimate (A.32), we can write

$$\left\| \sum_{i=1}^{2}\int_{t'}^{t''} \int_{\eta(s)}^{s} d\Gamma_u^n \, dJ_i(s) \right\|_p \le K n^{-4\beta}$$

for $t', t'' \in [t_k, t_{k+1}]$. Therefore, we have

$$\left\| \sum_{i=1}^{2}\int_0^t \int_{\eta(s)}^{s} d\Gamma_u^n \, dJ_i(s) \right\|_p \le \sum_{k=0}^{\lfloor nt/T \rfloor} \left\| \sum_{i=1}^{2}\int_{t_k}^{t_{k+1}\wedge t} \int_{\eta(s)}^{s} d\Gamma_u^n \, dJ_i(s) \right\|_p$$

$$\le K n^{1-4\beta}.$$

On the other hand, applying (A.32) to

$$\sum_{i=1}^{2}\int_{\eta(t)}^{t} \Gamma_{\eta(s)}^n \, dJ_i(s) = \sum_{i=1}^{2} \Gamma_{\eta(t)}^n (J_i(t) - J_i(\eta(t))),$$

we obtain

$$\left\| \sum_{i=1}^{2}\int_{\eta(t)}^{t} \Gamma_{\eta(s)}^n \, dJ_i(s) \right\|_p \le K n^{-3\beta} \le K n^{1-4\beta}.$$

This completes the proof.

A.6. L^p-Convergence results. In this subsection we denote by B and $\tilde{B}$ two independent fBm's on $[0,T]$ with $H>1/2$. As before, we adopt the notations for increment: $B_{st}=B_t-B_s$ and $\tilde{B}_{st}=\tilde{B}_t-\tilde{B}_s$. We start with the following processes defined on $[0,T]$:

$$\begin{aligned} z_n^1(t) &= 2\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}\int_{t_k}^{u}\int_{t_k}^{v} d\tilde{B}_r\,d\tilde{B}_v\,dB_u,\\ z_n^2(t) &= 2\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}\int_{t_k}^{u}\int_{t_k}^{v} dB_r\,d\tilde{B}_v\,d\tilde{B}_u,\\ z_n^3(t) &= \sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}(\tilde{B}_{t_kt_{k+1}})^2\,dB_u=\sum_{k=0}^{\lfloor nt/T\rfloor}(\tilde{B}_{t_kt_{k+1}})^2\cdot B_{t_kt_{k+1}}. \end{aligned}\tag{A.33}$$

Observe that, by an elementary application of the chain rule and the exchange of integrals, we have

$$z_n^1(t)=\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}(\tilde{B}_{t_ku})^2\,dB_u,\qquad z_n^2(t)=\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}(\tilde{B}_{ut_{k+1}})^2\,dB_u.\tag{A.34}$$

PROPOSITION A.3. *Let z_n^i, $i=1,2,3$ be defined in (A.33). Then there exists a constant K depending on H and T such that for $i=1,2,3$ we have*

$$n^{4H}\mathbb{E}\big(\big|z_n^i(t)-z_n^i(s)\big|^2\big)\le K(t-s),\quad t,s\in\Pi, s\le t.\tag{A.35}$$

Furthermore, for each $t\in[0,T]$ and $i=1,2$, we have that $n^{2H}z_n^i(t)$ converges in L^2 to $T^{2H}(2H+1)^{-1}B_t$ and $n^{2H}z_n^3(t)$ converges in L^2 to $T^{2H}B_t$ as $n\to\infty$.

PROOF. The proof is divided into several steps.

Step 1. L^2 estimate of z_n^1. We start with the decomposition

$$\begin{aligned} z_n^1(t) &= \sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}(u-t_k)^{2H}\,dB_u+\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}\big[(\tilde{B}_{t_ku})^2-(u-t_k)^{2H}\big]\,dB_u\\ &:= z_n^{11}(t)+z_n^{12}(t). \end{aligned}$$

With some elementary computations similar to Lemma 4.3 in [13] we can show that

$$\big(\mathbb{E}\big(\big|z_n^{12}(t)-z_n^{12}(s)\big|^2\big)\big)^{1/2}\le K\big(n^{-2}\vee n^{1/2-3H}\big)(t-s)^{1/2}.$$

In order to show (A.35) for z_n^1 it then remains to prove that (A.35) holds for $z_n^{11}(t)$.

Let us apply the covariance formula (2.1) to get

$$\begin{aligned} \mathbb{E}\big(z_n^{11}(t)^2\big) &= \sum_{k,k'=0}^{\lfloor nt/T\rfloor}\int_{t_{k'}}^{t_{k'+1}}\int_{t_k}^{t_{k+1}}(s-t_k)^{2H}\big(s'-t_{k'}\big)^{2H}\mu\big(ds\,ds'\big)\\ &= \left(\frac{T}{n}\right)^{6H}\sum_{k,k'=0}^{\lfloor nt/T\rfloor}\int_{k'}^{k'+1}\int_{k}^{k+1}(s-k)^{2H}\big(s'-k'\big)^{2H}\mu\big(ds\,ds'\big), \end{aligned}\tag{A.36}$$

where μ is defined by (3.2), and in the second equation we have replaced the variables (s,s') by $(\frac{T}{n}s,\frac{T}{n}s')$. Denoting

$$\bar{Q}(p)=\int_0^1\int_p^{p+1}(s-p)^{2H}t^{2H}\mu(ds\,dt).$$

Noticing that $\bar{Q}(p) = \bar{Q}(-p)$ for $p \in \mathbb{Z}$, equation (A.36) becomes

$$
\begin{aligned}
\mathbb{E}\big(z_n^{11}(t)^2\big) &= \left(\frac{T}{n}\right)^{6H} \sum_{k,k'=0}^{\lfloor nt/T \rfloor} \bar{Q}(k-k') \\
&= \left(\frac{T}{n}\right)^{6H} \bar{Q}(0)\big(\lfloor nt/T \rfloor + 1\big) + 2\left(\frac{T}{n}\right)^{6H} \sum_{0 \le k' < k \le \lfloor nt/T \rfloor} \bar{Q}(k-k') \\
&= \left(\frac{T}{n}\right)^{6H} \bar{Q}(0)\big(\lfloor nt/T \rfloor + 1\big) + 2\left(\frac{T}{n}\right)^{6H} \sum_{p=1}^{\lfloor nt/T \rfloor} \bar{Q}(p)\big(\lfloor nt/T \rfloor + 1 - p\big).
\end{aligned} \tag{A.37}
$$

In order to bound $\mathbb{E}(z_n^{11}(t)^2)$, we use the fact that $\lfloor nt/T \rfloor \le nt/T \le n^{2H}t/T$, $\bar{Q}(p) \le (p-1)^{2H-2}$ for $p > 1$ and $\lfloor nt/T \rfloor + 1 - p \le nt/T$. We end up with

$$
\mathbb{E}\big(z_n^{11}(t)^2\big) \le K n^{-4H} t, \quad t > 0.
$$

Taking into account that $\mathbb{E}(|z_n^{11}(t) - z_n^{11}(s)|^2) = \mathbb{E}(z_n^{11}(t - s - \frac{T}{n})^2)$ for $s, t \in \Pi$, we obtain the desired estimate (A.35) for z_n^{11}.

Step 2. L^2-convergence of z_n^1 (i). We first observe that to show the L^2-convergence of $n^{2H} z_n^1$, it suffices to show the following two convergences:

$$
\begin{aligned}
n^{4H}\mathbb{E}\big(z_n^1(t)^2\big) &\to T^{4H}(2H+1)^{-2} t^{2H}, \\
n^{2H}\mathbb{E}\big(z_n^1(t) B_t\big) &\to T^{2H}(2H+1)^{-1} t^{2H}.
\end{aligned} \tag{A.38}
$$

We consider the two convergences in (A.38) respectively in this and next steps.

Notice that the mean value theorem implies that for $|p| > 1$ there exists $\tilde{p} \in [p-1, p+1]$ such that

$$
\begin{aligned}
\bar{Q}(p) &= \int_0^1 \int_p^{p+1} (s-p)^{2H} t^{2H} \mu(ds\,dt) \\
&= \alpha_H \tilde{p}^{2H-2} \int_0^1 \int_p^{p+1} (s-p)^{2H} t^{2H}\, ds\, dt = c_H \tilde{p}^{2H-2},
\end{aligned}
$$

where $\alpha_H = H(2H-1)$ and $c_H = H(2H-1)(2H+1)^{-2}$. Therefore, we can write

$$
\begin{aligned}
&n^{-2H} \sum_{p=1}^{\lfloor nt/T \rfloor} \bar{Q}(p)\big(\lfloor nt/T \rfloor + 1 - p\big) \\
&\quad = c_H n^{-1} \sum_{p=1}^{\lfloor nt/T \rfloor} \left(\frac{\tilde{p}}{n}\right)^{2H-2} \frac{\lfloor nt/T \rfloor + 1 - p}{n} + n^{-2H} \bar{Q}(1) \lfloor nt/T \rfloor \\
&\quad = \frac{c_H}{T^{2H}} \frac{T}{n} \sum_{p=1}^{\lfloor nt/T \rfloor} \left(\frac{T}{n}\tilde{p}\right)^{2H-2} \left(\frac{T}{n}\big(\lfloor nt/T \rfloor + 1 - p\big)\right) + n^{-2H} \bar{Q}(1) \lfloor nt/T \rfloor,
\end{aligned}
$$

which is the Riemann sum of the function $s \to \frac{c_H}{T^{2H}} \cdot s^{2H-2}(t-s)$ from 0 to t plus a remainder term. Sending $n \to \infty$ we obtain

$$
\begin{aligned}
n^{-2H} \sum_{p=1}^{\lfloor nt/T \rfloor} \bar{Q}(p)\big(\lfloor nt/T \rfloor + 1 - p\big) &\to \frac{c_H}{T^{2H}} \int_0^t s^{2H-2}(t-s)\, ds \\
&= \frac{(2H+1)^{-2}}{2T^{2H}} t^{2H}.
\end{aligned} \tag{A.39}
$$

Applying (A.39) to the second term on the right-hand side of (A.37) and bounding the first term by n^{1-6H} we obtain the first convergence in (A.38).

Step 3. L^2-convergence of z_n^1 (ii). Take $t \in [t_l, t_{l+1})$ for $l = 0, \dots, n$. In order to show the second convergence in (A.38), we write $B_t = \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{t_k}^{t_{k+1}} dB_s - B_{t,t_{l+1}}$. Applying the covariance formula (2.1) as in (A.36) and using this expression of B_t, we obtain

$$\begin{aligned}\mathbb{E}(z_n^1(t)B_t) &= \sum_{k,k'=0}^{\lfloor nt/T \rfloor} \int_{t_{k'}}^{t_{k'+1}} \int_{t_k}^{t_{k+1}} (s-t_k)^{2H} \mu(ds\,ds') - \mathbb{E}(z_n^1(t) B_{t,t_{l+1}}) \\ &= \left(\frac{T}{n}\right)^{4H} \sum_{k,k'=0}^{\lfloor nt/T \rfloor} \int_{k'}^{k'+1} \int_{k}^{k+1} (s-k)^{2H} \mu(ds\,ds') - \mathbb{E}(z_n^1(t) B_{t,t_{l+1}}).\end{aligned}$$

In a similar way as for the convergence of $n^{4H}\mathbb{E}(z_n^1(t)^2)$, we can now show that the convergence of $n^{2H}\mathbb{E}(z_n^1(t)B_t)$ in (A.38) holds.

This completes the proof of the theorem for z_n^1. The proof for z_n^2 and z_n^3 can be shown in a similar way and is left to the reader. □

COROLLARY A.4. *Let B and $\tilde{B}$ be as in Proposition A.3. Let*

$$z_n^4(t) = \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \int_{t_k}^{v} d\tilde{B}_r\, dB_v\, d\tilde{B}_u. \tag{A.40}$$

Then the estimate (A.35) holds for z_n^4. Furthermore, for any $t \in [0,T]$ we have that $z_n^4(t)$ converges in L^2 to $T^{2H}(2H-1)(4H+2)^{-1}B_t$.

PROOF. We notice that by an exchange of integrals we obtain

$$\begin{aligned} z_n^3(t) &= \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{t_k}^{t_{k+1}} \tilde{B}_{t_k v} \tilde{B}_{v t_{k+1}}\, dB_v \\ &= \frac{1}{2} \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{t_k}^{t_{k+1}} ((\tilde{B}_{t_k t_{k+1}})^2 - (\tilde{B}_{t_k v})^2 - (\tilde{B}_{v t_{k+1}})^2)\, dB_v \\ &= \frac{1}{2}(z_n^3(t) - z_n^1(t) - z_n^2(t)).\end{aligned}$$

The corollary now follows from the application of Proposition A.3. □

In the following we consider the processes

$$\begin{aligned} z_n^6(t) &= \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \int_{t_k}^{v} dB_r\, dB_v\, du, \\ z_n^7(t) &= \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \int_{t_k}^{v} dr\, dB_v\, dB_u, \\ z_n^8(t) &= \sum_{k=0}^{\lfloor nt/T \rfloor} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \int_{t_k}^{v} dB_r\, dv\, dB_u. \end{aligned} \tag{A.41}$$

PROPOSITION A.5. *Let z_n^i $i=6,7,8$ be defined in* (A.41). *Then there exists a constant K depending on H and T such that for $i=6,7,8$ we have*

$$n^{4H}\mathbb{E}\big(\big|z_n^i(t)-z_n^i(s)\big|^2\big)\le K(t-s),\quad t,s\in\Pi. \tag{A.42}$$

Furthermore, for each $t\in[0,T]$ and $i=6,7$, we have that $n^{2H}z_n^i(t)$ converges in L^2 to $\frac{1}{2}T^{2H}(2H+1)^{-1}t$ and $n^{2H}z_n^8(t)$ converges in L^2 to $(2H-1)(4H+2)^{-1}T^{2H}t$ as $n\to\infty$.

PROOF. The proof follows the same arguments as in the proof of Proposition A.3 and Corollary A.4 and the details are omitted. □

A.7. Further L^p-convergence results. In this subsection we denote by $B=(B^1,\dots,B^m)$ an m-dimensional fBm. Let us introduce the following index sets:

$$\Xi=\{1,\dots,m\},\qquad \bar\Xi=\Xi\cup\{0\},\qquad \Xi_0=\Xi\times\Xi\times\Xi,\qquad \bar\Xi_0=\bar\Xi\times\bar\Xi\times\bar\Xi,$$

$$\Xi_{11}=\{(j,j',j'')\in\Xi_0: j\ne j'=j''\},\qquad \Xi_{12}=\{(j,j',j'')\in\Xi_0: j'\ne j=j''\},$$

$$\Xi_{13}=\{(j,j',j'')\in\Xi_0: j''\ne j=j'\},\qquad \Xi_{21}=\{(j,j',j'')\in\bar\Xi_0: j'=j''\ne j=0\},$$

$$\Xi_{22}=\{(j,j',j'')\in\bar\Xi_0: j=j''\ne j'=0\},$$

$$\Xi_{23}=\{(j,j',j'')\in\bar\Xi_0: j=j'\ne j''=0\}.$$

We also denote

$$\Xi_1=\Xi_{11}\cup\Xi_{12}\cup\Xi_{13}\quad\text{and}\quad \Xi_2=\Xi_{21}\cup\Xi_{22}\cup\Xi_{23}.$$

The following is a consequence of the results in the previous subsection:

COROLLARY A.6. *Consider the following processes on* $[0,T]$:

$$Z_n^1(t)=\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}dB_v^{j''}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}dB_u^{j'}\,dB_s^{j}$$

and

$$Z_n^2(t)=\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}dB_v^{j''}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}dB_u^{j'}\,dB_s^{j}.$$

(i) *If $(j,j',j'')\in\Xi_1$, then the estimate* (A.35) *holds for Z_n^1 and Z_n^2, and for $t\in[0,T]$ and $i=1,2$, $Z_n^i(t)$ converges in L^2 to $\frac{1}{2}T^{2H}B_t$.*

(ii) *If $(j,j',j'')\in\Xi_2$, then the estimate* (A.42) *holds for Z_n^1 and Z_n^2, and for $t\in[0,T]$ and $i=1,2$, $Z_n^i(t)$ converges in L^2 to $\frac{1}{2}T^{2H}t$.*

PROOF. In the case $(j,j',j'')\in\Xi_1$, the corollary follows by a decomposition of Z_n^i, $i=1,2$ into the sum of the processes of the forms of z_n^1, z_n^2 and z_n^4 in (A.33) and (A.40). In the case $(j,j',j'')\in\Xi_2$, the corollary can be shown by a decomposition of Z_n^i, $i=1,2$ into the sum of the processes of the forms of z_n^7, z_n^8 and z_n^9 in (A.41). □

LEMMA A.7. *Take $(j,j',j'')\in\bar\Xi_0\setminus(\Xi_0\cup\Xi_1\cup\Xi_2)$. Denote*

$$Z_n^3(t)=\sum_{k=0}^{\lfloor nt/T\rfloor}\int_{t_k}^{t_{k+1}}\int_{t_k}^{u}\int_{t_k}^{v}dB_r^{j''}\,dB_v^{j'}\,dB_u^{j}.$$

Then the following inequality holds:

$$\mathbb{E}\big(\big|Z_n^3(t)-Z_n^3(s)\big|^2\big)^{1/2}\le K\big(n^{-2}\vee n^{\frac{1}{2}-3H}\big)(t-s)^{1/2}.$$

PROOF. It follows from Proposition 5.5 in [11] that

$$\mathbb{E}\left(\left|Z_n^3(t)-Z_n^3(s)\right|^2\right) \le \mathbb{E}\left(\left|\sum_{k=\lfloor ns/T\rfloor+1}^{\lfloor nt/T\rfloor} B^{j}_{t_k t_{k+1}} B^{j'}_{t_k t_{k+1}} B^{j''}_{t_k t_{k+1}}\right|^2\right). \tag{A.43}$$

The lemma then follows from some elementary computations of the right-hand side of (A.43) similar to Lemma 4.3 in [13]. □

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