# RANDOM WALK ON RANDOM WALKS: LOW DENSITIES

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We consider a random walker in a dynamic random environment given
by a system of independent discrete-time simple symmetric random walks.
We obtain ballisticity results under two types of perturbations: low particle
density, and strong local drift on particles. Surprisingly, the random walker
may behave very differently depending on whether the underlying environ-
ment particles perform lazy or nonlazy random walks, which is related to a
notion of permeability of the system. We also provide a strong law of large
numbers, a functional central limit theorem and large deviation bounds under
an ellipticity condition.

**1. Introduction and main results.** Random walks on random environments are models
for the movement of a tracer particle in a disordered medium, and have been the subject of in-
tense research for over 40 years. The seminal works [23, 28, 29], concerning one-dimensional
random walk in static random environment (i.e., constant in time), established a rich spec-
trum of asymptotic behaviors that can be very different from that of usual random walks. In
higher dimensions, important questions remain open despite much investigation. For excel-
lent expositions on this topic, see [15, 31]. The *dynamic* version of the model, that is, when
the random environment is allowed to evolve in time, has also been studied for over three
decades (see, e.g., [13, 24]). However, models with both space and time correlations have
only been considered relatively recently. For an overview, we refer to the PhD theses [1, 19].
We will abbreviate "RWRE" for random walk in static random environment, and "RWDRE"
for random walk in dynamic random environment.

The setup of the present paper fits in the context of RWDRE on one-dimensional inter-
acting particle systems, as introduced in [4, 5]. One motivation for studying RWDRE in one
dimension comes from the static counterpart which is known to exhibit, in some regimes,
anomalous behavior such as transience with zero speed [29] and nondiffusive scalings [23,
28], in sharp contrast to usual homogeneous random walks. These phenomena are related to
*trapping effects*, whereby regions of the lattice with atypical environment configurations tend
to hold the random walker for abnormally large times. Since in the dynamic case the trapping
regions may disappear as time passes by, the question of whether these phenomena still take
place is naturally raised. This question is up to now only partially answered in the literature,
mostly by identifying regimes with no anomalous behavior. For example, [3, 5, 11, 16, 27]
identify general conditions and [2, 6, 17, 22, 25] study particular examples where laws of
large numbers and central limit theorems hold.

Let us introduce the environment on which we will define our random walker. It will be
a Markov chain in $(\mathbb{Z}_+)^{\mathbb{Z}}$, where $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Fix $\rho > 0$ and let $(N(x,0))_{x \in \mathbb{Z}}$ be an i.i.d.

collection of Poisson($\rho$) random variables. We take this as the initial state of the chain, that is, for every $x \in \mathbb{Z}$, we regard $N(x, 0)$ as an initial amount of particles placed at the site $x$. We next define the evolution of the chain. Although this could be done by writing down its transition kernel, let us give a more descriptive construction. Assume that every particle present at time 0 performs, independently, a simple symmetric random walk in discrete time (possibly lazy) and define $N(x, t)$ as being the number of particles present at the space-time point $(x, t)$.

More precisely, we associate an independent random walk trajectory to each particle in the initial configuration. For that, let $(S^{z,i})_{z \in \mathbb{Z}, i \in \mathbb{N}}$ be a collection of independent discrete-time simple symmetric random walks $S^{z,i} = (S_t^{z,i})_{t \in \mathbb{Z}_+}$ on $\mathbb{Z}$ which are independent of $(N(x, 0))_{x \in \mathbb{Z}}$. We assume that $S^{z,i}$ starts at $z$ almost surely, so that the random walks $(S_t^{z,i} - z)_{t \in \mathbb{Z}_+}$ take steps in $\{-1, 0, 1\}$, are centered, independent and identically distributed. For $i \leq N(z, 0)$, $S^{z,i}$ represents the trajectory of the $i$th particle that started at $z$. Now, for each $t \in \mathbb{N}$, we define $N(x, t) := \sum_{z \in \mathbb{Z}, i \leq N(z,0)} \mathbb{1}_{\{S_t^{z,i} = x\}}$.

It is standard to show that $\{N(x, t) : x \in \mathbb{Z}\}_{t \in \mathbb{Z}_+}$ indeed defines an homogeneous, discrete-time, Markov chain. The assumption that $(N(x, 0))_{x \in \mathbb{Z}}$ are i.i.d. Poisson($\rho$) random variables implies that this chain is stationary, that is, for each $t \in \mathbb{N}$, $(N(x, t))_{x \in \mathbb{Z}}$ are also independent Poisson($\rho$) random variables. We are going to regard the parameter $\rho > 0$ as the density of the environment. In Section 2, we revisit the construction of this chain and show how it can be seen as a Poisson point process in the space of doubly-infinite random walk trajectories in $\mathbb{Z}$. This will be useful for exploring independence of disjoint set of trajectories.

To define the random walker $X = (X_t)_{t \in \mathbb{Z}_+}$, fix $p_\circ, p_\bullet \in [0, 1]$. For a given realization of $N = (N(x, t))_{x \in \mathbb{Z}, t \in \mathbb{Z}_+}$, $X$ is defined as the time-inhomogeneous Markov chain on $\mathbb{Z}$ that starts at 0 and, when it reaches position $x$ at time $t$, jumps to $x + 1$ with probability

$$(1.1) \qquad p_\circ \quad \text{if } N(x, t) = 0, \quad \text{or} \quad p_\bullet \quad \text{if } N(x, t) \geq 1,$$

and jumps to $x - 1$ otherwise. The parameters $p_\circ, p_\bullet \in [0, 1]$ thus represent the chance for the random walker to jump to the right in the absence (respectively, presence) of particles. We revisit the definition of the random walker $X$ in Section 2, where a convenient graphical construction is given.

It will be also convenient to define the local drifts

$$(1.2) \qquad v_\circ := 2p_\circ - 1, \qquad v_\bullet := 2p_\bullet - 1.$$

The case $v_\circ v_\bullet > 0$ is called *nonnestling* and has already been treated in [21]. Here, we will focus on the *nestling* case

$$(1.3) \qquad v_\bullet \leq 0 < v_\circ,$$

meaning that the random walker experiences a local drift to the right on empty sites, and no drift to the right on sites occupied by particles.

An important parameter in our analysis will be

$$(1.4) \qquad q_0 := P(S_1^{0,1} = 0) \in [0, 1).$$

When $q_0 > 0$ we say that the random walks $S^{z,i}$ are *lazy*. (Here, $P$ stands for the probability measure on a space supporting the random elements $(S^{z,i})_{z \in \mathbb{Z}, i \in \mathbb{N}}$.) Surprisingly, the asymptotic behavior of the random walker may strongly depend on whether $q_0 = 0$ or $q_0 > 0$. Indeed, for small values of $p_\bullet$, the random walker may develop a positive speed if $q_0 > 0$ and a negative one if $q_0 = 0$. This is related to a notion of *permeability*: if $p_\bullet = q_0 = 0$, the random walker cannot cross any particles that it meets to the right, and we say that the system is *impermeable* to the random walker. If either $p_\bullet$ or $q_0$ are positive, it is possible for the walker to cross particles in both directions, and we call the system *permeable*.

Let $\mathbb{P}^\rho$ denote the joint law of $N$ and $X$ for a fixed density $\rho > 0$. In order to describe our results, we introduce the following condition:

DEFINITION 1.1 (Ballisticity condition).   Fixed $\rho$, $p_\circ$, $p_\bullet$, $q_0$ and given $v_\star \neq 0$, we say that the *ballisticity condition with speed* $v_\star$ is satisfied if there exist $\gamma > 1$ and $c_1, c_2 \in (0, \infty)$ such that

$$(1.5) \qquad \mathbb{P}^\rho\left(\exists n \in \mathbb{N}: \frac{v_\star}{|v_\star|} X_n < |v_\star| n - L\right) \leq c_1 \exp\{-c_2 (\log L)^\gamma\} \quad \forall L \in \mathbb{N}.$$

Condition (1.5) is reminiscent of ballisticity conditions from the literature of random walks in static random environments such as Sznitman's $(T')$ condition (cf. [30]). Such a condition provides control on the backtracking probability of the random walker that can be very useful in obtaining finer asymptotic results, see for example, Theorem 1.4 below.

Note that, if $\rho = 0$ (i.e., if no particles are present), the random walker has a global drift $v_\circ$, which is positive under (1.3). Our first result states that, in the permeable case, perturbations around $\rho = 0$ still lead to ballisticity with a positive speed $v_\star$.

THEOREM 1.2.   *Assume* (1.3) *and* $p_\bullet \vee q_0 > 0$. *Then there exist* $\rho_\star = \rho_\star(p_\circ, p_\bullet, q_0) > 0$ *and* $v_\star = v_\star(p_\circ, p_\bullet, q_0) > 0$ *such that, for any* $\rho \leq \rho_\star$, (1.5) *holds with* $\gamma = 3/2$.

Our second ballisticity result shows a radically distinct behavior for perturbations around $p_\bullet = 0$ in the impermeable case.

THEOREM 1.3.   *Assume* $q_0 = 0$. *For any* $p_\circ \in [0, 1]$, $\rho > 0$ *and* $\gamma \in (1, 3/2)$, *there exist* $v_\star = v_\star(\rho) < 0$ *and* $p_\star = p_\star(p_\circ, \rho, \gamma) \in (0, 1)$ *such that, if* $p_\bullet \leq p_\star$, *then* (1.5) *holds.*

Theorem 1.3 may be seen as a manifestation of particle conservation in our dynamic random environment. Indeed, when $q_0 = 0$, this conservation forces the random walker to interact with environment particles that it crosses; see Section 4.2. The difference in the ballistic behavior of the two cases is illustrated by the phase diagrams in Figure 1.

As already mentioned, the ballisticity condition (1.5) can be used to study further asymptotic properties of the random walker. The following theorem summarizes new results as well as previous results from [21].

THEOREM 1.4.   *Fix* $0 \leq p_\bullet < p_\circ \leq 1$, $\rho \geq 0$, $q_0 \in [0, 1)$ *and assume that* (1.5) *holds for some* $v_\star \neq 0$ *and some* $\gamma > 1$. *Assume additionally that*

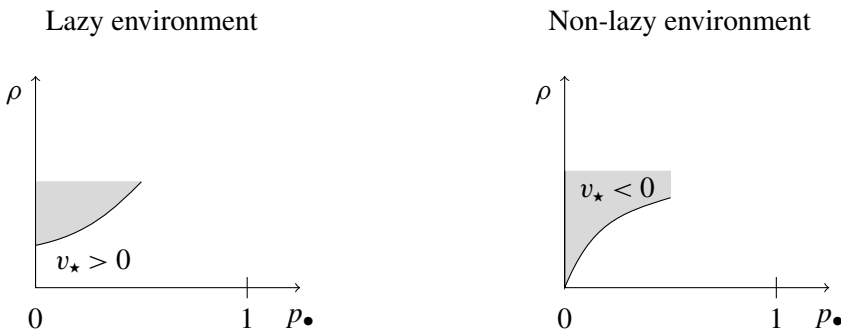$$(1.6) \qquad\qquad (a) \quad p_\bullet > 0 \quad if \ v_\star > 0$$



FIG. 1.   *Phase diagrams corresponding to lazy and nonlazy particles.*

*or*

(1.7) $\qquad\qquad\qquad\qquad (b) \quad p_\circ < 1 \quad \text{if } v_\star < 0.$

*Then there exist* $v = v(p_\circ, p_\bullet, q_0, \rho) \in \mathbb{R}$ *and* $\sigma = \sigma(p_\circ, p_\bullet, q_0, \rho) \in (0, \infty)$ *satisfying* $vv_\star > 0$, $|v| \geq |v_\star|$ *and such that the following hold*:

1. (*Strong law of large numbers*)

(1.8) $$\lim_{n \to \infty} \frac{X_n}{n} = v \quad \mathbb{P}^\rho\text{-}a.s.$$

2. (*Functional central limit theorem*) *Under* $\mathbb{P}^\rho$, *the sequence of processes*

(1.9) $$\left( \frac{X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v}{\sigma \sqrt{n}} \right)_{t \geq 0}, \quad n \in \mathbb{N},$$

*converges in distribution as* $n \to \infty$ (*with respect to the Skorohod topology*) *to a standard Brownian motion.*

3. (*Large deviation bounds*) *For any* $\varepsilon > 0$, *there exist constants* $c_1, c_2 > 0$ *such that*

(1.10) $$\mathbb{P}^\rho \left( \left| \frac{X_n}{n} - v \right| > \varepsilon \right) \leq c_1 e^{-c_2 (\log n)^\gamma} \quad \forall n \in \mathbb{N}.$$

Asymptotic results for RWDRE such as law of large numbers (LLN) and central limit theorems (CLT) under general conditions were derived for example, in [4, 5, 14, 16, 26, 27], often requiring *uniform mixing* conditions on the random environment (implying, e.g., that the conditional distribution of the environment at the origin given the initial state approaches a fixed law for large times uniformly). This uniformity can be relaxed in particular examples, for example, [9, 17, 25] (supercritical contact process), or under additional assumptions, for example, [2, 3] (spectral gap, weakly noninvariant) and [10] (attractivity). In [11] by exploiting some properties of the environment as seen by the random walker the authors are able to prove a LLN for a class of RWDRE imposing mixing conditions that do not require uniformity. There, a CLT is also proved under the assumption of uniform mixing.

The random environment we consider here does not fit the setup considered in the papers mentioned above. Indeed, being a conservative particle system, it exhibits poor mixing properties which complicates the usage of most of the available general techniques. For this reason, random walks on such conservative particle systems are challenging models. They have been studied in [6–8, 20, 22] (simple symmetric exclusion), and in [18, 21] (independent random walks). Each of these works imposes additional conditions and explores very specific properties of the environment in question. In particular, the works [18, 21, 22] introduce *perturbative* approaches, where parameters of the system are driven to a limiting value where the behavior is known. Although in this paper we also study perturbative regimes, it is important to observe that we not always recover the limiting behavior for small densities as it is shown by Theorem 1.3. This indicates that nontrivial adaptations of the techniques are needed.

At this point, a few remarks are in order:

1. Note that the assumption $p_\circ > p_\bullet$ in Theorem 1.4 imposes no loss of generality, by possibly reflecting the system from left to right. The conditions on $p_\circ, p_\bullet$ in items (a) and (b) can be seen as *ellipticity assumptions*, as they allow the random walk to take jumps in the direction of $v_\star$ independently of the environment.

2. Under the conditions of Theorem 1.2 and Theorem 1.4 in case (a), it is possible to show that the speed $v$ in (1.8) above is a continuous function of $\rho$ in the interval $[0, \rho_\star]$, see Remark 5.3 below. In particular, for fixed $p_\bullet > 0$, $v$ converges to $v_\circ$ as $\rho \to 0$. When $p_\bullet = 0$, we also expect that $v_\star$ in Theorem 1.2 may be taken arbitrarily close to $v_\circ$ by making $\rho_\star$ sufficiently small, but we are currently unable to prove this.

3. In [21] it has been proved that, when the environment has large particle density and $v_\bullet \neq 0$, the random walker obeys a LLN with a speed that has the same sign as $v_\bullet$. This could suggest that, for small densities, the speed should always have the same sign as that of $v_\circ$. However, this does not hold for perturbations around the impermeable case, as seen in Theorem 1.3. We can regard this fact as a discontinuity of the speed with respect to the density of particles as it approaches zero. This contrasts with the usual perturbative results where the behavior of the system is expected to mimic the behavior obtained at the limiting case. In our model, this feature is a product of the very specific microscopic dynamics of the environment and the random walk in the nonlazy, nonelliptic case, and is not stable under simple modifications (e.g., in continuous time). However, we still consider it an interesting cautionary example, as similar effects could arise in other situations. Moreover, we believe that, to some extent, it also reflects the slow mixing of the random environment. More precisely, we believe that no such discontinuities would be possible when the environment mixes sufficiently fast and uniformly (with respect to both initial configuration and density). This remains an interesting problem for future research.

4. A crossover from positive to negative speed of a RWDRE is also obtained in [22], where the random environment is a simple symmetric exclusion process. The transition is observed when varying the jump rate of the exclusion particles. We also mention [2], where very interesting symmetry properties of the speed are obtained (in particular for the case where the environment is given by the East model).

5. Theorems 1.2 and 1.3 are proved with the help of a renormalization scheme taken from [12]. However, the application of this scheme here is much more involved than in the high-density regime considered in [12, 21]. The extra difficulty comes from the fact that the mixing rate of the environment becomes worse as the density of particles is decreased, which is one of the reasons why the proof presented in [21] would not work. Furthermore, the discontinuity we observe as $\rho$ approaches zero in the case $q_0 = 0$ is an indication that the low-density regime is indeed more delicate.

6. Theorem 1.4 is proved via a regeneration argument as in [21]. Under b), the conclusion already follows from [21], Theorem 1.4, (and reflection symmetry); in this case, the ellipticity condition $p_\circ < 1$ can be in fact dropped using techniques from the proof of [12], Theorem 5.2. The proof of the theorem under (a) will be given in Section 5 below. The control of the regeneration time is here different, as the asymmetry in law of occupied/empty sites in the random environment leads to different monotonicity properties once the roles of $p_\circ$ and $p_\bullet$ are exchanged (cf. Section 5.1). We are presently unable to extend this analysis to the nonelliptic case, that is, when $p_\bullet = 0$.

7. It is important to comment on the flexibility of our techniques with respect to changes in the model. First of all, it would have been straightforward, if more technical, to extend our results to more general transition kernels for the underlying random environment; for simplicity, we do not pursue this here. On the other hand, extensions to $d \geq 2$ or different transition kernels for the random walker would require new ideas. The main source of difficulty in these cases is the lack of monotonicity properties that are used crucially in the present paper, specially in the proofs of Theorem 5.2 (control of the regeneration time) and Theorem 3.1 (triggering). Unfortunately, the technique used in [12] would no longer work, the reason being the asymmetry between particles and holes in the random environment. Despite of this, we believe that such extensions are not beyond the reach of current techniques, and that our contributions in the present paper could be helpful in future efforts to answer them. For example, in the case of 2-state transition kernels, we believe that the approach in [10] could be made to work, however several technical steps would need to be adapted.

The rest of the paper is organized as follows. Technical statements start in Section 2, where we provide a convenient construction of our model. Theorems 1.2–1.3 are proved in Section 3

by application of a renormalization setup from [12]; the proof relies on two triggering theorems that are in turn proved in Section 4. Finally, in Section 5 we prove Theorem 1.4 by means of a regeneration argument. Appendix contains the results from [12] that are used in Section 3.

**2. Construction.** In this section, we provide a convenient construction of our random environment and our random walker by means of a point process of trajectories as in [21].

Define the set of doubly-infinite trajectories

$$(2.1) \qquad W = \{w : \mathbb{Z} \to \mathbb{Z} : |w(i+1) - w(i)| \leq 1 \ \forall i \in \mathbb{Z}\}.$$

Note that trajectories in $W$ are allowed to jump to the left, jump to the right, or stay put. We endow the set $W$ with the $\sigma$-algebra $\mathcal{W}$ generated by the canonical coordinates $w \mapsto w(i)$, $i \in \mathbb{Z}$.

Let $(S^{z,i})_{z \in \mathbb{Z}, i \in \mathbb{N}}$ be a collection of independent random elements of $W$, with each $S^{z,i} = (S^{z,i}_\ell)_{\ell \in \mathbb{Z}}$ distributed as a double-sided simple symmetric random walk on $\mathbb{Z}$ started at $z$, that is, the past $(S^{z,i}_{-\ell})_{\ell \geq 0}$ and future $(S^{z,i}_\ell)_{\ell \geq 0}$ are i.i.d. and distributed as a simple symmetric random walk satisfying (1.4).

For a subset $K \subset \mathbb{Z}^2$, denote by $W_K$ the set of trajectories in $W$ that intersect $K$, that is, $W_K := \{w \in W : \exists i \in \mathbb{Z}, (w(i), i) \in K\}$. We define the space of point measures

$$(2.2) \qquad \Omega = \left\{\omega = \sum_i \delta_{w_i}; w_i \in W \text{ and } \omega(W_{\{y\}}) < \infty \text{ for every } y \in \mathbb{Z}^2\right\},$$

endowed with the $\sigma$-algebra generated by the evaluation maps $\omega \mapsto \omega(W_K)$, $K \subset \mathbb{Z}^2$.

For a fixed initial configuration $\eta = (\eta(x))_{x \in \mathbb{Z}} \in \mathbb{Z}_+^{\mathbb{Z}}$, we define the random element

$$(2.3) \qquad \omega := \sum_{z \in \mathbb{Z}} \sum_{i \leq \eta(z)} \delta_{S^{z,i}} \in \Omega$$

and, for $y \in \mathbb{Z}^2$, we set

$$(2.4) \qquad N(y) := \omega(W_{\{y\}})$$

and let

$$(2.5) \qquad U = (U_y)_{y \in \mathbb{Z}^2} \quad \text{be i.i.d. Uniform}[0, 1] \text{ random variables}$$

independent of $\omega$.

We define the space-time processes $Y^y = (Y^y_n)_{n \in \mathbb{Z}_+}$, $y \in \mathbb{Z}^2$ by setting

$$Y^y_0 = y,$$

$$(2.6) \qquad Y^y_{n+1} = Y^y_n + \begin{cases} (2\mathbb{1}_{\{U_{Y^y_n} \leq p_\circ\}} - 1, 1) & \text{if } N(Y^y_n) = 0, \\ (2\mathbb{1}_{\{U_{Y^y_n} \leq p_\bullet\}} - 1, 1) & \text{if } N(Y^y_n) \geq 1, \end{cases} \quad n \in \mathbb{Z}_+.$$

For $y = (x, t) \in \mathbb{Z}^2$, we define the random walkers $X^y = (X^y_n)_{n \in \mathbb{Z}_+}$ by the relation $Y^y_n = (X^y_n, n + t)$, that is, $X^y_n$ is the spatial projection of $Y^y_n$. Writing $X = X^0$, one may check that the pair $(N, X)$ has indeed the distribution described in Section 1.

For $\eta \in \mathbb{Z}_+^{\mathbb{Z}}$ fixed, we denote by $\mathbb{P}_\eta$ the joint law of $\omega$ and $U = (U_y)_{y \in \mathbb{Z}^2}$. For $\rho > 0$, denote by $\nu_\rho$ the product Poisson($\rho$) law on $\mathbb{Z}_+^{\mathbb{Z}}$. We write $\mathbb{P}^\rho = \int \mathbb{P}_\eta \nu_\rho(d\eta)$, that is, $\mathbb{P}^\rho$ is the joint law of $\omega$ and $U$ when $\eta$ is distributed as $\nu_\rho$. Our configuration space will be taken as

$$(2.7) \qquad \overline{\Omega} := \Omega \times [0, 1]^{\mathbb{Z}^2},$$

equipped with the product $\sigma$-algebra.

An important observation is that, under $\mathbb{P}^\rho$, $\omega$ is a Poisson point process on $\Omega$ with intensity measure $\rho\mu$, where

$$(2.8) \qquad \mu = \sum_{z \in \mathbb{Z}} P_z$$

and $P_z$ is the law of $S^{z,1}$ as an element of $W$.

For $y = (x, n) \in \mathbb{Z}^2$ and $w \in W$, define the space-time translation $\theta_y w$ as

$$(2.9) \qquad \theta_y w(i) := w(i - n) + x, \quad i \in \mathbb{Z}.$$

The translations of a measurable function $g : \overline{\Omega} \to E$ are then defined by setting

$$(2.10) \qquad g_y = \theta_y g := g \circ \theta_y.$$

Note that, under $\mathbb{P}^\rho$, the law of $(\omega, U)$ is invariant with respect to space-time translations; in particular, the law of $Y^y - y$ does not depend on $y$.

We will also need the following definition.

DEFINITION 2.1. For $\omega, \omega' \in \Omega$, we say that $\omega \le \omega'$ when $\omega(A) \le \omega'(A)$ for all $A \in \mathcal{W}$. We say that a random variable $f : \overline{\Omega} \to \mathbb{R}$ is *nondecreasing* when $f(\omega, \xi) \le f(\omega', \xi)$ for all $\omega \le \omega'$ and all $\xi \in [0, 1]^{\mathbb{Z}^2}$. We say that $f$ is *nonincreasing* if $-f$ is nondecreasing. We extend these definitions to events $A$ in $\sigma(\omega, U)$ by considering $f = \mathbb{1}_A$. Standard coupling arguments imply that $\mathbb{E}^\rho(f) \le \mathbb{E}^{\rho'}(f)$ for all nonincreasing random variables $f$ and all $\rho \le \rho'$.

REMARK 2.2. The above construction provides two forms of monotonicity:

(i) Initial position: If $x \le x'$ have the same parity (i.e., $x' - x \in 2\mathbb{Z}$), then

$$(2.11) \qquad X_i^{(x,n)} \le X_i^{(x',n)} \quad \forall n \in \mathbb{Z} \; \forall i \in \mathbb{Z}_+.$$

(ii) Environment: If $v_\circ \ge v_\bullet$, then $X_n^y$ is nonincreasing (in the sense of Definition 2.1) for any $y \in \mathbb{Z}^2$, $n \in \mathbb{Z}_+$.

**3. Renormalization: Proof of Theorems 1.2–1.3.** In this section, we apply the renormalization setup from Section 3 of [12] to reduce the proof of our main results to two *triggering statements*, Theorems 3.1 and 3.2. The relevant results from [12] that we use here are stated in the Appendix.

THEOREM 3.1. *Assume* $p_\bullet \vee q_0 > 0$. *There exists* $c = c(p_\circ, p_\bullet, q_0) > 0$ *such that*

$$(3.1) \qquad \mathbb{P}^{L^{-1/16}}(X_L < L^{15/16}) \le c \exp\{-c^{-1}(\log L)^2\} \quad \forall L \in \mathbb{N}.$$

THEOREM 3.2. *Assume* $q_0 = 0$. *For any* $\hat{\rho} > 0$, *there exist* $\hat{v} = \hat{v}(\hat{\rho}) < 0$ *and* $c > 0$ *such that the following holds. For any* $\hat{L} \in \mathbb{N}$, *there exists* $p_\star = p_\star(\hat{\rho}, p_\circ, \hat{L}) \in (0, 1)$ *such that, if* $p_\bullet \le p_\star$, *then*

$$(3.2) \qquad \mathbb{P}^{\hat{\rho}}(X_{\hat{L}} > \hat{v}\hat{L}) \le c \exp\{-c^{-1}(\log \hat{L})^{3/2}\}.$$

The proof of Theorems 3.1–3.2 will be given in Section 4. Next we use Corollary A.1 from the Appendix (which corresponds to [12], Corollary 3.11) to show how these two theorems respectively imply Theorems 1.2 and 1.3.

Before, we recall some definitions taken from [12] that will be useful during the proofs.

DEFINITION 3.3. A function $\Gamma : \mathbb{Z}_+ \to \mathbb{Z}$ is said 1-Lipschitz if $|\Gamma(t+1) - \Gamma(t)| \leq 1$ for every $t \in \mathbb{Z}_+$. Let $H : \overline{\Omega} \times \mathbb{Z} \to \{0, 1\}$ and $L \in \mathbb{N}$. We say that a 1-Lipschitz function $\Gamma : \mathbb{Z}_+ \to \mathbb{Z}$ is an $(L, H)$-crossing if $\Gamma(0) \in [0, L) \cap \mathbb{Z}$ and for every $t \in [0, L) \cap \mathbb{Z}$, $(\omega, U) \in \overline{\Omega}$,

$$H\big(\theta_{(\Gamma(t), t)}(\omega, U), \Gamma(t+1) - \Gamma(t)\big) = 1.$$

DEFINITION 3.4. Let $H : \overline{\Omega} \times \mathbb{Z} \to \{0, 1\}$ and $L \in \mathbb{N}$. Given $g : \overline{\Omega} \to [-1, 1]$ and a $(L, H)$-crossing $\Gamma : \mathbb{Z}_+ \to \mathbb{Z}$, we define the average $\chi_\Gamma^g$ of $g$ along $\Gamma$ by

$$(3.3) \qquad \chi_\Gamma^g(\omega, U) := \frac{1}{L} \sum_{i=0}^{L} g_{(\Gamma(i), i)}(\omega, U),$$

where $g_y = \theta_y g$ as in (2.10).

PROOF OF THEOREM 1.2. Define a local function $g : \overline{\Omega} \to \{-1, 1\}$ by setting

$$(3.4) \qquad g(\omega, U) = \begin{cases} 1 & \text{if } U_0 < p_\bullet, \text{ or if } N(0) = 0 \text{ and } U_0 < p_\circ, \\ -1 & \text{if } U_0 \geq p_\circ \text{ or if } N(0) > 0 \text{ and } U_0 \geq p_\bullet, \end{cases}$$

that is, the function $g$ returns the first step of the random walker $X^0$ for a given realization of $\omega, U$. Then we define a function $H : \overline{\Omega} \times \mathbb{Z} \to \{0, 1\}$ by

$$(3.5) \qquad H\big((\omega, U), z\big) = 1_{\{g(\omega, U) = z\}}.$$

In words, $H$ decides whether a jump $z$ is correct ($H = 1$) or not ($H = 0$) for a given realization of $\omega, U$ according to whether the actual random walk $X^0$ would take $z$ as its first jump or not. Note that

$$(3.6) \qquad \begin{aligned} &\Gamma : [0, \infty) \cap \mathbb{Z} \to \mathbb{Z} \quad \text{is a } (L, H)\text{-crossing if and only if} \\ &\Gamma_t = X_t^y \text{ for every } t \in [0, L) \cap \mathbb{Z} \text{ and some} \\ &y \in \{0, \dots, L-1\} \times \{0\}, \end{aligned}$$

that is, the $(L, H)$-crossings are the trajectories of the RWDRE with initial position in $\{0, \dots, L-1\}$. Also note the following correspondence between events: for any $L \in \mathbb{N}$, $\hat{v} > 0$,

$$(3.7) \qquad \begin{aligned} &\{\exists (L, H)\text{-crossing } \Gamma : \chi_\Gamma^g \leq \hat{v}\} \\ &\quad = \{\exists x \in \{0, \dots, L-1\} : X_L^{(x, 0)} - x \leq \hat{v} L\}. \end{aligned}$$

Since, for $v_\star \in (0, 1)$,

$$(3.8) \qquad \mathbb{P}^{L^{-1/16}}(\exists n \geq 1 : X_n^0 < v_\star n - L) \leq \mathbb{P}^{L^{-1/16}}(\exists n \geq L : X_n^0 \leq v_\star n),$$

we only need to bound the right-hand side for some $v_\star \in (0, 1)$. Now, by (3.7), translation invariance and Theorem 3.1, for all $\hat{L}$ large enough,

$$(3.9) \qquad \begin{aligned} &\mathbb{P}^{\hat{L}^{-1/16}}(\exists \text{ a } (\hat{L}, H)\text{-crossing } \Gamma \text{ with } \chi_\Gamma^g \leq \hat{L}^{-1/16}) \\ &\leq \hat{L} \mathbb{P}^{\hat{L}^{-1/16}}(X_{\hat{L}}^0 \leq \hat{L}^{15/16}) \\ &\overset{\text{Theorem 3.1}}{\leq} c\hat{L} \exp\{-c^{-1}(\log \hat{L})^2\} < \exp(-(\log \hat{L})^{3/2}). \end{aligned}$$

Noting that the events in (3.7) are measurable in $\sigma(N(y), U_y : y \in B_{0,L})$ (where $B_{0,L} := ([-2L, 3L) \times [0, L)) \cap \mathbb{Z}^2$), and are nondecreasing by (1.3), we verify the assumptions of

Corollary A.1 (taking $v(L) = \rho(L) = L^{-15/16}$, and $\hat{L} = L_{\hat{k}}$ for some $\hat{k}$ large enough), obtaining $v_\star \in (0,1)$, $\rho_\star > 0$ and $c > 0$ such that, for all $\rho \leq \rho_\star$,

$$\mathbb{P}^\rho(X_n^0 \leq v_\star n) \leq \mathbb{P}^\rho(\exists \text{ a } (n, H)\text{-crossing } \Gamma \text{ with } \chi_\Gamma^g \leq v_\star)$$
(3.10)
$$\leq c^{-1} \exp(-c(\log n)^{3/2})$$

for all $n \in \mathbb{Z}_+$. To conclude, sum over $n \geq L$ and apply the union bound to (3.8). $\square$

PROOF OF THEOREM 1.3.    This time, we define $g : \overline{\Omega} \to \{-1, 1\}$ as

$$(3.11) \qquad g(\omega, U) = \begin{cases} -1 & \text{if } U_0 < p_\bullet \wedge p_\circ, \text{ or if } \omega(W_0) = 0 \text{ and } U_0 < p_\circ, \\ 1 & \text{otherwise.} \end{cases}$$

For $y \in \mathbb{Z}^2$, define a space-time process $\widetilde{Y}_t^y$, $t \in \mathbb{Z}_+$ by setting, analogously to (2.6),

$$(3.12) \qquad \widetilde{Y}_0^y = y \quad \text{and} \quad \widetilde{Y}_{t+1}^y = \widetilde{Y}_t^y + (g(\theta_{\widetilde{Y}_t^y}(\omega, U)), 1), \quad t \in \mathbb{Z}_+.$$

Denote by $\widetilde{X}_t^y$ the first coordinate of $\widetilde{Y}_t^y$. Note that, by invariance in law of $\omega$ under reflection through the origin, $\widetilde{X}^y$ has the same distribution as $-X^y$. Setting $H : \overline{\Omega} \times \mathbb{Z} \to \{0, 1\}$ as in (3.5), we analogously obtain (3.6)–(3.7) with $X$ substituted by $\widetilde{X}$.

Fix now $\gamma \in (1, 3/2)$ and take $k_o$ as in Corollary A.1. Fix $\rho > 0$ and consider an auxiliary density $\hat{\rho} > 0$, to be fixed later. For this $\hat{\rho}$, let $\hat{v} < 0$ as in Theorem 3.2; we may assume that $|\hat{v}| < 1$. Fix $\hat{k} \geq k_o$, $p_\circ \in [0, 1]$ and let $p_\star$ be as in Theorem 3.2 for $\hat{L} = L_{\hat{k}}$. Reasoning as in the proof of Theorem 1.2, we see that, if $p_\bullet \leq p_\star$, then

$$\mathbb{P}^{\hat{\rho}}(\exists \text{ a } (\hat{L}, H)\text{-crossing } \Gamma \text{ with } \chi_\Gamma^g \leq |\hat{v}|) \leq \hat{L}\mathbb{P}^{\hat{\rho}}(X_{\hat{L}}^0 \geq \hat{L}\hat{v})$$
(3.13)
$$\overset{\text{Theorem 3.2}}{\leq} c\hat{L} \exp\{-c^{-1}(\log \hat{L})^{3/2}\} < \exp(-(\log \hat{L})^\gamma)$$

whenever $\hat{k}$ (and thus $\hat{L}$) is large enough. The events in (3.7) (with $X$ replaced by $\widetilde{X}$) are again measurable in $\sigma(N(y), U_y : y \in B_{0,L})$, and are either always nondecreasing, or always nonincreasing (depending on whether $p_\circ \geq p_\bullet$ or not). Applying Corollary A.1 (with $v(L) = |\hat{v}|$, $\rho(L) = \hat{\rho}$) we obtain $\rho_\infty, c > 0$ and $v_\star < 0$ depending on $\hat{\rho}$ such that

$$(3.14) \qquad \mathbb{P}^{\rho_\infty}(X_n^0 \geq v_\star n) \leq c^{-1} \exp(-c(\log n)^\gamma)$$

for all $n \in \mathbb{Z}_+$. Now we note that, using the explicit expression for $\rho_\infty$ (which is mentioned in the proof of [12], Corollary 3.11, see Remark A.2 in the Appendix), we may choose $\hat{\rho}$ in such a way that (3.14) is still valid with $\rho$ in place of $\rho_\infty$. To conclude, sum (3.14) over $n \geq L/2$ and use $\{\exists n \geq 1 : X_n^0 > v_\star n + L\} \subset \{\exists n \geq L/2 : X_n^0 \geq v_\star n\}$ together with a union bound. $\square$

**4. Triggering: Proof of Theorems 3.1–3.2.**    Here we give the proofs of Theorem 3.1 (Section 4.1) and Theorem 3.2 (Section 4.2).

4.1. *Permeable systems at low density.*    Throughout this section, we assume $p_\bullet \vee q_0 > 0$ (and $v_\circ > 0 \geq v_\bullet$). As mentioned in the Introduction, we call this case permeable since the random walker is able to cross over particles of the environment. The usefulness of this condition comes from the fact that $X$ may be coupled with an independent homogeneous random walk $\bar{X}$ with drift $v_\circ$ (which we call a "ghost walker") such that, whenever the initial configuration $\eta$ consists of at most one particle that is not at the origin, there is a positive probability that $X_n = \bar{X}_n$ for all $n \in \mathbb{Z}_+$. In fact, we will show that this probability decays at most exponentially in the number of particles of the environment. This suggests the following

strategy: whenever a "ghost walker" is started to the left of $X$, it can "push" $X$ to the right. This may happen with small probability but, if enough time is given, many trials are possible and so there is a large probability that at least one of them succeeds.

In order to implement this idea, we work first in a time scale at which typical empty regions in the initial configuration remain empty, and the number of particles between such regions is relatively small. This ensures that $X$ does not move very far to the left, and that the "ghost walkers" do not meet too many particles on their way. The original scale is then reached via translation-invariance and a union bound.

We proceed to formalize the strategy outlined above. In the following, we state two propositions which will then be used to prove Theorem 3.1. Their proofs are postponed to Sections 4.1.1–4.1.2 below.

First of all we define the ghost walkers. For $(x, t) \in \mathbb{Z}^2$, put

$$\bar{X}_0^{(x,t)} := x,$$

(4.1)
$$\bar{X}_{s+1}^{(x,t)} := \bar{X}_s^{(x,t)} + \begin{cases} 1 & \text{if } U_{(\bar{X}_s^{(x,t)}, s+t)} \le p_\circ, \\ -1 & \text{otherwise,} \end{cases} \quad s \in \mathbb{Z}_+.$$

Then $\bar{X}^{(x,t)}$ is a simple random walk with drift $v_\circ$ started at $x$. For $T \in [0, \infty]$, let

(4.2)
$$G_T^{(x,t)} := \{X_s^{(x,t)} = \bar{X}_s^{(x,t)} \; \forall s \in [0, T]\}$$

be the good event where the random walk $X^{(x,t)}$ follows $\bar{X}^{(x,t)}$ up to time $T$. A comparison between $X$ and $\bar{X}^{(x,t)}$ on this event is given by the next lemma.

LEMMA 4.1. *Fix $(x, t) \in \mathbb{Z}^2$ with $x \in 2\mathbb{Z}$. If $X_t \ge x$ and $G_T^{(x,t)}$ occurs, then*

$$X_{t+s} \ge \bar{X}_s^{(x,t)} \quad \text{for all } s \in [0, T].$$

PROOF. Follows from Remark 2.2(i) and the definitions of $X$, $\bar{X}$, $G_T^{(x,t)}$. □

To set up the scales for our proof, we fix $\alpha, \beta, \beta' \in (0, 1)$ satisfying

(4.3)
$$0 < \frac{\alpha}{2} < \beta' < \beta < \alpha < 2\beta < \frac{1}{8}$$

and we let

(4.4)
$$T_i := i2\lfloor 2v_\circ^{-1} L^\beta \rfloor, \quad i \in [0, M_L] \cap \mathbb{Z} \text{ where } M_L := \frac{1}{4} v_\circ L^{\alpha - \beta},$$

(4.5)
$$\ell_L := \lfloor L^{\beta'} \rfloor.$$

We assume that $L$ is large enough so that $\ell_L, M_L \ge 1$.

Let us comment on the conditions in (4.3). The goal will be to drive our random walker a distance of order $L^\beta$ to the right in a time $L^\alpha$. To do so, we will launch $M_L$ "ghost walkers" for time intervals of length $L^\beta$ (so $\beta < \alpha$). What we need is an empty region of size $L^{\beta'}$ within distance $L^\beta$ to the left of the origin (so $\beta' < \beta < 1/16$), which the symmetric walks of the environment will not visit for a time $L^\alpha$ (hence $\alpha/2 < \beta'$). This is what we define now.

If $p_\bullet = 0$, it is not possible to couple $X_1^{(x,t)}$ and $\bar{X}_1^{(x,t)}$ if there is a particle at $(x, t)$. Thus, if we aim to control $G_T^{(x,t)}$, we should have $N(x, t) = 0$. To that end, define

(4.6)
$$\hat{Z} := \max\{z < -2\ell_L : N(x, 0) = 0 \; \forall x \in \mathbb{Z}, |x - z| \le 2\ell_L\}$$

to be the center of the first interval of $4\ell_L + 1$ empty sites to the left of the origin in the initial configuration. Then set

(4.7)
$$X_- := \begin{cases} \hat{Z} - \ell_L & \text{if } \hat{Z} - \ell_L \in 2\mathbb{Z}, \\ \hat{Z} - \ell_L + 1 & \text{otherwise.} \end{cases}$$

Note that $X_- \in 2\mathbb{Z}$.

In order to use Lemma 4.1, we must control the probability that $X$ crosses $X_-$ before time $L^\alpha$. This is the content of the following proposition, whose proof relies on standard properties of simple random walks and Poisson random variables.

PROPOSITION 4.2.   *There exist $c, \varepsilon > 0$ such that, for all large enough $L \in \mathbb{N}$,*

(4.8)
$$\mathbb{P}^{L^{-\frac{1}{16}}}\left(\min_{0 \le s \le L^\alpha} X_s < X_-\right) \le ce^{-c^{-1}L^\varepsilon}.$$

The next proposition shows that, with large probability, one of the events $G_{T_1}^{(X_-, T_i)}$'s occurs. Its proof depends crucially on the permeability of the system.

PROPOSITION 4.3.   *There exists $c > 0$ such that, for all large enough $L \in \mathbb{N}$,*

(4.9)
$$\mathbb{P}^{L^{-\frac{1}{16}}}\left(\bigcup_{i \in [0, M_L-1]} G_{T_1}^{(X_-, T_i)} \cap \{\bar{X}_{T_1}^{(X_-, T_i)} \ge L^\beta\}\right) \ge 1 - ce^{-c^{-1}(\log L)^2}.$$

We are now ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1.   First we argue that, for some constant $c > 0$,

(4.10)
$$\mathbb{P}^{L^{-\frac{1}{16}}}\left(\sup_{0 \le s \le L^\alpha} X_s < L^\beta\right) \le ce^{-c^{-1}(\log L)^2} \quad \forall L \in \mathbb{N}.$$

Indeed, by Lemma 4.1, the complement of the event in (4.10) contains the event

$$\left\{\min_{0 \le s \le L^\alpha} X_s \ge X_-\right\} \cap \bigcup_{i \in [0, M_L-1]} G_{T_1}^{(X_-, T_i)} \cap \{\bar{X}_{T_1}^{(X_-, T_i)} \ge L^\beta\},$$

which by Propositions 4.2–4.3 has probability at least $1 - ce^{-c^{-1}(\log L)^2}$.

Now let $\tau_k$ be the sequence of random times when the increments of $X$ are at least $L^\beta$, that is, $\tau_0 := 0$ and recursively

(4.11)
$$\tau_{k+1} := \inf\{s > \tau_k : X_s - X_{\tau_k} \ge L^\beta\}, \quad k \ge 0.$$

Setting $K := \sup\{k \ge 0 : \tau_k \le L\}$, we obtain

(4.12)
$$X_L = \sum_{i=0}^{K-1} X_{\tau_{i+1}} - X_{\tau_i} + X_L - X_{\tau_K} \ge KL^\beta - (\tau_{K+1} - \tau_K).$$

On the event

(4.13)
$$B_L := \{\tau_{k+1} - \tau_k \le L^\alpha \ \forall k = 0, \ldots, K\},$$

we have $K \ge L^{1-\alpha} - 1$. Therefore, by (4.12), on $B_L$ we have

(4.14)
$$X_L \ge L^{1-\alpha+\beta} - L^\beta - L^\alpha \ge L^{\frac{15}{16}}$$

for large $L$ since $1 - \alpha + \beta > 15/16 > \alpha > \beta$. Thus we only need to control the probability of $B_L$. But, by the definition of $X$,

$$\mathbb{P}^{L^{-\frac{1}{16}}}(B_L^c) \leq \mathbb{P}^{L^{-\frac{1}{16}}}\left(\exists (x, t) \in [-L, L] \times [0, L]: \sup_{s \in [0, L^\alpha]} X_s^{(x,t)} < L^\beta\right)$$

(4.15)

$$\leq cL^2 \mathbb{P}^{L^{-\frac{1}{16}}}\left(\sup_{0 \leq s \leq L^\alpha} X_s < L^\beta\right) \leq ce^{-c^{-1}(\log L)^2},$$

where we used a union bound, translation-invariance and (4.10). This completes the proof of Theorem 3.1. $\square$

4.1.1. *Proof of Proposition* 4.2. Recall the definition of $\hat{Z}$ in (4.6). The idea behind the proof of Proposition 4.2 is that, with our choice of scales, the interval $[\hat{Z} - \ell_L, \hat{Z} + \ell_L]$ remains empty throughout the time interval $[0, L^\alpha]$. Since inside this interval $X$ behaves as a random walk with a positive drift, it avoids $X_- \leq \hat{Z} - \ell_L + 1$ with large probability.

We first show that $\hat{Z} - 2\ell_L \geq -L^\beta$ with large probability.

LEMMA 4.4.

(4.16)
$$\mathbb{P}^{L^{-\frac{1}{16}}}(\hat{Z} - 2\ell_L < -L^\beta) \leq ce^{-c^{-1}L^{\beta - \beta'}}.$$

PROOF. We may assume that $L$ is large enough. Let $E_0 := 0$ and recursively

(4.17)
$$E_{k+1} := \max\{z < E_k: N_0(z) > 0\}, \quad k \geq 0.$$

Then $(E_k - E_{k+1})_{k \geq 0}$ are i.i.d. $\text{Geom}(1 - e^{-L^{-\frac{1}{16}}})$ random variables. Let

(4.18)
$$K := \inf\{k \geq 0: |E_{k+1} - E_k| > 4\ell_L\}.$$

Then $K + 1$ has a geometric distribution with parameter $e^{-4\ell_L L^{1/16}}$. Thus

$$\mathbb{P}^{L^{-\frac{1}{16}}}\left(K + 1 > \frac{1}{4}L^{\beta - \beta'}\right) \leq \left(1 - e^{-4L^{-(1/16 - \beta')}}\right)^{\frac{1}{4}L^{\beta - \beta'}}$$

(4.19)

$$\leq 4^{\frac{1}{4}L^{\beta - \beta'}} e^{-\frac{1}{4}(1/16 - \beta')L^{\beta - \beta'} \log L}$$

$$\leq ce^{-c^{-1}L^{\beta - \beta'}}.$$

Since $|\hat{Z} - 2\ell_L| \leq 4\ell_L(K + 1)$,

(4.20) $\quad \mathbb{P}^{L^{-\frac{1}{16}}}(\hat{Z} - 2\ell_L < -L^\beta) \leq \mathbb{P}^{L^{-\frac{1}{16}}}\left(K + 1 > \frac{1}{4}L^{\beta - \beta'}\right) \leq ce^{-c^{-1}L^{\beta - \beta'}}$

by (4.19). This finishes the proof. $\square$

Next we show that, with large probability, the particles of the random environment do not penetrate deep inside the empty region up to time $L^\alpha$. Let

(4.21)
$$\mathcal{E}_L := \{N(y) = 0 \,\forall y \in [\hat{Z} - \ell_L, \hat{Z} + \ell_L] \times [0, L^\alpha]\}.$$

LEMMA 4.5. *There exists $c > 0$ such that*

(4.22)
$$\mathbb{P}^{L^{-\frac{1}{16}}}(\mathcal{E}_L^c) \leq ce^{-\frac{1}{c}L^{(\beta - \beta') \wedge (2\beta' - \alpha)}}.$$

PROOF.    For $x \in \mathbb{Z}$, the random variable

$$
(4.23) \qquad \widehat{N}_L(x) := \sum_{z \notin [x-2\ell_L, x+2\ell_L]} \sum_{i \leq N(z,0)} \mathbb{1}_{\{\exists s \in [0, L^\alpha]:\, S_s^{z,i} \in [x-\ell_L, x+\ell_L]\}}
$$

has a Poisson distribution with parameter

$$
(4.24) \qquad \lambda_L(x) := L^{-\frac{1}{16}} \sum_{z \notin [x-2\ell_L, x+2\ell_L]} P\big(\exists s \in [0, L^\alpha]:\, S_s^{z,1} \in [x-\ell_L, x+\ell_L]\big),
$$

where $S^{z,1}$ is a simple symmetric random walk started at $z$ as defined in the Introduction. By standard random walk estimates, we have

$$
(4.25) \quad \lambda_L(x) \leq 2 \sum_{k > \ell_L} P\Big(\sup_{s \in [0, L^\alpha]} |S_s^{0,1}| \geq k\Big) \leq c \sum_{k > \lfloor L^{\beta'} \rfloor} e^{-\frac{k^2}{cL^\alpha}} \leq cL^\alpha e^{-c^{-1} L^{2\beta' - \alpha}}.
$$

Therefore, by Lemma 4.4 and (4.25),

$$
(4.26) \qquad \mathbb{P}^{L^{-\frac{1}{16}}}\big(\widehat{N}_L(\hat{Z}) > 0\big)
$$

$$
\leq \mathbb{P}^{L^{-\frac{1}{16}}}\big(\hat{Z} < -L^\beta\big) + \mathbb{P}^{L^{-\frac{1}{16}}}\big(\exists x \in [-L^\beta, 0]:\, \widehat{N}_L(x) > 0\big)
$$

$$
(4.27) \qquad \leq ce^{-c^{-1} L^{(\beta - \beta')}} + cL^\beta \sup_x \lambda_L(x) \leq ce^{-\frac{1}{c} L^{(\beta - \beta') \wedge (2\beta' - \alpha)}}.
$$

Since $N(z,0) = 0$ for all $z \in [\hat{Z} - 2\ell_L, \hat{Z} + 2\ell]$ by definition, $\widehat{N}_L(\hat{Z})$ is equal to the total number of particles that enter $[\hat{Z} - \ell_L, \hat{Z} + \ell_L] \times [0, L^\alpha]$. This completes the proof.  □

Let now, for $t \in \mathbb{N}$,

$$
(4.28) \qquad
\begin{aligned}
H_+^{(t)} &:= \inf\{s \geq 0:\, X_s^{(\hat{Z}, t)} - \hat{Z} = \ell_L\}, \\
H_-^{(t)} &:= \inf\{s \geq 0:\, X_s^{(\hat{Z}, t)} - \hat{Z} = -\ell_L + 1\}
\end{aligned}
$$

be the times when the random walk $X^{(\hat{Z}, t)}$ hits the sites $\hat{Z} + \ell_L$ or $\hat{Z} - \ell_L + 1$. Let

$$
(4.29) \qquad \mathcal{D}_L := \big\{H_-^{(t)} > H_+^{(t)} \wedge (L^\alpha - t) \; \forall t \in [0, L^\alpha]\big\}.
$$

The last lemma of this section shows that also $\mathcal{D}_L$ has large probability.

LEMMA 4.6.

$$
(4.30) \qquad \mathbb{P}^{L^{-\frac{1}{16}}}\big(\mathcal{D}_L^c \mid \mathcal{E}_L\big) \leq ce^{-c^{-1} L^{\beta'}}.
$$

PROOF.    Fix $t \in [0, L^\alpha]$ and note that, on the event $\mathcal{E}_L$, $X_s^{(\hat{Z}, t)} - \hat{Z}$ is up to time $H_+^{(t)} \wedge H_-^{(t)} \wedge (L^\alpha - t)$ equal to $\bar{X}_s^{(\hat{Z}, t)} - \hat{Z}$. The latter is a random walk with drift $v_\circ > 0$, so by standard estimates we obtain

$$
(4.31) \qquad
\begin{aligned}
\mathbb{P}^{L^{-\frac{1}{16}}}\big(H_-^{(t)} \leq H_+^{(t)} \wedge (L^\alpha - t) \mid \mathcal{E}_L\big) &\leq \mathbb{P}^{L^{-\frac{1}{16}}}\Big(\inf_{s \geq 0} \bar{X}_s^{(\hat{Z}, t)} - \hat{Z} \leq -\ell_L + 1\Big) \\
&\leq ce^{-c^{-1} \ell_L} \leq ce^{-c^{-1} L^{\beta'}}.
\end{aligned}
$$

The proof is completed using (4.31) and a union bound over $t \in [0, L^\alpha]$.  □

With Lemmas 4.4–4.6 at hand we can finish the proof of Proposition 4.2.

PROOF OF PROPOSITION 4.2.    By Lemmas 4.5–4.6,

$$\mathbb{P}^{L^{-\frac{1}{16}}}(\mathcal{D}_L) \geq 1 - ce^{-c^{-1}L^{\varepsilon}},$$ (4.32)

where $\varepsilon := \beta' \wedge (\beta - \beta') \wedge (2\beta' - \alpha)$. The proof is finished by noting that, since $X$ must hit $\hat{Z}$ in order to reach $\hat{Z} - \ell_L + 1 \geq X_-$, if $\mathcal{D}_L$ occurs then $X_s \geq X_- \ \forall s \in [0, L^{\alpha}]$.    □

4.1.2. *Proof of Proposition* 4.3.   For $(x, t) \in \mathbb{Z}^2$ and $T \in [0, \infty)$, let

$$\Lambda_T^{(x,t)} := \left\{ \bar{X}_s^{(x,t)} - x \geq \frac{1}{2}v_\circ s \ \forall s \in [0, T] \right\}, \qquad \Lambda_\infty^{(x,t)} := \bigcap_{T>0} \Lambda_T^{(x,t)}.$$ (4.33)

When $(x, t) = (0, 0)$, we omit it from the notation of both $G_T^{(x,t)}$ and $\Lambda_T^{(x,t)}$.

The proof of Proposition 4.3 follows two steps that are presented in Lemmas 4.7 and 4.9. We first show a lower bound on the probability of $G_\infty \cap \Lambda_\infty$. This lower bound is provided in Lemma 4.7 and decays exponentially in the number of particles in $\eta$. Intuitively speaking this can be interpreted as if the walker had to pay a constant price to ignore each particle. Then in Lemma 4.9 we show that, if the initial configuration has a logarithmic number of particles and we are given enough attempts, the walker is very likely to ignore all of them.

For $\eta \in \mathbb{Z}_+^{\mathbb{Z}}$, denote by

$$|\eta| := \sum_{z \in \mathbb{Z}} \eta(z) \in [0, \infty]$$ (4.34)

the total number of particles in $\eta$. Note that $|N(\cdot, t)| = |\eta|$ a.s. under $\mathbb{P}_\eta$.

The first goal of the section is the following key lemma, providing a lower bound on the probability of $G_\infty \cap \Lambda_\infty$ when $|\eta| < \infty$ and $\eta(0) = 0$.

LEMMA 4.7.    *There exists $p_* \in (0, 1)$ such that*

$$\inf_{\eta: \, |\eta| \leq k, \eta(0)=0} \mathbb{P}_\eta(G_\infty \cap \Lambda_\infty) \geq p_*^k \quad \forall k \geq 0.$$ (4.35)

In order to prove Lemma 4.7, we will need an auxiliary result. For a set $B \subset \mathbb{Z}$ and two configurations $\eta, \xi \in \mathbb{Z}_+^{\mathbb{Z}}$ satisfying $\xi \leq \eta$ (i.e., $\xi(x) \leq \eta(x) \ \forall x \in \mathbb{Z}$), let

$$\eta^{B,\xi}(x) := \begin{cases} \eta(x) - \xi(x) & \text{if } x \in B, \\ \eta(x) & \text{otherwise.} \end{cases}$$ (4.36)

For $A \subset \mathbb{Z}^2$, we write $N(A) = (N(y))_{y \in A}$ and $U_A = (U_y)_{y \in A}$. The following lemma is a consequence of the i.i.d. nature of the particles in the environment.

LEMMA 4.8.    *Let $A \subset \mathbb{Z}^2$ and $B \subset \mathbb{Z}$. For any two configurations $\xi \leq \eta \in \mathbb{Z}_+^{\mathbb{Z}}$ and any measurable bounded function $f$,*

$$\mathbb{E}_\eta[f(N(A), U_A) \mid (S^{z,i})_{i \leq \xi(z), z \in B}] = \mathbb{E}_{\eta^{B,\xi}}[f(N(A), U_A)]$$

(4.37)

$$\text{a.s. on the event } \{S_n^{z,i} \cap A = \varnothing \ \forall n \in \mathbb{Z}, i \leq \xi(z), z \in B\}.$$

PROOF.    For $(x, t) \in \mathbb{Z}^2$, let

$$N^{B,\xi}(x, t) := \sum_{z \notin B} \sum_{1 \leq i \leq \eta(z)} \mathbb{1}_{\{S_t^{z,i}=x\}} + \sum_{z \in B} \sum_{\xi(z) < i \leq \eta(z)} \mathbb{1}_{\{S_t^{z,i}=x\}}.$$ (4.38)

On the event in the second line of (4.37), $f(N(A), U_A) = f(N^{B,\xi}(A), U_A)$ and the latter is independent of $(S^{(z,i)})_{i \leq \xi(z), z \in B}$. To conclude, note that $N^{B,\xi}$ has under $\mathbb{P}_\eta$ the same distribution of $N$ under $\mathbb{P}_{\eta^{B,\xi}}$. $\quad\square$

We can now give the proof of Lemma 4.7.

PROOF OF LEMMA 4.7. We start with the case $q_0 > 0$. We claim that one may assume $\eta(z) = 0$ for all $z \leq 0$. Indeed, apply Lemma 4.8 with $A = \{(x, t) \in \mathbb{Z}_+^2 : x \geq \frac{1}{2}v_0 t\}$, $B = (-\infty, -1] \cap \mathbb{Z}$ and $\xi(z) = \eta(z)\mathbb{1}_{\{z<0\}}$ to obtain

$$\mathbb{P}_\eta(G_\infty \cap \Lambda_\infty) \geq P(S_n^{0,1} - 1 \notin A \,\forall n \in \mathbb{Z}_+)^{|\xi|} \mathbb{P}_{\eta^{B,\xi}}(G_\infty \cap \Lambda_\infty),$$

where $\eta^{B,\xi}(z) = 0$ for all $z \leq 0$ and $|\eta^{B,\xi}| = |\eta| - |\xi|$. We thus let

$$(4.39) \qquad p_k := \inf_{|\eta|=k, \eta(z)=0 \,\forall z \leq 0} \mathbb{P}_\eta(G_\infty \cap \Lambda_\infty).$$

It is clear that

$$(4.40) \qquad p_0 = \mathbb{P}_0(\Lambda_\infty) = P\left(\bar{X}_n \geq \frac{1}{2}v_0 n \,\forall n \in \mathbb{Z}_+\right) > 0.$$

Let $A' = (\bigcup_{i=0}^2 \{(i,i)\}) \cup \{(x,t) \in \mathbb{Z}^2 : t \geq 3, x \geq \frac{1}{2}v_0 t\}$ and $B = \{1, 2\}$. We say that "$S^{z,i}$ avoids $A'$" if $S_n^{z,i} \notin A'$ for all $n \in \mathbb{Z}$. Since $q_0 > 0$,

$$(4.41) \qquad \tilde{p} := \inf_{z \in B} P(S^{z,1} \text{ avoids } A') > 0.$$

We will prove that, for all $k \geq 0$,

$$(4.42) \qquad p_k \geq p_{**}^k \quad \text{where } p_{**} := p_0 \tilde{p}$$

by induction on $k$. Let $|\eta| \geq 1$, $\eta(z) = 0$ for all $z \leq 0$, and assume that (4.42) has been shown for all $k < |\eta|$.

Assume first that $\eta(1) + \eta(2) \geq 1$ and put $\xi(z) = \eta(z)\mathbb{1}_{\{1,2\}}(z)$. Noting that $G_\infty \cap \Lambda_\infty$ is measurable in $\sigma(N(A'), U_{A'})$, use Lemma 4.8 and the induction hypothesis to write

$$(4.43) \qquad \begin{aligned} \mathbb{P}_\eta(G_\infty, \Lambda_\infty) &\geq \mathbb{E}_\eta\left[\prod_{z \in B, i \leq \xi(z)} \mathbb{1}_{\{S^{z,i} \text{ avoids } A'\}} \mathbb{P}_\eta\left(G_\infty \cap \Lambda_\infty \mid (S^{z,i})_{z \in B, i \leq \xi(z)}\right)\right] \\ &\geq \tilde{p}^{|\xi|} p_{|\eta|-|\xi|} \geq \tilde{p} p_{**}^{|\eta|-1} \geq p_{**}^{|\eta|}. \end{aligned}$$

If $\eta(1) + \eta(2) = 0$, let

$$(4.44) \qquad \tau := \inf\{n \in \mathbb{N} : N(\bar{X}_n + 1, n) + N(\bar{X}_n + 2, n) \geq 1\}.$$

Note that $\tau < \infty$ a.s. since $\bar{X}$ has a positive drift while the environment particles are symmetric. Let $\bar{\eta}_\tau(x) = N(\bar{X}_\tau + x, \tau)$ and note that, since the random walks are all 1-Lipschitz, $\bar{\eta}_\tau(z) = 0$ for all $z \leq 0$. Furthermore, $X$ is equal to $\bar{X}$ until time $\tau$ since it meets no environment particles up to this time. Thus, using the Markov property and (4.43) we can write

$$(4.45) \qquad \begin{aligned} \mathbb{P}_\eta(G_\infty \cap \Lambda_\infty) &\geq \mathbb{P}_\eta\left(\Lambda_\tau \cap G_\infty^{(\bar{X}_\tau, \tau)} \cap \Lambda^{(\bar{X}_\tau, \tau)}\right) \\ &= \mathbb{E}_\eta\left[\mathbb{1}_{\Lambda_\tau} \mathbb{P}_{\bar{\eta}_\tau}(G_\infty \cap \Lambda_\infty)\right] \\ &\geq \tilde{p} p_{**}^{|\eta|-1} \mathbb{P}_\eta(\Lambda_\tau) \geq p_{**}^{|\eta|}, \end{aligned}$$

completing the induction step.

We turn now to the case $q_0 = 0$, $p_\bullet > 0$. In this case, we can actually control

$$(4.46) \qquad p_k := \inf_{|\eta|=k} \mathbb{P}_\eta(G_\infty \cap \Lambda_\infty) = \inf_{y \in \mathbb{Z}^2} \inf_{|\eta|=k} \mathbb{P}_\eta(G_\infty^y \cap \Lambda_\infty^y),$$

where the second equality holds by the Markov property, particle conservation and translation invariance. Let $p_{**} := p_\bullet p_0 \hat{p}$ where $p_0$ is as in (4.40) and

$$(4.47) \qquad \hat{p} := P(S^{0,1} \text{ avoids } A''), \qquad A'' := \left\{ (x,t) \in \mathbb{Z}^2 : t \geq 1, x \geq \frac{1}{2} v_\circ t \right\}.$$

Then we can prove (4.42) by induction in a similar way as for the previous case.

Indeed, suppose first that $\eta(0) > 0$. Note that, since $X_1 = 1$ when $U_0 \leq p_\bullet$,

$$(4.48) \qquad \mathbb{P}_\eta(G_\infty \cap \Lambda_\infty)$$

$$\geq \mathbb{P}_\eta\big(U_0 \leq p_\bullet, G_\infty^{(1,1)} \cap \Lambda_\infty^{(1,1)}\big) = p_\bullet \mathbb{P}_\eta\big(G_\infty^{(1,1)} \cap \Lambda_\infty^{(1,1)}\big)$$

$$(4.49) \qquad \geq p_\bullet \mathbb{E}_\eta\bigg[ \prod_{i \leq \eta(0)} \mathbb{1}_{\{S^{0,i} \text{ avoids } A''\}} \mathbb{P}_\eta\big(G_\infty^{(1,1)} \cap \Lambda_\infty^{(1,1)} \mid (S^{0,i})_{i \leq \eta(0)}\big) \bigg].$$

Noting that $G_\infty^{(1,1)} \cap \Lambda_\infty^{(1,1)}$ is measurable in $\sigma(N(A''), U_{A''})$, we may apply Lemma 4.8 with $B = \{0\}$, $\xi = \eta \mathbb{1}_0$ followed by the induction hypothesis to obtain

$$(4.50) \qquad \mathbb{P}_\eta(G_\infty \cap \Lambda_\infty) \geq p_\bullet \hat{p}^{|\eta(0)|} p_{|\eta|-|\eta(0)|} \geq p_\bullet \hat{p} p_{**}^{|\eta|-1}.$$

If $\eta(0) = 0$, define

$$(4.51) \qquad \tau := \inf\{n \in \mathbb{N} : N(\bar{X}_n, n) \geq 1\} \in [1, \infty].$$

Setting $\bar{\eta}_\tau(x) = N(\bar{X}_\tau + x, \tau)$, use the Markov property and (4.50) to write

$$\mathbb{P}_\eta(\tau < \infty, G_\infty \cap \Lambda_\infty) \geq \mathbb{E}_\eta\big[\mathbb{1}_{\{\tau < \infty\}} \mathbb{1}_{\Lambda_\tau} \mathbb{P}_{\eta_\tau}(G_\infty \cap \Lambda_\infty)\big]$$

$$(4.52) \qquad \qquad \geq p_\bullet \hat{p} p_{**}^{|\eta|-1} \mathbb{P}_\eta(\tau < \infty, \Lambda_\tau).$$

Now note that $G_\infty$ occurs if $\tau = \infty$ and use (4.52) to obtain

$$\mathbb{P}_\eta(G_\infty \cap \Lambda_\infty) = \mathbb{P}_\eta(\tau = \infty, \Lambda_\infty) + \mathbb{P}_\eta(\tau < \infty, G_\infty \cap \Lambda_\infty)$$

$$(4.53) \qquad \geq p_\bullet \hat{p} p_{**}^{|\eta|-1}\{\mathbb{P}_\eta(\tau = \infty, \Lambda_\infty) + \mathbb{P}_\eta(\tau < \infty, \Lambda_\infty)\} = p_{**}^{|\eta|},$$

concluding the proof. $\quad\square$

Next we use Lemma 4.7 to show that, if $|\eta|$ is sufficiently small and is empty in an interval of radius $\ell_L$ around 0, then one of the $G_{T_1}^{(0,T_i)}$'s occurs with large probability.

LEMMA 4.9. *There exist $\delta, \varepsilon, c > 0$ such that*

$$(4.54) \qquad \inf_{\substack{\eta: |\eta| \leq \delta \log L, \\ \eta(z)=0 \,\forall z \in [-\ell_L, \ell_L]}} \mathbb{P}_\eta\bigg( \bigcup_{i \in [0, M_L-1]} G_{T_1}^{(0,T_i)} \cap \Lambda_{T_1}^{(0,T_i)} \bigg) \geq 1 - ce^{-c^{-1}L^\varepsilon}.$$

PROOF. For $p_*$ is as in Lemma 4.7, fix $\delta > 0$ such that $\delta \log \frac{1}{p_*} < \alpha - \beta$. Fix $\eta$ with $|\eta| \leq \delta \log L$, $\eta(z) = 0$ for all $z \in [-\ell_L, \ell_L]$.

Put $\eta_t(x) := N(x,t)$ and use the Markov property to write, for $k \geq 0$,

$$(4.55) \qquad \mathbb{P}_\eta\bigg( \bigcap_{i=0}^{k+1} (G_{T_1}^{(0,T_i)} \cap \Lambda_{T_1}^{(0,T_i)})^c \cap \{\eta_{T_{i+1}}(0) = 0\} \bigg)$$

$$\leq \mathbb{E}_\eta\bigg[ \prod_{i=0}^{k} \mathbb{1}_{(G_{T_1}^{(0,T_i)} \cap \Lambda_{T_1}^{(0,T_i)})^c \cap \{\eta_{T_{i+1}}(0)=0\}} \mathbb{P}_{\eta_{T_{k+1}}}\big((G_{T_1} \cap \Lambda_{T_1})^c\big) \bigg].$$

Since $|\eta_{T_{k+1}}| = |\eta| \leq \delta \log L$ and $\eta_{T_{k+1}}(0) = 0$ inside the integral, by Lemma 4.7 we may bound (4.55) from above by

$$(4.56) \qquad (1 - L^{\delta \log p_*}) \mathbb{P}_\eta \left( \bigcap_{i=0}^{k} (G_{T_1}^{(0,T_i)} \cap \Lambda_{T_1}^{(0,T_i)})^c \cap \{\eta_{T_i}(0) = 0\} \right).$$

We conclude by induction that

$$(4.57) \qquad \mathbb{P}_\eta \left( \bigcap_{i=0}^{\lfloor M_L \rfloor - 1} (G_{T_1}^{(0,T_i)} \cap \Lambda_{T_1}^{(0,T_i)})^c \cap \{\eta_{T_i}(0) = 0\} \right) \leq (1 - L^{\delta \log p_*})^{\lfloor M_L \rfloor}$$

$$\leq c e^{-\frac{1}{c} L^{\varepsilon_*}},$$

where $\varepsilon_* := \alpha - \beta + \delta \log p_* > 0$ by our choice of $\delta$. Now, using standard random walk estimates as in the proof of Lemma 4.5, we obtain

$$(4.58) \qquad \mathbb{P}_\eta (\exists t \in [0, L^\alpha] : \eta_t(0) > 0) \leq c e^{-c^{-1} L^{\varepsilon'}}$$

for some $\varepsilon' > 0$, so we may take $\varepsilon := \varepsilon' \wedge \varepsilon_*$. $\quad\square$

Finally, we gather all results of this section to prove Proposition 4.3.

PROOF OF PROPOSITION 4.3. Note that, if $X_- \geq -L^\beta + 1$, then $\Lambda_{T_1}^{(X_-, T_i)} \subset \{\bar{X}_{T_1}^{(X_-, T_i)} \geq L^\beta\}$. Therefore, by Lemma 4.4, it is enough to show that

$$(4.59) \qquad \mathbb{P}^{L^{-\frac{1}{16}}} \left( \bigcap_{i \in [0, M_L - 1]} (G_{T_1}^{(X_-, T_i)} \cap \Lambda_{T_1}^{(X_-, T_i)})^c \cap \{X_- \geq -L^\beta + 1\} \right) \leq c e^{-c^{-1}(\log L)^2}.$$

By a union bound and translation invariance, the left-hand side of (4.59) is at most

$$(4.60) \qquad L^\beta \mathbb{P}^{L^{-\frac{1}{16}}} \left( \bigcap_{i \in [0, M_L - 1]} (G_{T_1}^{(0,T_i)} \cap \Lambda_{T_1}^{(0,T_i)})^c \cap \mathcal{E}_L \right)$$

where $\mathcal{E}_L := \{N(z, 0) = 0 \; \forall z \in [-\ell_L, \ell_L]\}$.

Recalling the definition of $T_i$, $\ell_L$ in (4.4), we note that, since all our random walks are 1-Lipschitz, there exists $c_1 > 0$ such that the indicator functions of $G_{T_1}^{(0,T_i)}$, $\Lambda_{T_1}^{(0,T_i)}$ and $\mathcal{E}_L$ are functionals of $U_A, N(A)$ with $A := [-c_1 L^\beta, c_1 L^\beta] \times [0, L^\alpha] \cap \mathbb{Z}^2$.

Let $B := \mathbb{Z} \setminus [-(c_1 + 1)L^\beta, (c_1 + 1)L^\beta]$, put

$$(4.61) \qquad \widehat{N}_L := \sum_{z \in B} \sum_{i \leq N(z,0)} \mathbb{1}_{\{\exists s \in [0, L^\alpha] : S_s^{z,i} \in [-c_1 L^\beta, c_1 L^\beta]\}}$$

and, analogously to (4.36),

$$(4.62) \qquad \eta^B(x) := \begin{cases} N(x, 0) & \text{if } x \notin B, \\ 0 & \text{otherwise.} \end{cases}$$

Lemmas 4.8 and 4.9 imply that

$$\mathbb{P}^{L^{-\frac{1}{16}}} \left( \bigcap_{i=0}^{\lfloor M_L \rfloor - 1} (G_{T_1}^{(0,T_i)} \cap \Lambda_{T_1}^{(0,T_i)})^c \cap \mathcal{E}_L \right)$$

$$(4.63) \qquad \leq \mathbb{P}_{L^{-\frac{1}{16}}} (\widehat{N}_L > 0) + \mathbb{E}^{L^{-\frac{1}{16}}} \left[ \mathbb{1}_{\mathcal{E}_L} \mathbb{P}_{\eta^B} \left( \bigcap_{i=0}^{\lfloor M_L \rfloor - 1} (G_{T_1}^{(0,T_i)} \cap \Lambda_{T_1}^{(0,T_i)})^c \right) \right]$$

$$\leq \mathbb{P}^{L^{-\frac{1}{16}}} (\widehat{N}_L > 0) + \mathbb{P}^{L^{-\frac{1}{16}}} (|\eta^B| > \delta \log L) + c e^{-c^{-1} L^\varepsilon}.$$

Reasoning as in the proof of Lemma 4.5 (see (4.23)–(4.25)), we obtain

$$\mathbb{P}^{L^{-\frac{1}{16}}}(\widehat{N}_L > 0) \le ce^{-c^{-1}L^{2\beta-\alpha}},$$

(4.64)

while, since $|\eta^B|$ has under $\mathbb{P}^{L^{-\frac{1}{16}}}$ a Poisson law with parameter at most $cL^{-(1/16-\beta)}$,

$$\mathbb{P}^{L^{-\frac{1}{16}}}(|\eta^B| > \delta \log L) \le (cL^{-(1/16-\beta)})^{\delta \log L} \le ce^{-c^{-1}(\log L)^2}.$$

(4.65)

Combining (4.60)–(4.65), we obtain (4.59) and finish the proof. $\square$

### 4.2. Perturbations of impermeable systems.

In this section, we assume $q_0 = 0$. As already mentioned, the main strategy in the proof of Theorem 3.2 is a comparison with an infection model, which we now describe.

Recall the random walks $S^{z,i}$ from Section 2. Define recursively a random process $\xi(z, i, n) \in \{0, 1\}$, $z \in \mathbb{Z}$, $i \in \mathbb{N}$, $n \in \mathbb{N}$ by setting

(4.66)
$$\begin{aligned}
\xi(z, i, 0) &= 1 \quad \text{if } z \ge 0, z \in 2\mathbb{Z} \text{ and } i \le N(z, 0),\\
\xi(z, i, 0) &= 0 \quad \text{otherwise,}
\end{aligned}$$

and, supposing that $\xi(z, i, n)$ is defined for all $z \in \mathbb{Z}$, $i \in \mathbb{N}$,

(4.67)
$$\xi(z, i, n+1) = \begin{cases} 1 & \begin{aligned} &\text{if } i \le N(z, 0) \text{ and}\\ &\exists z' \in \mathbb{Z}, i' \in \mathbb{N} \text{ with } \eta(z', i', n) = 1, S_n^{z',i'} = S_n^{z,i},\end{aligned}\\ 0 & \text{otherwise.} \end{cases}$$

The interpretation is that, if $\xi(z, i, n) = 1$, then the particle $S^{z,i}$ is *infected* at time $n$, and otherwise it is *healthy*. Then (4.67) means that, whenever a group of particles shares a site at time $n$, if one of them is infected then all will be infected at time $n + 1$.

We are interested in the process $\bar{X} = (\bar{X}_n)_{n \in \mathbb{Z}_+}$ defined by

(4.68)
$$\bar{X}_n = \min\{S_n^{z,i} : z \in \mathbb{Z}, i \le N(z, 0) \text{ and } \xi(z, i, n) = 1\},$$

that is, $\bar{X}_n$ is the leftmost infected particle at time $n$. We call $\bar{X}$ the *front of the infection*.

Note that, by (4.66) and since $q_0 = 0$, all infected particles live on $2\mathbb{Z}$. In particular, $\bar{X}_n \in 2\mathbb{Z}$ for all $n \ge 0$. This implies the following.

LEMMA 4.10. *If $p_\bullet = 0$, then $X_n \le \bar{X}_n$ for all $n \ge 0$.*

PROOF. Since the processes are one-dimensional, proceed by nearest-neighbour jumps, are ordered at time 0 and the difference in their positions lies in $2\mathbb{Z}$, we only need to consider what happens at times $s$ when $X_s = \bar{X}_s$. For such times, $X_{s+1} = X_s - 1$ since $p_\bullet = 0$, and thus $X_{s+1} \le \bar{X}_{s+1}$. $\square$

The advantage of the comparison above becomes clear in light of the following.

PROPOSITION 4.11. *For any $\hat{\rho} > 0$, there exist $\hat{v} < 0$, $c > 0$ such that*

(4.69)
$$\mathbb{P}^{\hat{\rho}}(\bar{X}_L > \hat{v}L) \le c \exp\{-(\log L)^{3/2}/c\} \quad \forall L \in \mathbb{N}.$$

PROOF. This is just an adaptation of the statement of Proposition 1.2 of [12] once we map $2\mathbb{Z}$ to $\mathbb{Z}$ and apply reflection symmetry. Its proofs follow exactly the same lines as in there. $\square$

We are now ready to finish the:

PROOF OF THEOREM 3.2. Fix $\hat{\rho} > 0$ and $\hat{L} \in \mathbb{N}$. Suppose first that $p_\bullet = 0$. By Lemma 4.10 and Proposition 4.11, there exist $\hat{v} < 0$, $c > 0$ independent of $\hat{L}$ such that

$$(4.70) \qquad \mathbb{P}^{\hat{\rho}}(X_{\hat{L}} > \hat{v}\hat{L}) \leq \mathbb{P}^{\hat{\rho}}(\bar{X}_{\hat{L}} > \hat{v}\hat{L}) \leq ce^{-(\log \hat{L})^{3/2}/c}.$$

Note now that, since $X_{\hat{L}}$ is supported in a finite space-time box, the probability in the left-hand side of (4.70) is a continuous function of $p_\bullet$. Thus we can find $p_\star > 0$ such that, if $p_\bullet \leq p_\star$, then (4.70) holds with $c$ replaced by $2c$, concluding the proof. $\square$

**5. Regeneration: Proof of Theorem 1.4.** In this section, we extend the results of Section 4 of [21] to the case $v_\bullet < v_\circ$ and give the proof of Theorem 1.4 under the conditions of item (a). The definition of the regeneration time $\tau$ given next is exactly as in [21]. Differences appear only in Section 5.1, where the tail of $\tau$ is controlled.

Fix $\rho > 0$. We assume that (1.5) holds with $v_\star > 0$ and some $\gamma > 1$. We assume additionally that $p_\bullet > 0$. In the sequel, we abbreviate $\mathbb{P} = \mathbb{P}^\rho$.

Define $\bar{v} = \frac{1}{3}v_\star$. For $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, let $\angle(x, n)$ be the cone in the first quadrant based at $(x, n)$ with angle $\bar{v}$, that is,

$$(5.1) \qquad \angle(x, n) = \angle(0, 0) + (x, n), \quad \text{where } \angle(0, 0) = \{(x, n) \in \mathbb{Z}_+^2; x \geq \bar{v}n\},$$

and $\diagdown(x, n)$ the cone in the third quadrant based at $(x, n)$ with angle $\bar{v}$, that is,

$$(5.2) \qquad \diagdown(x, n) = \diagdown(0, 0) + (x, n), \quad \text{where } \diagdown(0, 0) = \{(x, n) \in \mathbb{Z}_-^2 : x < \bar{v}n\}.$$

(See Figure 2.) Note that $(0, 0)$ belongs to $\angle(0, 0)$ but not to $\diagdown(0, 0)$.

Fixed $y \in \mathbb{Z}^2$, define the following sets of trajectories in $W$:

$$
\begin{aligned}
W_y^{\angle} &= \text{trajectories that intersect } \angle(y) \text{ but not } \diagdown(y), \\
(5.3) \qquad W_y^{\diagdown} &= \text{trajectories that intersect } \diagdown(y) \text{ but not } \angle(y), \\
W_y^+ &= \text{trajectories that intersect both } \angle(y) \text{ and } \diagdown(y).
\end{aligned}
$$

Note that $W_y^{\angle}$, $W_y^{\diagdown}$ and $W_y^+$ form a partition of $W$. We write $Y_n$ to denote $Y_n^0$. For $y \in \mathbb{Z}^2$, define the sigma-algebras

$$(5.4) \qquad \mathcal{G}_y^I = \sigma(\omega(A) : A \subset W_y^I, A \in \mathcal{W}), \quad I = \angle, \diagdown, +,$$

and note that these are jointly independent under $\mathbb{P}$. Define also the sigma-algebras

$$(5.5) \qquad
\begin{aligned}
\mathcal{U}_y^{\angle} &= \sigma(U_z : z \in \angle(y)), \\
\mathcal{U}_y^{\diagdown} &= \sigma(U_z : z \in \diagdown(y)),
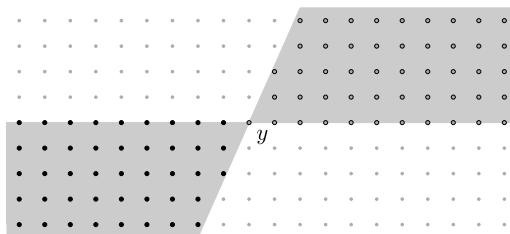\end{aligned}
$$



FIG. 2. *An illustration of the sets $\angle(y)$ (represented by white circles) and $\diagdown(y)$ (represented by filled black circles), with $y = (x, n) \in \mathbb{Z}^2$.*

and set

(5.6) $$\mathcal{F}_y = \mathcal{G}_y^{\nearrow} \vee \mathcal{G}_y^+ \vee \mathcal{U}_y^{\nearrow}.$$

Next, define the *record times*

(5.7) $$R_k = \inf\{n \in \mathbb{Z}_+ : X_n \geq (1 - \bar{v})k + \bar{v}n\}, \quad k \in \mathbb{N},$$

that is, the time when the walk first enters the cone

(5.8) $$\angle_k := \angle((1 - \bar{v})k, 0).$$

Note that, for any $k \in \mathbb{N}$, $y \in \angle_k$ if and only if $y + (1, 1) \in \angle_{k+1}$. Thus $R_{k+1} \geq R_k + 1$, and $X_{R_{k+1}} - X_{R_k} = 1$ if and only if $R_{k+1} = R_k + 1$.

Define a filtration $\mathcal{F} = (\mathcal{F}_k)_{k \in \mathbb{N}}$ by setting

(5.9) $$\mathcal{F}_k = \{B \in \sigma(\omega, U) : \forall y \in \mathbb{Z}^2, \exists B_y \in \mathcal{F}_y \text{ s.t. } B \cap \{Y_{R_k} = y\} = B_y \cap \{Y_{R_k} = y\}\},$$

that is, $\mathcal{F}_k$ is the sigma-algebra generated by $Y_{R_k}$, all $U_z$ with $z \in \nearrow(Y_{R_k})$ and all $\omega(A)$ such that $A \subset W_{Y_{R_k}}^{\nearrow} \cup W_{Y_{R_k}}^+$. In particular, $(Y_i)_{0 \leq i \leq R_k} \in \mathcal{F}_k$.

Finally, define the event

(5.10) $$A^y = \{Y_i^y \in \angle(y) \; \forall i \in \mathbb{Z}_+\},$$

in which the walker remains inside the cone $\angle(y)$, the probability measure

(5.11) $$\mathbb{P}^{\angle}(\cdot) = \mathbb{P}(\cdot | \omega(W_0^+) = 0, A^0),$$

the *regeneration record index*

(5.12) $$\mathcal{I} = \inf\{k \in \mathbb{N} : \omega(W_{Y_{R_k}}^+) = 0, A^{Y_{R_k}} \text{ occurs}\}$$

and the *regeneration time*

(5.13) $$\tau = R_{\mathcal{I}}.$$

The following two theorems are our key results for the regeneration time.

THEOREM 5.1. *Almost surely on the event* $\{\tau < \infty\}$, *the process* $(Y_{\tau+i} - Y_\tau)_{i \in \mathbb{Z}_+}$ *under either the law* $\mathbb{P}(\cdot \mid \tau, (Y_i)_{0 \leq i \leq \tau})$ *or* $\mathbb{P}^{\angle}(\cdot \mid \tau, (Y_i)_{0 \leq i \leq \tau})$ *has the same distribution as that of* $(Y_i)_{i \in \mathbb{Z}_+}$ *under* $\mathbb{P}^{\angle}(\cdot)$.

THEOREM 5.2. *There exists a constant* $c_0 > 0$ *such that*

(5.14) $$\mathbb{E}[e^{c_0 (\log \tau)^\gamma}] < \infty$$

*and the same holds under* $\mathbb{P}^{\angle}$.

Theorem 5.1 is proved exactly as in [21]. Theorem 5.2 was proved in [21] in the nonnestling case and in the case $v_\bullet \geq v_\circ$. In the following Section 5.1, we will fill the remaining gap by showing that it also holds when $v_\circ > 0 \geq v_\bullet$.

We may now conclude the:

PROOF OF THEOREM 1.4.    Defining

(5.15) $$v = \frac{\mathbb{E}^{\angle}[X_\tau]}{\mathbb{E}^{\angle}[\tau]}, \qquad \sigma^2 = \frac{\mathbb{E}^{\angle}[(X_\tau - \tau v)^2]}{\mathbb{E}^{\angle}[\tau]},$$

one may follow the proof of Theorem 1.4 in [21] (Section 4.3 therein).    □

REMARK 5.3 (Continuity of $v$ and $\sigma$). Careful inspection of the proof of Theorem 5.2 will reveal the following uniformity property: fix $q_0$, $p_\circ$, $p_\bullet > 0$ and assume that (1.5) holds for some $\rho_\star, v_\star > 0$ (e.g., as given by Theorem 1.2). Then the expectation in (5.14) is uniformly bounded over $\rho \in [0, \rho_\star]$. Together with (5.15), this may be used to show that both $v$ and $\sigma$ are continuous functions of $\rho$ in the interval $[0, \rho_\star]$. In the interest of brevity, we will not pursue this here but only sketch a proof strategy. For a full proof of similar statements, see [17], Section 6.4. The key points are as follows: first note that the expectation of any bounded local function of $N, U$ is continuous in $\rho$ (where local means that the function only depends on the values of $N, U$ inside a finite space-time box). This can be shown by simultaneously coupling systems with all values of $\rho$, and applying the dominated convergence theorem. The crux of the argument is thus to approximate (uniformly over $\rho \in [0, \rho_\star]$) the expectations in (5.15) by expectations of bounded local functions, which can be done using the uniform version of (5.14) together with the Lipschitz property of all random walks involved.

5.1. *Proof of Theorem 5.2.* In this section, we control the tail of the regeneration time $\tau$ in the case $p_\bullet > 0$, $v_\bullet < v_\circ$, in particular proving Theorem 5.2. The proof strategy is the same as in [21]; the main technical difference is the definition of good record times given in (5.26)–(5.29) below. Otherwise, the proof proceeds almost identically as in [21]. For completeness, we provide all details here.

Let us motivate why a different definition of good record times is necessary. The reason stems from the different monotonicity properties of the model with respect to the presence of particles. Indeed, when $v_\bullet > v_\circ$, adding particles to the system always pushes the random walker to the right, so that, when estimating from below conditional probabilities for the random walker to move ballistically, one may ignore particles it met in the past, thus losing memory. When $v_\circ > v_\bullet$, monotonicity is reversed, so this strategy is no longer feasible. Instead, we control conditional probabilities for auxiliary random walkers that, by construction, ignore all particles from the distant past. Since meeting such particles is extremely unlikely, it turns out that, when the record time is good, the true random walker equals the auxiliary random walker with overwhelming probability. Details are carried out below.

In what follows, constants may depend on $v_\circ$, $v_\bullet$, $v_\star$ and $\rho$. Define the *influence field at a point $y \in \mathbb{Z}^2$* as

$$(5.16) \qquad h(y) = \inf\{l \in \mathbb{Z}_+ : \omega(W_y^+ \cap W_{y+(l,l)}^+) = 0\}.$$

LEMMA 5.4 (Lemma 4.3 of [21]). *There exist constants $c_1, c_2 > 0$ (depending on $v_\star, \rho$ only) such that, for all $y \in \mathbb{Z}^2$,*

$$(5.17) \qquad \mathbb{P}[h(y) > l] \le c_1 e^{-c_2 l}, \quad l \in \mathbb{Z}_+.$$

Set

$$(5.18) \qquad \delta = \frac{1}{4\log(\frac{1}{p_\bullet})}, \qquad \epsilon = \frac{1}{4}(c_2\delta \wedge 1),$$

and put, for $T > 1$,

$$(5.19) \qquad T' = \lfloor T^\epsilon \rfloor, \qquad T'' = \lfloor \delta \log T \rfloor.$$

Define the *local influence field at $(x, n)$* as

$$(5.20) \qquad h^T(x, n) = \inf\{l \in \mathbb{Z}_+ : \omega(W_{x-\lfloor (1-\bar{v})\rfloor T',n}^{\angle} \cap W_{x,n}^+ \cap W_{x+l,n+l}^+) = 0\}.$$

Then we have the following.

LEMMA 5.5 (Lemma 4.4 of [21]).   *For all $T > 1$ it holds $\mathbb{P}$-a.s. that*

$$(5.21) \qquad \mathbb{P}\big(h^T(y) > l \mid \mathcal{F}_{y-(\lfloor(1-\bar{v})\rfloor T',0)}\big) \le c_1 e^{-c_2 l} \quad \forall y \in \mathbb{Z}^2, l \in \mathbb{Z}_+,$$

*where $c_1$, $c_2$ are the same constants of Lemma 5.4.*

For $y \in \mathbb{Z}^2$, denote by

$$(5.22) \qquad \kappa(y) := \max\{k \in \mathbb{N}: y \in \angle_k\}$$

the index of the last cone containing $y$. Note that $\kappa(Y_{R_k}) = k$. Then define, for $t \in \mathbb{N}$, the space-time parallelogram

$$(5.23) \qquad \mathcal{P}_t(y) = \big(\angle(y) \setminus \angle_{\kappa(y)+t}\big) \cap \big(y + \{(x, n) \in \mathbb{Z}^2: n \le t/\bar{v}\}\big)$$

and its right boundary

$$(5.24) \qquad \partial^+\mathcal{P}_t(y) = \{z \in \mathbb{Z}^2 \setminus \mathcal{P}_t(y): z - (1,0) \in \mathcal{P}_t(y)\}.$$

We say that "$Y^y$ exits $\mathcal{P}_t(y)$ through the right" when the first time $i$ at which $Y_i^y \notin \mathcal{P}_t(y)$ satisfies $Y_i^y \in \partial^+\mathcal{P}_t(y)$. Note that, if $y = Y_{R_k}$, this implies $Y_i^y = Y_{R_{k+t}}$.

In order to adapt the argument in [21], we will need to modify the definition of good record times given there. For this, we need some additional definitions.

For $y \in \mathbb{Z}^2$, let

$$(5.25) \qquad \widetilde{W}_y := \bigcup_{z \in \partial^+\mathcal{P}_{T'}(y)} W^{\angle}_{z-(\lfloor(1-\bar{v})\rfloor T',0)} \cap W^+_z \cap W^+_{z+(T'',T'')}$$

and, for $y_1, y_2 \in \mathbb{Z}^2$, denote by $\widetilde{\mathcal{T}}_{y_1,y_2}$ the trace of all trajectories in $\omega$ that do not belong to $\widetilde{W}_{y_1}$ or intersect $\diagup(y_2)$. Let $\widetilde{Y}^{y_1,y_2}$ be the analogous of $Y^{y_2}$ defined using $\widetilde{\mathcal{T}}_{y_1,y_2}$ instead of $\mathcal{T}$. Note that, since $v_\circ > v_\bullet$, by monotonicity we have $\widetilde{X}_t^{y_1,y_2} \ge X_t^{y_2}$ for all $y_1, y_2 \in \mathbb{Z}^2$ and $t \in \mathbb{Z}_+$.

We say that $R_k$ is a *good record time (g.r.t.)* when

$$(5.26) \qquad h^T(y) \le T'' \quad \forall y \in \partial^+\mathcal{P}_{T'}(Y_{R_{k-T'}}),$$

$$(5.27) \qquad U_{Y_{R_k}+(l,l)} \le p_\bullet \quad \forall l = 0, \dots, T''-1,$$

$$(5.28) \qquad \omega\big(W^{\angle}_{\widetilde{Y}_{R_k}} \cap W^+_{Y_{R_k}+(T'',T'')}\big) = 0,$$

$$(5.29) \qquad \qquad \widetilde{Y}^k \text{ exits } \mathcal{P}_{T'}(Y_{R_k+(T'',T'')}) \text{ through the right},$$

where $\widetilde{Y}^k := \widetilde{Y}^{y_1,y_2}$ with $y_1 = Y_{R_{k-T'}}$, $y_2 = Y_{R_k+(T'',T'')}$. Note that (5.26) is the same as $\{\omega(\widetilde{W}_{Y_{R_{k-T'}}}) = 0\}$ and that, when (5.27) happens, $Y_{R_{k+T''}} = Y_{R_k} + (T'',T'')$. Figure 3 illustrates a good record time. The main differences with respect to the analogous definition in [21] are:

1. In (5.26), we require a small local field not exactly at $Y_{R_k}$ but in every point of $\partial^+\mathcal{P}_{T'}(Y_{R_{k-T'}})$, a set to which $Y_{R_k}$ belongs with large probability.

2. We do not require (5.29) for $Y$ but only for $\widetilde{Y}$; we will see that, if the record time is good, then the same holds for $Y$ with large probability.

We will need the following consequence of (1.5).

LEMMA 5.6.

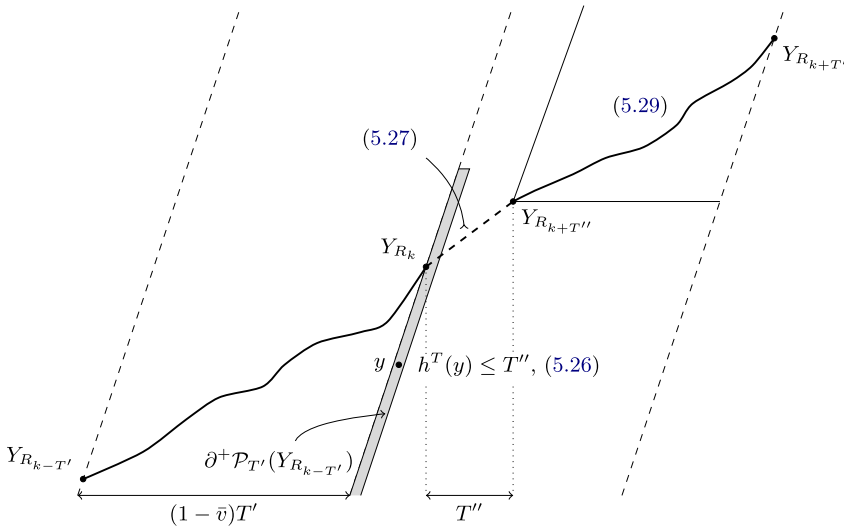$$(5.30) \qquad \mathbb{P}(X_n \ge nv_\star \ \forall n \in \mathbb{Z}_+) > 0.$$

FIG. 3.   *Illustration of a good record time $R_k$. Note the validity of the conditions* (5.26), (5.27) *and* (5.29). *The gray region illustrates the set* $\partial^+\mathcal{P}_{T'}(Y_{R_{k-T'}})$.

PROOF.   Fix $L > 1$ large enough such that

$$(5.31) \qquad \mathbb{P}\big(\exists n \in \mathbb{Z}_+ \colon X_n < nv_\star - L(1 - v_\star)\big) \leq \frac{1}{2},$$

which is possible by (1.5). If $t > L$, then

(5.32)

$$\mathbb{P}(X_n \geq nv_\star \ \forall n \in \mathbb{Z}_+)$$

$$\geq \mathbb{P}\big(U_{(i,i)} \leq p_\bullet \ \forall i = 0, \ldots, L-1, \ X_n^{(L,L)} - L \geq nv_\star - (1-v_\star)L \ \forall n \in \mathbb{Z}_+\big)$$

$$= p_\bullet^L\{1 - \mathbb{P}\big(\exists n \in \mathbb{Z}_+ \colon X_n < nv_\star - (1-v_\star)L\big)\}$$

$$\geq \frac{1}{2}p_\bullet^L > 0$$

as desired.   □

As in [21], the following proposition is the main step to control the tail of the regeneration time.

PROPOSITION 5.7.   *There exists a constant $c_3 > 0$ such that, for all $T > 1$ large enough,*

$$(5.33) \qquad \mathbb{P}[R_k \text{ is not a g.r.t. for all } 1 \leq k \leq T] \leq e^{-c_3\sqrt{T}}.$$

PROOF.   First we claim that there exists a $c > 0$ such that, for any $k \geq T'$,

$$(5.34) \qquad \mathbb{P}[R_k \text{ is a g.r.t.}|\mathcal{F}_{k-T'}] \geq cT^{\delta \log(p_\bullet)} \quad \text{a.s.}$$

To prove (5.34), we will find $c > 0$ such that

$$(5.35) \qquad \mathbb{P}\big[(5.26)|\mathcal{F}_{k-T'}\big] \geq c \quad \text{a.s.,}$$

$$(5.36) \qquad \mathbb{P}\big[(5.27)|(5.26), \mathcal{F}_{k-T'}\big] \geq T^{\delta \log(p_\bullet)} \quad \text{a.s.,}$$

$$(5.37) \qquad \mathbb{P}\big[(5.28)|(5.26), (5.27), \mathcal{F}_{k-T'}\big] \geq c \quad \text{a.s.,}$$

$$(5.38) \qquad \mathbb{P}\big[(5.29)|(5.26), (5.27), (5.28), \mathcal{F}_{k-T'}\big] \geq c \quad \text{a.s.}$$

(5.35): Fix $B \in \mathcal{F}_{k-T'}$. Summing over the values of $Y_{R_{k-T'}}$ and using a union bound we may write

$$(5.39) \qquad \mathbb{P}((5.26)^c, B) \leq \sum_{y_1 \in \mathbb{Z}^2} \sum_{y_2 \in \partial^+ \mathcal{P}_{T'}(y_1)} \mathbb{P}(h^T(y_2) > T'', Y_{R_{k-T'}} = y_1, B_{y_1}).$$

Noting that $y_2 - (\lfloor(1 - \bar{v})\rfloor)T', 0) - y_1 \in \mathbb{Z}_+^2$ for large enough $T$, we may use Lemma 5.5 and $|\partial^+ \mathcal{P}_t(y)| \leq t/\bar{v}$ to further bound (5.39) by

$$(5.40) \qquad \frac{c_1}{\bar{v}} T' e^{-c_2 T''} \mathbb{P}(B) \leq \frac{c_1}{\bar{v}} e^{c_2} T^{-\frac{3}{4}\delta c_2} \mathbb{P}(B),$$

where the last inequality uses the definition of $\epsilon$. Thus, for $T$ large enough, (5.35) is satisfied with for example, $c = 1/2$.

(5.36): This follows from the fact that $(U_{Y_{R_k}+(l,l)})_{l \in \mathbb{N}_0}$ is independent of the sigma-algebra $\sigma(\omega(A) : A \subset \widetilde{W}_{Y_{R_{k-T'}}}) \vee \mathcal{F}_k$ with respect to which (5.26) is measurable.

(5.37): We may ignore the conditioning on (5.27) since this event is independent of the others. Since (5.26) is equivalent to $\omega(\widetilde{W}_{Y_{R_{k-T'}}}) = 0$, for $B \in \mathcal{F}_{k-T'}$ we may write

$$
\begin{aligned}
\mathbb{P}((5.28), (5.26), B) &= \mathbb{P}(\omega(W_{Y_{R_k}}^{\swarrow} \cap W_{Y_{R_{k+T''}}}^+ \setminus \widetilde{W}_{Y_{R_{k-T'}}}) = 0, (5.26), B) \\
&= \sum_{\substack{y_1, y_2 \in \mathbb{Z}^2: \\ y_2 - y_1 \in \mathbb{N}^2}} \mathbb{P}(\omega(W_{y_2}^{\swarrow} \cap W_{y_2+(T'',T'')}^+ \setminus \widetilde{W}_{y_1}) = 0, \\
&\qquad\qquad Y_{R_k} = y_2, Y_{R_{k-T'}} = y_1, \omega(\widetilde{W}_{y_1}) = 0, B_{y_1}) \\
&= \sum_{\substack{y_1, y_2 \in \mathbb{Z}^2: \\ y_2 - y_1 \in \mathbb{N}^2}} \mathbb{P}(\omega(W_{y_2}^{\swarrow} \cap W_{y_2+(T'',T'')}^+ \setminus \widetilde{W}_{y_1}) = 0) \\
&\qquad\qquad \times \mathbb{P}(Y_{R_k} = y_2, Y_{R_{k-T'}} = y_1, \omega(\widetilde{W}_{y_1}) = 0, B_{y_1}) \\
&\geq \mathbb{P}(\omega(W_0^+) = 0)\mathbb{P}((5.26), B),
\end{aligned}
$$

(5.41)

where the second equality uses the independence between $\sigma(\omega(A) : A \subset W_{y_2}^{\swarrow} \setminus \widetilde{W}_{y_1})$ and $\mathcal{F}_{y_2} \vee \sigma(\omega(A) : A \subset \widetilde{W}_{y_1})$, and the last step uses the monotonicity and translation invariance of $\omega$.

(5.38): We may again ignore (5.27) in the conditioning since this event is independent of all the others. Note that $(5.26) \cap \{Y_{R_{k-T'}} = y\} = (5.26)_y \cap \{Y_{R_{k-T'}} = y\}$ where $(5.26)_y \in \sigma(\omega(A) : A \subset \widetilde{W}_y)$, and similarly $(5.28) \cap \{Y_{R_k} = y\} = (5.28)_y \cap \{Y_{R_k} = y\}$ with $(5.28)_y \in \mathcal{F}_{y+(T'',T'')}$. Now take $B \in \mathcal{F}_{k-T'}$ and write

$$
\begin{aligned}
&\mathbb{P}((5.29), (5.28), (5.26), B) \\
&\qquad = \sum_{\substack{y_1, y_2 \in \mathbb{Z}^2: \\ y_2 - y_1 \in \mathbb{N}^2}} \mathbb{P}(\widetilde{Y}^{y_1, y_2+(T'',T'')} \text{ exits } \mathcal{P}_{T'}(y_2 + (T'', T'')) \text{ through the} \\
&\qquad\qquad \text{right}, Y_{R_k} = y_2, Y_{R_{k-T'}} = y_1, (5.28)_{y_2}, (5.26)_{y_1}, B_{y_1}).
\end{aligned}
$$

(5.42)

Since $\widetilde{Y}^{y,z}$ is independent of $\mathcal{F}_z \vee \sigma(\omega(A)\colon A \subset \widetilde{W}_y)$, the last line equals

$$\sum_{\substack{y_1,y_2 \in \mathbb{Z}^2: \\ y_2-y_1 \in \mathbb{N}^2}} \mathbb{P}\bigl(\widetilde{Y}^{y_1,y_2+(T'',T'')} \text{ exits } \mathcal{P}_{T'}\bigl(y_2 + (T'',T'')\bigr) \text{ through the right}\bigr)$$

(5.43)
$$\times \mathbb{P}\bigl(Y_{R_k} = y_2,\, Y_{R_{k-T'}} = y_1,\, (5.28)_{y_2},\, (5.26)_{y_1},\, B_{y_1}\bigr)$$

$$\geq \mathbb{P}(X_n \geq n v_\star \forall n \in \mathbb{Z}_+)\mathbb{P}\bigl((5.28),(5.26),B\bigr),$$

where for the last step we use $\widetilde{X}_t^{y,z} \geq X_t^z$ and translation invariance. Now (5.38) follows from (5.43) and Lemma 5.6.

Thus, (5.34) is verified. To conclude, note that $\{R_k \text{ is a g.r.t.}\} \in \mathcal{F}_{k+\bar{c}T'}$ for some $\bar{c} \in \mathbb{N}$ independent of $T$. Indeed, this can be verified for each (5.26)–(5.29) using the observation that, if an event $A \in \mathcal{F}_\infty$ satisfies $A \cap \{Y_{R_k} = y\} = A_y \cap \{Y_{R_k} = y\}$ with $A_y \in \mathcal{F}_{y+(t,t)}$, then $A \in \mathcal{F}_{k+t+1}$. Hence we obtain

$$\mathbb{P}(R_k \text{ is not a g.r.t. for any } k \leq T)$$

(5.44)
$$\leq \mathbb{P}\biggl(R_{(\bar{c}+1)kT'} \text{ is not a g.r.t. for any } k \leq \frac{T}{(\bar{c}+1)T'}\biggr)$$

$$\leq \exp\biggl\{-\frac{c}{\bar{c}+1}\frac{T^{1+\delta}\log(p_\circ \wedge p_\bullet)}{T'}\biggr\} \leq \exp\biggl\{-\frac{c}{\bar{c}+1}T^{\frac{1}{2}}\biggr\}$$

by our choice of $\epsilon$ and $\delta$. $\quad\square$

With the previous results in place, only few modifications to [21] are necessary to conclude the proof of Theorem 5.2. The details are given next.

PROOF OF THEOREM 5.2. Since $\mathbb{P}^{\angle}(\cdot) = \mathbb{P}(\cdot|\Gamma_0)$ with $\mathbb{P}(\Gamma_0) > 0$, it is enough to prove the statement under $\mathbb{P}$. To that end, define the subsets

$$\mathcal{H}_{v_\star,t} := \bigl\{(x,n) \in \mathbb{Z}^2\colon x < -t + v_\star n\bigr\}, \quad t \in \mathbb{N},$$

and, for $y \in \mathbb{Z}^2$, the events

$$\bigl\{Y^y \text{ touches } y + \mathcal{H}_{v_\star,t}\bigr\} := \bigl\{\exists n \in \mathbb{N}\colon Y_n^y - y \in \mathcal{H}_{v_\star,t}\bigr\}.$$

Note that, by (1.1), there is a constant $c > 0$ such that

(5.45)
$$\mathbb{P}^\rho\bigl(Y^y \text{ touches } y + \mathcal{H}_{v_\star,\lfloor T'/2\rfloor}\bigr) \leq c^{-1}e^{-c(\log T)^\gamma} \quad \text{for all } T > 1.$$

Define the events

(5.46)
$$E_1 = \bigl\{\exists y \in [-T,T] \times [0,T] \cap \mathbb{Z}^2\colon h(y) \geq T'/2\bigr\},$$
$$E_2 = \bigl\{\exists y \in [-T,T] \times [0,T] \cap \mathbb{Z}^2\colon Y^y \text{ touches } y + \mathcal{H}_{v_\star,\lfloor T'/2\rfloor}\bigr\}.$$

By Lemma 5.4, (5.45) and a union bound, there is a $c > 0$ such that

(5.47)
$$\mathbb{P}(E_1 \cup E_2) \leq c^{-1}e^{-c(\log T)^\gamma} \quad \forall T > 1.$$

Let us show that, for $T$ large enough and on the event $E_1^c \cap E_2^c$, if $R_k$ is a g.r.t. with $k \leq \bar{v}T$ then $\tau \leq R_{k+T''} \leq T$. Indeed, if $T'' < T' < \bar{v}T/2$, then on $E_2^c$ it must be that $R_{\lfloor \bar{v}T\rfloor+T''} \leq T$, as otherwise $Y$ would touch $\mathcal{H}_{v_\star,\lfloor T'/2\rfloor}$. Thus we only need to verify that, under the conditions stated,

(5.48)
$$\omega\bigl(W_{Y_{R_{k+T''}}}^+\bigr) = 0$$

and

(5.49) $$A^{Y_{R_k+T''}} \quad \text{occurs.}$$

Note that, on $E_2^c$, $Y_{R_k} \in \partial^+ \mathcal{P}_{T'}(Y_{R_{k-T'}})$. Together with (5.26) this implies that $\widetilde{\mathcal{T}}_{y,z}$ coincides with $\mathcal{T}$ inside $\angle(z)$, where $y = Y_{R_{k-T'}}$ and $z = Y_{R_{k+T''}}$. It follows that, on $E_2^c$ and by (5.29), $Y_t^z = \widetilde{Y}_t^k \in \angle(z)$ for all $t \in \mathbb{Z}_+$, that is, $A^{Y_{R_k+T''}}$ occurs. Now note that, given (5.28), (5.48) is equivalent to

(5.50) $$\omega\left(W_{Y_{R_k}}^+ \cap W_{Y_{R_k+(T'',T'')}}^+\right) = 0.$$

But since

$$W_{Y_{R_k}}^+ \cap W_{Y_{R_k+(T'',T'')}}^+ = \left(W_{Y_{R_k}}^+ \cap W_{Y_{R_k}-(\lfloor(1-\bar{v})T'\rfloor,0)}^{\angle} \cap W_{Y_{R_k+(T'',T'')}}^+\right)$$
$$\cup \left(W_{Y_{R_k}}^+ \cap W_{Y_{R_k}-(\lfloor(1-\bar{v})T'\rfloor,0)}^+ \cap W_{Y_{R_k+(T'',T'')}}^+\right),$$

(5.50) follows on $E_1^c$ from the facts that $Y_{R_k} \in \partial^+ \mathcal{P}_{T'}(Y_{R_{k-T'}})$, and that (5.26) is equivalent to $\omega(\widetilde{W}_{Y_{R_{k-T'}}}) = 0$ (recall (5.25)).

To conclude, use (5.46) and Proposition 5.7 to write, for $T$ large enough,

$$\mathbb{P}(\tau > T) \leq \mathbb{P}(E_1 \cup E_2) + \mathbb{P}(R_k \text{ is not a g.r.t. } \forall k \leq \bar{v}T)$$

(5.51) $$\leq c^{-1} e^{-c(\log T)^\gamma} + e^{-c_3(\bar{v}T)^{1/2}}. \qquad \square$$

## APPENDIX

In this Appendix we state a result taken from [12] that is used as a tool in the proofs of Theorems 1.2 and 1.3. Its proof is presented in [12] and will not be reproduced here.

First we need to define a sequence of scales which is used in the renormalization scheme in [12]:

(A.1) $$L_0 = 10^{50} \quad \text{and} \quad L_{k+1} = \lfloor L_k^{1/2} \rfloor L_k, \quad \text{for } k \geq 0.$$

Recall the definition of nonincreasing and nondecreasing events in (2.1).

COROLLARY A.1 (Corollary 3.11 from [12]). *Let* $(L_k)_{k \in \mathbb{Z}_+}$ *be given by* (A.1). *For any* $\gamma \in (1, 3/2]$, *there exists an index* $k_o = k_o(\gamma, d) \in \mathbb{N}$ *satisfying the following. Fix two functions* $g : \overline{\Omega} \to [-1, 1]$, $H : \overline{\Omega} \times \mathbb{Z}^d \to \{0, 1\}$ *and two nonnegative sequences* $v(L), \rho(L)$. *Assume that, for some* $L_* \in \mathbb{N}$ *and all* $L \geq L_*$, $v(L) \wedge \rho(L) \geq L^{-1/16}$ *and, for any* $\hat{v} > 0$, *the event*

(A.2) $$\{\text{there exists an } (L, H)\text{-crossing } \Gamma \text{ such that } \chi_\Gamma^g \leq \hat{v}\}$$

*is measurable in* $\sigma(N(y), U_y : y \in [-L, 2L) \times [0, L))$ *and nonincreasing (respectively nondecreasing). Assume additionally that, for some* $\hat{k} \geq k_o$ *such that* $L_{\hat{k}} \geq L_*$,

$$\mathbb{P}^{\rho(L_{\hat{k}})}(\exists \text{ an } (L_{\hat{k}}, H)\text{-crossing } \Gamma \text{ such that } \chi_\Gamma^g \leq v(L_{\hat{k}})) \leq \exp(-(\log L_{\hat{k}})^\gamma).$$

*Then there exist (explicit)* $\rho_\infty, v_\infty > 0$ *such that, for each* $\varepsilon > 0$,

$$\mathbb{P}^\rho(\exists \text{ an } (L, H)\text{-crossing } \Gamma \text{ such that } \chi_\Gamma^g \leq v_\infty - \varepsilon) \leq c^{-1} \exp(-c(\log L)^\gamma)$$

*for some* $c \in (0, \infty)$, *all* $L \in \mathbb{N}$ *and all* $\rho \geq \rho_\infty$ *(respectively, all* $\rho \leq \rho_\infty$).

REMARK A.2. The expression of $\rho_\infty$ is given explicitly in the proof of [12], Corollary 3.11). For this article, all we need to know is that $\rho_\infty \leq \alpha \rho(L_{\hat{k}})$ in the nonincreasing case and $\rho_\infty \geq \alpha^{-1} \rho(L_{\hat{k}})$ in the nondecreasing case, where $\alpha$ is an explicit universal constant.

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